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B.A. PART - III **PAPER-IV**
SEMESTER-VI **MATHEMATICAL METHODS-I**

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FOURIER SERIES–I

Structure :

I. **Introduction**
II. **Some Important Results**
III. **Dirichlet's Conditions**
IV. **Some Important Examples**
V. **Self Check Exercise**

I. Introduction

A series of the form $\frac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x) + ... + (a_n \cos nx + b_n \sin nx) + ...$ is known to be a **Fourier Series** . The numbers a_0, a_n, b_n (n = 1, 2, 3,....) which are constants independent of x are known as **Fourier coefficients.**

Further, Fourier series for any function f(x) for $x \in (\alpha, \alpha + 2\pi)$ can be expressed as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where a_0, a_n, b_n are evaluated by using '**Euler's formulae**'. While finding the values of a_0, a_n, b_n we assume that the series on the right hand side is uniformly convergent for $x \in (\alpha, \alpha + 2\pi)$ and can be integrated term by term in the given interval. We take

$$a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\, dx$$

(a) If $\alpha = 0$, then the interval reduces to $(0, 2\pi)$

(b) If $\alpha = -\pi$, then the interval reduces to $(-\pi, \pi)$, and we have the following cases:

Case I : When f(x) is an odd function

(i) $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = 0 \qquad [\because f(x)\cos nx \text{ is an odd function}]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\, dx \qquad [\because f(x)\sin nx \text{ is an even function}]$$

Therefore, if f(x) is an odd function then the Fourier expansion contains only sine terms as

$$f(x) = \sum_{n=0}^{\infty} b_n \sin nx, \text{where } b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\, dx$$

Case II : When f(x) is an even function

(i) $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\, dx \qquad [\because f(x)\cos nx \text{ is an even function}]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = 0 \qquad [\because f(x)\sin nx \text{ is an odd function}]$$

Therefore, if f(x) is an even function then the Fourier expansion contains only consine terms as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx\, dx \cdot$$

II. Some Important Results

To determine a_0, a_n and b_n, following results should be kept in mind (m and n are integers)

(i) $$\int_{\alpha}^{\alpha+2\pi} \sin nx\, dx = 0 = \int_{\alpha}^{\alpha+2\pi} \cos nx\, dx \text{ if } n \neq 0$$

$$\left[\because \int_{\alpha}^{\alpha+2\pi} \sin nx = -\left[\frac{\cos nx}{n}\right]_{\alpha}^{\alpha+2\pi} = -\frac{1}{n}\left[\cos n\alpha - \cos n\alpha\right] = 0\right]$$

(ii) $$\int_{\alpha}^{\alpha+2\pi} \sin mx \cos nx\, dx = \frac{1}{2}\int_{\alpha}^{\alpha+2\pi} [\sin(m+n)x + \sin(m-n)x]\, dx$$

$$= \frac{-1}{2}\left|\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n}\right|_{\alpha}^{\alpha+2\pi} = 0\text{'}m \neq n$$

(iii) $$\int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx\, dx = 0 \text{ if } m \neq n$$

(iv) $$\int_{\alpha}^{\alpha-2\pi} \sin mx \sin nx\, dx = 0 \; if \; m \neq n$$

(v) $$\int_{\alpha}^{\alpha-2\pi} cos^2 nx\, dx = \left[\frac{x}{2} + \frac{sin2nx}{4n}\right]_{\alpha}^{\alpha-2\pi} = \pi$$

$$\int_{\alpha}^{\alpha-2\pi} sin^2 nx\, dx = \left[\frac{x}{2} - \frac{sin2nx}{4x}\right]_{\alpha}^{\alpha-2\pi} = \pi, n \neq 0$$

(vi) $$\int_{\alpha}^{\alpha-2\pi} \sin nx \cos nx\, dx = 0, n \neq 0$$

(vii) $$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx)$$

(viii) $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx)$.

III. Dirichlet's Conditions

A function f(x) can be expressed as a Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

where a_0, a_n, b_n are constants, provided.

(i) f (x) is periodic, single valued and finite.

(ii) f (x) has a finite number of discontinuities in any one period.

(iii) f (x) has at the most a finite number of maxima and minima.

The above mentioned conditions are known as Dirichlet Conditions.

The integrals of the form $\int_0^a f(x) \frac{\sin nx}{x} dx$ and $\int_0^a f(x) \frac{\sin nx}{x} dx$ are known as Dirichlet Integrals. Fourier series converges to f(x) at every point of continuity. Further, at a point of discontinuity, the sum of the series is equal to the mean of the limit on the right and left.

i.e. $\frac{1}{2}$ [f (x + 0) + f(x – 0)]

where f(x + 0) and f(x – 0) denote the limit on the right and the limit on the left respectively.

IV. Some Important Examples

Example 1 : Expand f (x) = x sin x, $0 < x < 2\pi$ as a Fourier series.

Sol. Let f (x) = x sin x

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{(By def.)}$$

By Euler's formulae, we have

$$a_0 = \frac{1}{n}\int_0^{2\pi} f(x)\, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \, dx$$

$$= \frac{1}{\pi}\left[x(-\cos x) - 1 \, . \left(-\sin x\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[-2\pi\right] = -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} (x)\cos nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\; 92 \sin x \cos nx)\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin (n+1)\,x - \sin (n-1)\,x\right] dx$$

$$= \frac{1}{2\pi}\left[x\left\{-\frac{\cos (n+1)\,x}{n+1} + \frac{\cos (n-1)\,x}{n-1}\right\} - 1\left\{-\frac{\sin (n+1)\,x}{(n+1)^2} + \frac{\sin (n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left\{\frac{-\cos 2(n+1)\,\pi}{n+1} + \frac{\cos 2(n-1)\,\pi}{n-1}\right\}\right], n \neq 1$$

$$= \frac{-1}{n+1} + \frac{1}{n-1} = \frac{2}{n^2-1}, n \neq 1$$

dx

π

In particular when n = 1, we have

$$a_1 = \frac{1}{n}\int_0^{2\pi} x \sin x \cos x\, dx = \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x$$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi} = \frac{1}{2\pi}\left[-\pi\right] = -\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[2 \sin nx \sin x\right] dx$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x\left[\cos (n-1)\,x - \cos (n+1)\,x\right] dx\right.$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right\}\right.$$

$$\left. -1\left\{-\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right] = 0,\ n \neq 1$$

In particular when n = 1, we have

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x\, dx = \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)\, dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1.\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi.2\pi - \frac{4\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right] = \frac{1}{2\pi}(2\pi^2) = \pi$$

$\therefore$
$$f(x) = \frac{a_0}{2} + a_1\cos x + b_1 \sin x + \sum_{n=2}^{\infty} a_n \cos nx + \sum_{n=2}^{\infty} b_n \sin nx$$

$$= -1 - \frac{1}{2}\cos x + \pi \sin x + \sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx + 0$$

$$= -1 + \pi \sin x - \frac{1}{2}\cos x + \frac{2}{2^2-1}\cos 2x + \frac{2}{3^2-1}\cos 3x + \ldots.$$

Example 2 : Obtain the Fourier Series expansion of f(x) = x sin x in (–π, π). Hence deduce that

$$\frac{\pi}{4} = \frac{1}{1.2} + \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} + \ldots$$

OR

$$\frac{1}{1.3}+\frac{1}{3.5}-\frac{1}{5.7}+\ldots=\frac{\pi-2}{4}$$

Sol. Let $f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$

Because f (x) = x sinx is an even function, $\therefore\ b_n = 0$

Now $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin x\,dx=\frac{2}{\pi}\int_0^{\pi}x\sin x\,dx$

$$=\frac{2}{\pi}\left[x(-\cos x)\Big|_0^{\pi}+\int_0^{\pi}\cos x\,dx\right]=\frac{2}{\pi}\left[-\pi\cos\pi\right]=2$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin x\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}x\sin x\cos nx\,dx$$

$$=\frac{1}{\pi}\int_0^{\pi}\left[x\sin(n+1)x-x\sin(n-1)x\right]dx$$

$$=\frac{1}{\pi}\left[-\frac{x\cos(n+1)x}{n+1}\Bigg|_0^{\pi}+\int_0^{\pi}\frac{\cos(n+1)x}{n+1}dx+\frac{x\cos(n-1)x}{n-1}\Bigg|_0^{\pi}-\int_0^{\pi}\frac{\cos(n-1)x}{n-1}dx\right]$$

$$=\frac{1}{\pi}\left[-\frac{\pi\cos(n+1)\pi}{n+1}+\frac{\pi\cos(n-1)\pi}{n-1}\right],\ n\neq 1$$

$$=\frac{\pi}{\pi}\left[-\frac{(-1)^{n+1}}{n+1}+\frac{(-1)^{n-1}}{n-1}\right]=(-1)^n\left[\frac{1}{n+1}-\frac{1}{n-1}\right]$$

$$=(-1)^n\left[\frac{-2}{n^2-1}\right]=-\frac{2(-1)^n}{n^2-1},\ n\neq 1$$

In particular when n = 1

$$a_1=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin x\cos x\,dx=\frac{2}{\pi}\int_0^{\pi}x\sin x\cos x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right)\Big|_0^{\pi} - \int_0^{\pi}\left(-\frac{\cos 2x}{2}\right)dx\right]$$

$$= \frac{-1}{\pi}\left[\frac{\pi}{2}.\cos 2\pi\right] = -\frac{1}{2}$$

Thus $f(x) = \frac{1}{2}a_0 + a_1 \cos x + \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos nx$

$\therefore$ $$x \sin x = \frac{1}{2}(2) = \frac{\cos x}{2} - 2 . \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos nx$$

$$= 1 - \frac{\cos x}{2} - 2\left[\frac{\cos 2x}{3} - \frac{\cos 3x}{8} + \frac{\cos 4x}{15} + ...\right]$$

For $x = \frac{\pi}{2}$,

$$\frac{\pi}{2} = 1 - 2\left[\frac{-1}{3} + \frac{1}{15} - \frac{1}{35} + ...\right] = 1 + \frac{2}{3} - \frac{2}{3.5} + \frac{2}{5.7} ...$$

Dividing both sides by 2

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} + ...$$

OR

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} + ... = \frac{\pi}{4} - \frac{1}{2} = \frac{\pi - 2}{4}.$$

Example 3 : Expand in a series of sines and cosines of multiple of x, the function given by

$f(x) = x - \pi$ when $-\pi < x < 0$

$f(x) = \pi - x$ when $0 < x < \pi$

what is the sum of the series for $x = \pm \pi$ and $x = 0$?

Sol. Let $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

By Euler's Formuale :

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

(Reason-1)

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (x-\pi)\,dx + \int_{0}^{\pi} (\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left[-\int_{0}^{\pi} (x+\pi)\,dx + \int_{0}^{\pi} (\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left[-2\int_{0}^{\pi} x\,dx\right]$$

$$= -\frac{1}{\pi}\left[x^2\right]_0^{\pi} = -\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (x-\pi)\cos nx\,dx + \int_{0}^{\pi} (\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\int_{0}^{\pi} (x+\pi)\cos nx\,dx + \int_{0}^{\pi} (\pi - x)\cos nx\,dx\right]$$

Reason-1 :

In the first integral

Take $x = -t \Rightarrow dx = -dt$

$x = -\pi \Rightarrow t = \pi, x = 0 \Rightarrow t = 0$

$$\therefore \int_{-\pi}^{0} (x-\pi)\,dx$$

$$= -\int_{\pi}^{0} (-t-\pi)\,dt$$

$$= -\int_{0}^{\pi} (t+\pi)\,dt$$

$$= -\int_{0}^{\pi} (x+\pi)\,dx$$

By the above substitution

Reason-2 :

$$\int_{-\pi}^{0} (x-\pi)\cos nx\,dx$$

$$= \int_{\pi}^{0} (-t-\pi)\cos(-nt)(-dt)$$

$$= \int_{\pi}^{0} (t+\pi)\cos nt\,dt$$

$$= -\int_{0}^{\pi} (t+\pi)\cos nt\,dt$$

$$= -\int_{0}^{\pi} (x+\pi)\cos nx\,dx$$

$$= -\frac{2}{\pi}\int_0^{\pi} x \cos nx \, dx$$

$$= \frac{-2}{\pi}\left[x\frac{\sin nx}{n}\Big|_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\,dx\right]$$

$$= \frac{2}{n\pi}\left[-\frac{\cos nx}{n}\right]_0^{x}$$

$$= \frac{-2}{n^2\pi}\left[\cos n\pi - \cos 0\right]$$

$$= -\frac{2}{\pi n^2}\left[(-1)^n - 1\right] = \frac{2}{\pi n^2}[1-(-1)^n]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(x-\pi)\sin nx + \int_0^{\pi}(\pi - x)\sin nx\,dx\right]$$

(Reason-3)

$$= \frac{1}{\pi}\left[\int_0^{\pi}(x+\pi)\sin nx + \int_0^{\pi}(\pi - x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[2\pi\int_0^{\pi}\sin nx\,dx\right]$$

$$= 2\left[-\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= -\frac{2}{n}[\cos n\pi - \cos 0]$$

By the substitution mentioned in Reason-1 :

$$\int_{-\pi}^{0}(x-\pi)\sin nx\,dx$$

$$= \int_{\pi}^{0}(-t-\pi)\sin(-nt)(-dt)$$

$$= -\int_{\pi}^{0}(t+\pi)\sin nt\,dt$$

$$= \int_0^{\pi}(t+\pi)\sin nt\,dt$$

$$= \int_0^{\pi}(x+\pi)\sin nx\,dx$$

$$= \frac{2}{n}[1-(-1)^n]$$

The coeff. a_n and b_n are zero when n is even. In $[-\pi, \pi]$, the points, $x = 0$ and $x = \pm\pi$ are the only points of discontinuity of f.

∴ when x is different from 0 and $\pm\pi$, we have by using definition

$$f(x) = \frac{1}{2}(-\pi) + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}[1-(-1)^n]\cos nx + \sum_{n=1}^{\infty} \frac{2}{n}[1-(-1)^n]\sin nx$$

$$= -\frac{1}{2}\pi + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right] + 4\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \dots\right]$$

V. Self Check Exercise

1. Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.
2. Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$.

   Hence show that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$.
3. Expand $f(x) = |\cos x|$ as Fourier series in the interval $-\pi < x < \pi$. and $f(x) = |\sin x|$ in $-\pi < x < \pi$.
4. Obtain the Fourier series for the function.

   $$f(x) = \begin{cases} \cos x & \text{for } 0 < x < \pi \\ -\cos x & \text{for } -\pi < x < 0 \end{cases}$$
5. Find the fourier expansion of function $f(x) = \begin{cases} 1 & 0 < x < \pi \\ 2 & \pi < x < 2\pi \end{cases}$ and then

   deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

Suggested Readings :

1. Shanthi Narayan and P.K. Mittal, A Course of Mathematical Analysis by S. Chand and Company.

FOURIER SERIES-II

Structure :

I. Introduction

In this lesson, we firstly introduce the idea of finding the Fourier series of function f(x) in the interval $c < x < c + 2l$, given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Where $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi}{l}\,dx$$

Note 1. When^{l} c = 0, x ∈ (0, 2l) then

$$a_0 = \frac{1}{l}\int_0^{2t} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_0^{2t} f(x)\cos\frac{n\pi}{l}x\,dx$$

$$b_n = \frac{1}{l}\int_0^{2t} f(x)\sin\frac{n\pi}{l}x\,dx$$

Note 2. When c = $-l$, $x \in (-l, l)$ then

$$a_0 = \frac{1}{l}\int_{-l}^{1} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{1} f(x)\cos\frac{n\pi}{l}x\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{1} f(x)\sin\frac{n\pi}{l}x\,dx\cdot$$

II. Half Range Sine and Cosine Series

If a function f (x) is defined over the interval (0, l) then it is capable of two distinct half range series. One is cosine series and the other is sine series.

(1) We know that when f (–x) = f(x), the function is even or in other words f(x) is extended to reflect in y-axis. Since f (x) is even therefore f(x). Sin nx will be odd and as such there will be no terms containing ones therefore the half range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where $a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx,\ a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi\,x}{l}dx$

(2) Again if f(–x) = –f(x), the function is odd or in other words f(x) is extended to reflect in the origin. Since f(x) is odd therefore f(x) cos nx will also be odd and as such the terms a_0 and a_n will vanish and f(x) contain only sine terms. Therefore the half range sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} dx$$

where $b_n = \dfrac{2}{l} \displaystyle\int_0^l f(x) \sin \frac{n\pi x}{l} dx$.

III. Parseval's Theorem

Theorem : Fourier Series of f(x) over an interval $c < x < c + 2l$ is given by

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}\right)$$

then $\dfrac{1}{2} \displaystyle\int_c^{c+2l} [f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$

Proof : The Fourier Series of f(x) on $c < x < c + 2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}\right] \qquad \ldots (1)$$

where $a_0 = \dfrac{1}{l} \displaystyle\int_c^{c+2l} f(x)\, dx$,

$$a_0 = \frac{1}{l} \int_c^{c+2l} f(x) \frac{\cos n\pi x}{l} dx$$

$$b_0 = \frac{1}{l} \int_c^{c+2l} f(x) \sin \frac{n\pi x}{l} dx$$

Multiplying both sides of (1) by f(x) we get

$$[f(x)]^2 = \frac{a_0}{2} f(x) + \sum_{n=1}^{\infty} a_n f(x) \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n f(x) \sin \frac{n\pi x}{l} \qquad \ldots (2)$$

Integrating both sides of (1) on between the limits c to $c + 2l$ w.r.t. x we have

$$\int_c^{c+2l} [f(x)]^2 dx = \frac{a_0}{2} \int_c^{c+2l} f(x)\, dx + \sum_{n=1}^{\infty} a_n \int_c^{c+2l} f(x) \cos \frac{n\pi x}{l} dx$$

$$+\sum_{n=1}^{\infty} b_n \int_c^{c+2l} f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{a_0}{2} l\, a_0 = \sum_{n=1}^{\infty} a_n (l a_n) + \sum_{n=1}^{\infty} b_n\ (l b_n)$$

$$\int_c^{c+2l} [f(x)]^2\ dx = \frac{l\, a_0^2}{2} + l \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

$$\therefore \frac{1}{2l} \int_c^{c+2l} [f(x)]^2\ dx = \frac{1}{2l}\left[\frac{l\, a_0^2}{2} + l \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right]$$

or $$\frac{1}{2l} \int_c^{c+2\gamma} [f(x)]^2\ dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

Hence the Prove.

IV. Some Important Examples

Example 1 : Obtain Fourier series for the function

$f(x) = \pi x \qquad 0 \le x \le 1$

$= \pi(2 - x) \qquad 1 \le x \le 2$

Sol. Here, $l = 1$

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$a_0 = \int_0^2 f(x)\, dx = \int_0^1 \pi x\, dx + \int_1^2 \pi(2 - x)\, dx$$

$$= \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \pi\left[\frac{1}{2}\right] + \pi\left[(4-2) - \left(2 - \frac{1}{2}\right)\right] = \pi$$

$$a_0 = \int_0^2 f(x) \cos n\pi x \, dx$$

$$= \int_0^1 \pi x \cos nx\pi \, dx + \int_1^2 \pi (2 - x) \cos n\pi x \, dx$$

$$= \int_0^1 \pi x \cos n\pi x \, dx + \int_0^1 \pi x \cos n\pi x \, dx$$

$$= 2\int_0^1 \pi x \cos n\pi x \, dx$$

$$= 2\left[\pi x . \frac{\sin n\pi x}{n\pi} - \pi\left(-\frac{\cos n\pi x}{\pi^2 n^2}\right)\right]_0^1$$

$$= 2\left(\frac{\cos n\pi}{n^2\pi} - \frac{1}{n^2\pi}\right)$$

$$= \frac{2}{n^2\pi}(\cos n\pi - 1) = \frac{2}{n^2\pi}\left\{(-1)^n - 1\right\}$$

In the second integral

Take $2 - x = t \Rightarrow dx = -dt$

$x = 2 \Rightarrow t = 0$
$x = 1 \Rightarrow t = 1$

$$\therefore \int_1^2 \pi (2 - x) \cos n\pi x \, dx$$

$$= \int_1^0 \pi (t) \cos (2n\pi - n\pi t)(-dt)$$

$$= \pi \int_0^1 t \cos n\pi t \, dt$$

$$b_n = \int_0^2 f(x) \sin n\pi x \, dx$$

$$= \int_0^1 \pi x \sin n\pi x \, dx + \int_1^2 \pi (2 - x) \sin n\pi x \, dx$$

$$= \int_0^1 \pi x \sin \pi x \, dx - \int_0^1 n x \sin n\pi x \, dx = 0$$

$$= 0$$

By the above substitution

$$\therefore \int_1^2 \pi (2 - x) \sin n\pi x \, dx$$

$$= \int_1^0 \pi t \sin (2n\pi - n\pi t)(-dt)$$

$$= -\int_0^1 \pi t \sin n\pi t \, dt$$

$$= -\int_0^1 \pi x \sin n\pi x \, dx$$

$$\therefore f(x) = \frac{1}{2}(\pi) + \sum_{n=0}^{\infty} \frac{2}{n^2\pi}[(-1)^n - 1]\cos n\pi x$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + ...\right].$$

Example 2 : If $f(x) = x$ when $0 < x < \frac{\pi}{2} = \pi - x$ when $\frac{\pi}{2} < x < \pi$

Show that (i) $f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + ...\right]$

(ii) $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + ...\right]$

Sol. (i) For the half range sine series

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\int_0^{\pi/2} x\sin nx\,dx + \int_0^{\pi/2} x\sin(n\pi - nx)\,dx$$

In the second integral

put $\pi - x = t \Rightarrow dx = -dt$

$x = \pi \Rightarrow t = 0$

$x = \pi/2 \Rightarrow t = \pi/2$

$$\therefore \int_{\pi/2}^{\pi} (\pi - x)\sin nx\,dx$$

$$= -\int_{\pi/2}^{0} t\sin(n\pi - nt)\,dt$$

$$= \int_0^{\pi/2} x\sin(n\pi - nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} x\,[\sin nx + \sin(n\pi - nx)\,dx]$$

$$= \frac{2}{\pi} \int_0^{\pi/2} . \, x \, 2 \sin \frac{n\pi}{2} \cos\left(nx - \frac{n\pi}{2}\right) dx$$

$$= \frac{4}{\pi} \sin \frac{n\pi}{2} \int_0^{\pi/2} x \, . \cos\left(nx - \frac{n\pi}{2}\right) dx$$

$$= \frac{4 \sin n\pi/2}{\pi}\left[x \, . \frac{\sin(nx - n\pi/2)}{n} \Bigg|_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin\left(nx - \frac{n\pi}{2}\right)}{n} dx \right]$$

$$= \frac{4 \sin \frac{n\pi}{2}}{\pi}\left[0 + \frac{1}{\pi^2} \cos\left(nx - \frac{n\pi}{2}\right)\Bigg|_0^{\pi/2}\right]$$

$$= \frac{4 \sin \frac{n\pi}{2}}{\pi}\left[\frac{1}{n^2}\left(1 - \cos \frac{n\pi}{2}\right)\right]$$

$$= \frac{4 \sin \frac{n\pi}{2}}{\pi n^2} - \frac{4 \sin \frac{n\pi}{2} . \cos \frac{n\pi}{2}}{\pi n^2}$$

$$= \frac{4 \sin \frac{n\pi}{2}}{\pi n^2} - \frac{2 \sin n\pi}{n^2 \pi} = \frac{4 \sin \frac{n\pi}{2}}{\pi n^2} \qquad \{\because \sin n\pi = 0, n \in I\}$$

$\therefore$ Half range sine series is

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{\sin \frac{n\pi}{2}}{n^2} \sin nx \right]$$

$$= \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \ldots\right]$$

(ii) For the half range cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

By Euler's Formuale

$$a_0 = \frac{1}{\pi/2}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} (\pi - x)\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{8} + \frac{(\pi - x)^2}{-2}\Bigg|_{\pi/2}^{\pi}\right]$$

By the above substitutions

$$\int_{\pi/2}^{\pi} (\pi - x)\cos nx\,dx$$

$$= \int_{\pi/2}^{0} t \cos n(\pi - t)(-dt)$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{8} - \frac{1}{2}\left(0 - \frac{\pi^2}{4}\right)\right]$$

$$= \frac{2}{\pi}\left(\frac{\pi^2}{4}\right) = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi/2}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\,dx\right]$$

$$= \int_0^{\pi/2} t\cos(n\pi - nt)\,dt$$

$$= \frac{2}{\pi}\int_0^{\pi/2} x\,[\cos nx + \cos(n\pi - nx)]\,dx$$

$$= \int_0^{\pi} x\cos(n\pi - nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} x\left\{2\cos\frac{n\pi}{2}.\cos\left(nx - \frac{n\pi}{2}\right)\right\}dx$$

$$= \frac{4}{\pi}\cos\frac{n\pi}{2}\left[x\,\frac{\sin\left(nx - n\,\frac{\pi}{2}\right)}{n}\Bigg|_{0}^{\pi/2} - \int_{0}^{\pi/2}\frac{\sin\left(nx - n\,\frac{\pi}{2}\right)}{n}\,dx \right]$$

$$= \frac{4}{\pi}\cos\frac{n\pi}{2}\left[0 + \frac{1}{n^2}\cos\left(nx - n\,\frac{\pi}{2}\right)\Bigg|_{0}^{\pi/2}\right]$$

$$= \frac{4\cos n\,\frac{\pi}{2}}{\pi\, n^2}\left[\cos 0 - \cos n\,\frac{\pi}{2}\right]$$

$$= \frac{4\cos n\,\frac{\pi}{2}}{\pi\, n^2}\left(1 - \cos n\,\frac{\pi}{2}\right) = \frac{8}{\pi}\,.\,\frac{\cos n\,\frac{\pi}{2}\,.\,\sin^2\frac{n\pi}{4}}{n^2}$$

$\therefore$

$$f(x) = \frac{1}{2}\,a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{\pi}{4} + \frac{2}{\pi}\left\{\sum_{n=1}^{\infty}\frac{4\cos\frac{n\pi}{2}\,.\,\sin^2\frac{n\pi}{2}}{n^2}\cos nx\right\}$$

$$= \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{4(-1)\,(1)}{4}\cos 2x + \frac{4\,(-1)\,(1)^2}{6^2}\cos 6x + \frac{4\,(-1)\,(1)^2}{(10)^2}\cos 10x + \ldots\ldots\right]$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \ldots\right]$$

(By taking n = 2, 6, 10,.....)

V. Self Check Exercise

1. Find the Fourier Series for the function given by

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 1 - x, & 1 < x < 2 \end{cases}$$

2. Find the Fourier series in the interval (–2, 2) when

$$f(x) = \begin{cases} 0 & -2 < x < 0 \\ 1 & 0 < x < 2 \end{cases}$$

3. Express f(x) = x has half range sine series in the interval $0 < x < 2$.
4. Express sin x as cosine series when $0 < x < \pi$.

Suggested Readings :

1. Shanthi Narayan and P.K. Mittal, A Course of Mathematical Analysis by S. Chand and Company.

LESSON NO. 2.1 **Author : Dr. Chanchal**

FOURIER TRANSFORMS–I

Structure :

2.1.1 Introduction

As we are already familiar about the concept of Fourier series, So now we can easily understand the concept of Fourier transform which is of two types i.e., Fourier Sine transform and Fourier cosine transform. It has many applications in the areas of engineering and science as it is widelyused for solving practical integral and partial differential equations with suitable boundary conditions.

2.1.2 Dirichlet's Conditions

Let f(x) be a function which satisfies the following conditions :

(a) f (x) is defined on some interval say $x \in (-\lambda, \lambda)$.

(b) f (x) and f'(x) are piecewise continuous in $(-\lambda, \lambda)$.

(c) f (x) is periodic whose period is 2λ.

The above slated conditions are known as Dirichlet's conditions.

2.1.3 Fourier Integral Formula

Le f (x) be a function satisfying **Dirichlet's conditions** and is absolutely integrable in $(-\infty\ \infty)$ i.e. $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges,

Then $f(x) = \int_0^{\infty} \left(f(s)\cos sx + G(s)\sin sx\right) ds$

where $\pi F(s) = \int_{-\infty}^{\infty} f(w) \cos s\, w\, dw, \pi G(s) = \int_{-\infty}^{\infty} f(w) \sin s\, w\, dw$

Note : Here, f (x) can be re-written as

$$f(x) = \int_0^{\infty} \left(\left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} f(w) \cos s\, w\, dw \right\} \cos sx + \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} f(w) \sin s\, w\, dw \right\} \sin sx \right) ds$$

$$= \frac{1}{\pi} \int_{s=0}^{s=\infty} \int_{w=-\infty}^{w=\infty} f(w) \{\cos s\, w \cos s\, x + \sin s\, w \sin sx\}\, dw\, ds$$

$$= \frac{1}{\pi} \int_{s=0}^{s=\infty} \int_{w=-\infty}^{w=\infty} f(w) \cos s\,(x - w)\, dw\, ds$$

$$= \frac{1}{2\pi} \int_{s=-\infty}^{s=\infty} \int_{w-\infty}^{w=\infty} f(w) \cos s(x - w)\, dw\, ds \cdot$$

Further, the Fourier integral formula can be represented in different forms as:

I. **General Form :** $\left[f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(w) \left\{ \int_0^{\infty} \cos s\,(x - w)\, ds \right\} dw \right]$

II. **Fourier Sine Integral Formula :** If f(x) is an odd function,

then $\left[f(x) = \frac{2}{\pi} \int_0^{\infty} \sin sx \left\{ \int_0^{\infty} f(w) \sin s\, w\, dw \right\} ds \right]$

III. **Fourier Cosine Integral Formula :** If f(x) is an even function,

then $\left[f(x) = \frac{2}{\pi} \int_0^{\infty} \sin sx \left\{ \int_0^{\infty} f(w) \sin s\, w\, dw \right\} ds \right]$

IV. **Complex or Exponential Form :** $\left[f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\,sx} \left\{ \int_{-\infty}^{\infty} f(w)\, e^{-i\,sw} dw \right\} ds \right.$

Or

$$\left. f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-sx} \left\{ \int_{-\infty}^{\infty} f(w)\, e^{isw} dw \right\} ds \right].$$

2.1.4 Fourier Transform

Let f (t) be a function defined and piecewise continuous on (–∞ ∞) and is absolutely convergent on (–∞ ∞) i.e. $\int_{-\infty}^{\infty} |f(t)|\, dt$ converges,

Then **Fourier transform of f(t),** denoted by f (f(t)) may be defined as

$$F(f(t)) = \int_{-\infty}^{\infty} f(t)\, e^{ist} dt = G(s) \text{ (say)} \qquad \ldots (1)$$

Here, the function f (t) is inverse Fourier transform of G (s) and defined as

$$f(t) = F^{-1}(G(s)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(s)\, e^{-ist} dt \qquad \ldots (2)$$

This is called the **Inversion formula** for Fourier transform.

Further, there exist some other definitions of Fourier transform, given by

I. $$F(f(t)) = \int_{-\infty}^{\infty} f(t)\, e^{-ist} dt = G(s) \qquad \ldots (3)$$

and $$f(t) = F^{-1}(G(s)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(s)\, e^{ist} ds \qquad \ldots (4)$$

II. $$F(f(t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt = G(s) \qquad \ldots (5)$$

and $$f(t) = F^{-1}(G(s)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(s)\, e^{-ist} ds \qquad \ldots (6)$$

III. $$F(f(t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-ist} dt = G(s) \qquad \ldots (7)$$

and $$f(t) = F^{-1}(G(s)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(s)\, e^{ist} ds \qquad \ldots (8)$$

This is known as symmetric form of definition.

All the above definitions are equivalent and such Fourier transforms are called **infinite or complex Fourier transforms.**

2.1.5 Fourier Sine and Cosine Transforms with Inversion Formulae

1. Fourier Sine Transform : Let f(t) be a function defined and piecewise continuous on (0, ∞) and is absolutely convergent on (0 ∞) i.e. $\int_0^\infty |f(t)|\,dt$ converges

Then, Fourier sine transform of f(t), denoted by $F_s(f(t))$ may be defined as

$$F_s(f(t)) = \int_0^\infty f(t)\sin st\,dt = G_s(s) \qquad \ldots (1)$$

Here, the function f (t) is **inverse fourier sine transform** of $G_s(s)$ or $F_s(f(t))$ and defined as

$$f(t) = \frac{2}{\pi}\int_0^\infty G_s(s)\sin st\,ds \qquad \ldots (2)$$

This is known as **Inversion Formula for Fourier sine transform.**

Another Definition :

$$F_s(f(t) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin st\,dt = G_s(s)$$

and $$f(t) = F_s^{-1}(G_s(s)) = \sqrt{\frac{2}{\pi}}\int_0^\infty G_s(s)\,st\,ds$$

which is known as symmetric form of definition.

2. Fourier Cosine Transform : Let f (t) be a function defined and piecewise continuouson (0 ∞) and is absolutely convergent on (0 ∞) i.e. $\int_0^\infty |f(t)|\,dt$ converges.

Then, Fourier cosine transform of f(t), denoted by $F_c(f(t))$ may be defined as,

$$F_c(f(t)) = \int_0^\infty f(t)\cos st\,dt = G_c(s) \qquad \ldots (1)$$

Here, the function f(t) is **inverse Fourier cosine transform** of $G_c(s)$ or $F_c(f(t))$ and defined as

$$f(t) = \frac{2}{\pi}\int_0^\infty G_c(s)\cos s\,t\,ds \qquad \ldots (2)$$

This is known as **Inversion Formula for Fourier Cosine transform.**

Another Definition :

$$F_c\ (f(t)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos st\, dt = G_c(s)$$

and $$f(t) = F_c^{-1}(G_c(s)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos st\, dt = G_c(s)$$

This is known as symmetric form of definition.

2.1.6 Some Important Examples

Example 1 : Using Fourier cosine integral formula, show that

$$\frac{\pi}{2} e^{-x} \cos x = \int_0^\infty \frac{(w^2+2)\cos wx}{w^2+4}\, dw, x \geq 0 .$$

Sol. By Fourier cosine integral formula,

we have $$f(x) = \frac{2}{\pi} \int_0^\infty \cos w\, x \left\{ \int_0^\infty f(u) \cos wu\, du \right\} dw \qquad \ldots \text{(i)}$$

Here $$f(x) = \frac{\pi}{2} e^{-x} \cos x$$

$\Rightarrow$ $$f(u) = \frac{\pi}{2} e^{-u} \cos u$$

$\therefore$ $$\int_0^\infty f(u) \cos wu\, du = \int_0^\infty \frac{\pi}{2} e^{-u} \cos u \cos wu\, du$$

$$= \frac{\pi}{2} \times \frac{1}{2} \int_0^\infty e^{-u} (2 \cos wu \cos u)\, du$$

$$= \frac{\pi}{4} \int_0^\infty e^{-u} (\cos (w+1)\, u + \cos (w-1) u)\, du$$

$$= \frac{\pi}{4} \left(\int_0^\infty e^{-u} \cos(w+1)u\, du + \int_0^\infty e^{-u} \cos(w-1)u\, du \right)$$

$$= \frac{\pi}{4}\left(\frac{1}{(w+1)^2+1}+\frac{1}{(w-1)^2+1}\right)$$

$$= \frac{\pi}{4}\left(\frac{w^2-2w+2+w^2+2w+2}{(w^2+2w+2)\,(w^2-2w+2)}\right)$$

$$= \frac{\pi}{4}\,\frac{2\,(w^2+2)}{(w^2+2)^2-(2w)^2} = \frac{\pi(w^2+2)}{2(w^4+4)}$$

$\therefore$ (1) becomes,

$$\frac{\pi}{2}e^{-x}\cos x = \frac{2}{\pi}\int_0^{\infty}\cos wx\left(\frac{\pi}{2}\,\frac{w^2+2}{w^2+4}\right)dw$$

$$= \int_0^{\infty}\frac{(\cos\, wx)\,(w^2+2)}{w^2+4}\,dw\,.$$

Example 2 : Let $f(x)=\begin{cases}0, & x<1\\ e^{-x}, & x>0\end{cases}$. Find fourier integral of f (x).

Sol. By using fourier integral of f(x),

we have $f(x)=\dfrac{1}{\pi}\displaystyle\int_0^{\infty}\left(\int_{-\infty}^{\infty} f(w)\cos s\,(x-w)\,dw\right)ds$

$$\Rightarrow \quad f(x)=\frac{1}{\pi}\int_0^{\infty}\left(\int_{-\infty}^{\infty} f(w)\cos s(x-w)dw+\int_0^{\infty} f(w)\cos s\,(x-w)\,dw\right)ds$$

$$=\frac{1}{\pi}\int_0^{\infty}\left(\int_{-\infty}^{\infty} 0.\cos s\,(x-w)dw+\int_0^{\infty} e^{-w}\cos s\,(x-w)\,dw\right)ds$$

$$=\frac{1}{\pi}\int_0^{\infty}\left(\int_{-\infty}^{0} e^{-w}(\cos sx\cos sw+\sin sx\,\sin sw)\,dw\right)ds$$

$$=\frac{1}{\pi}\int_0^{\infty}\left(\cos\, sx\int_{-\infty}^{0} e^{-w}\cos\, s\, w\, dw+\sin\, sx\int_0^{\infty} e^{-w}\sin\, sw\, dw\right)ds$$

$$= \frac{1}{\pi}\int_0^\infty \left(\cos sx \left(\frac{1}{s^2+1} \right) + \sin s\, x \left(\frac{s}{s^2+1} \right) \right) ds$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\cos sx + s \sin sx}{s^2+1}\, ds\,.$$

Example 3 : Evaluate F (f (t)), where $f(t) = \begin{cases} 1-t^2, & -1 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$ and then find

$$\int_0^\infty \frac{t \cos t - \sin t}{t^3} \cos \frac{t}{2}\, dt\,.$$

Sol. $F(f(t)) = \int_{-\infty}^{\infty} e^{i s t} f(t)\, dt$

$$= \int_{-\infty}^{-1} e^{i s t} f(t)\, dt + \int_{-1}^{1} e^{i s t} f(t)\, dt + \int_{1}^{\infty} e^{i s t} f(t)\, dt$$

$$= \int_{-\infty}^{-1} e^{i s t} (0)\, dt + \int_{-1}^{1} (1-t^2)\, e^{i s t}\, dt + \int_{1}^{\infty} e^{i s t} (0)\, dt$$

$$= 0 + \left(\left\{ \frac{(1-t^2)\, e^{i s t}}{i\, s} \right\}_{-1}^{1} - \int_{-1}^{1} (-2\, t)\, \frac{e^{i s t}}{i\, s}\, dt \right) + 0 = (0-0) + \frac{2}{i\, s} \int_{-1}^{1} t e^{i s t} dt$$

$$= \frac{2}{i\, s} \left(\left(\frac{t e^{i s t}}{i\, s} \right)_{-1}^{1} - \int_{-1}^{1} 1 \,.\, \frac{e^{i s t}}{i\, s}\, dt \right)$$

$$= \frac{2}{i\, s} \left(\frac{1}{i\, s} \left(e^{i s} - (-1)\, e^{i s} \right) - \frac{1}{i\, s} \left(\frac{e^{i s t}}{i\, s} \right)_{-1}^{1} \right)$$

$$= \frac{2}{i\, s} \left(\frac{2 \cos s}{i\, s} + \frac{1}{i^2\, s^2} \left(e^{i s} - e^{i s} \right) \right)$$

$$= \frac{2}{i\,s}\left(\frac{2\cos s}{i\,s} + \frac{1}{s^2}(2\,i\sin s)\right)$$

$$= \frac{4}{i^2 s}\left(\frac{\cos s}{s} + \frac{i^2 \sin s}{s^2}\right) = \frac{-4}{s}\left(\frac{s\cos s - \sin s}{s^2}\right)$$

$$= -4\left(\frac{s\cos - \sin s}{s^3}\right) = G\,(s)\,(\text{say})$$

IInd Part : To evaluate $\int_{-\infty}^{\infty} \frac{t\cos t - \sin t}{t^3} \cos t/2\,dt$ use inversion formula of fourier transform

$\therefore$ we have $\frac{1}{2\pi}\int_{-\infty}^{\infty} G\,(s)\,e^{-i\,s\,t} ds = f(t)$

$$\Rightarrow \quad \int_{-\infty}^{\infty} -4\left(\frac{s\cos s - \sin s}{s^3}\right)(\cos s\ t - i\sin s\ t)\,ds = 2\,\pi\,f(t)$$

Equate real parts on both sides,

we get $\int_{-\infty}^{\infty} -4\left(\frac{s\cos s - \sin s}{s^3}\right)\cos s\ t\ ds = 2\pi\,f(t)$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \frac{\sin s - \cos s}{s^3}\cos s\ t\ ds = \frac{2\pi}{4} f(t)$$

$$= \begin{cases} \frac{\pi}{2}(1 - t^2), & -1 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

To find required integral, Put $t = \frac{1}{2}$

$\therefore$ we get $\int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3}\cos\frac{s}{2}\,ds = \frac{\pi}{2}\left(1 - \frac{1}{4}\right) = \frac{3\pi}{8}$

$$\Rightarrow \quad 2\int_0^\infty \frac{(-1)(s\cos s - \sin s)}{s^3}\cos\frac{s}{2}ds = \frac{3\pi}{8}$$

$$\Rightarrow \quad \int_0^\infty \frac{s\cos s - \sin s}{s^3}\cos\frac{s}{2}dt = \frac{-3\pi}{16}$$

we get $\int_0^\infty \frac{t\cos t - \sin t}{t^3}\cos\frac{t}{2}dt = \frac{-3\pi}{16}$.

Example 4 : Find the Fourier sine and cosine transform of f(t) where

$$f(t) = \begin{cases} 2t & , \quad 0 < t < 1 \\ 2 - 2t & , \quad 1 < t < 2 \\ 0 & , \quad t > 2 \end{cases}$$

Sol. **For sine transform**

$$F_s(f(t)) = \int_0^\infty f(t)\sin s\, t\, dt$$

$$= \int_0^1 f(t)\sin st\,dt + \int_1^2 f(t)\sin st\,dt + \int_2^\infty f(t)\sin st\,dt$$

$$= \int_0^1 2\,t\sin st\,dt + \int_1^2 (2-2t)\sin st\,dt + \int_2^\infty 0\,.\sin st\,dt$$

$$= 2\left(t\left(\frac{-\cos st}{s}\right)\Bigg\}_0^1 - \int_0^1 1\,.\left(\frac{-\cos st}{s}\right)dt\right) + \left((2-2t)\left(\frac{-\cos st}{s}\right)\Bigg\}_1^2 - \int_1^2 (-2)\left(\frac{-\cos st}{s}\right)dt\right)$$

$$= \left(\frac{-2}{s}\cos s + 0\right) + \frac{2}{s}\left(\frac{\sin s\,t}{s}\right)_0^1 + \left(-\frac{(-2)}{s}\cos 2s + 0\right) - \frac{2}{s}\left(\frac{\sin s\,t}{s}\right)_1^2$$

$$= -\frac{2}{s}\cos s + \frac{2}{s^2}(\sin s - 0) + \frac{2}{s}\cos 2s - \frac{2}{s^2}(\sin 2s - \sin s)$$

$$= -\frac{2}{s}\cos s + \frac{2}{s}\cos 2s - \frac{2}{s^2}\sin 2s + \frac{2}{s^2}\sin s + \frac{2}{s^2}\sin s$$

$$= -\frac{2}{s}\cos s + \frac{4}{s^2}\sin s + \frac{2}{s}\cos 2s - \frac{2}{s^2}\sin 2s.$$

For cosine transform

$$F_c(f(t)) = \int_0^\infty f(t)\cos s\,t\,dt$$

$$= \int_0^1 f(t)\cos st\,dt + \int_1^2 f(t)\cos s\,t\,dt + \int_2^\infty f\,(t)\cos s\,t\,dt$$

$$= \int_0^1 2t\cos s\,t\,dt + \int_1^2 (2-2t)\cos s\,t\,dt + \int_2^\infty 0\,.\cos s\,t\,dt$$

$$= \left(2t\frac{\sin s\,t}{s}\right)_0^1 - \int_0^1 \frac{2\sin s\,t}{s}dt + \left((2-2t)\frac{\sin s\,t}{s}\right)_1^2 - \int_1^2 \frac{-2\sin s\,t}{s}dt + 0$$

$$= \left(\frac{2\sin s}{s} - 0\right) - \frac{2}{s}\left(-\frac{\cos s\,t}{s}\right)_0^1 + \left(-\frac{2}{s}\sin 2s - 0\right) + \frac{2}{s}\left(-\frac{\cos s\,t}{s}\right)_1^2$$

$$= \frac{2}{s}\sin s + \frac{2}{s^2}(\cos s - 1) - \frac{2}{s}\sin 2s - \frac{2}{s^2}(\cos 2s - \cos s)$$

$$= \frac{2}{s}\sin s + \frac{4}{s^2}\cos s - \frac{2}{s}\sin 2s - \frac{2}{s^2}\cos 2s - \frac{2}{s^2}\cdot d$$

Example 5 : Find f(t) if $F_s(f(t)) = s^m e^{-\lambda s}$, $m \in N$
and $F_c(f(t)) = s^m e^{-\lambda s}$

Sol. (i) Given $F_s(f(t)) = s^m e^{-\lambda s}$

$$\Rightarrow \qquad f(t) = F_s^{-1}(s^m\ e^{-\lambda s}) = \frac{2}{\pi}\int_0^\infty s^m e^{-\lambda s}\sin st\,ds \qquad \ldots (i)$$

To evaluate $\int_0^\infty s^m e^{-\lambda s}\ \sin st\,ds$

We know $\int_0^\infty e^{-\lambda s}\sin st\,ds = \frac{t}{\lambda^2 + t^2}$ ... (ii)

Differentiate (i) m times w.r.t.λ, we get

$$(-1)^m \int_0^\infty e^{-\lambda s} s^m \sin st\, ds = \frac{(-1)^m \lfloor m}{(\lambda^2 + t^2)^{\frac{m+1}{2}}} \sin\left((m+1)\tan^{-1}\frac{t}{\lambda}\right)$$

$$\left(\because \frac{d^n}{dx^n}\left(\frac{a}{x^2+a^2}\right) = \frac{(-1)^n \lfloor n}{(x^2+a^2)^{\frac{n+1}{2}}} \sin\left((n+1)\tan^{-1}\frac{a}{x}\right)\right)$$

$$\Rightarrow \quad (-1)^m \left(\frac{\pi}{2} f(t)\right) = \frac{(-1)^m \lfloor m}{(\lambda^2 + t^2)^{\frac{m+1}{2}}} \sin\left((m+1)\tan^{-1}\frac{t}{\lambda}\right) \qquad \text{(Using i)}$$

$$\Rightarrow \quad f(t) = \frac{2\lfloor m}{n\,(t^2+\lambda^2)^{\frac{m+1}{2}}} \sin\left((m+1)\tan^{-1}\frac{t}{\lambda}\right)$$

(ii) Given $F_c\,(f(t)) = s^m e^{-\lambda s}$

$$\Rightarrow \quad f(t) = F_c^{-1}(s^m\ e^{-\lambda t}) = \frac{2}{\pi}\int_0^\infty s^m e^{-\lambda s} \cos st\, ds \qquad \ldots \text{(iii)}$$

To evaluate $\int_0^\infty s^m e^{-\lambda s} \cos st\, ds$

We know $\int_0^\infty e^{-\lambda s} \cos st\, ds = \frac{t}{\lambda^2 + t^2}$ ... (iv)

Differentiate (iv) m times w.r.t.λ, we get

$$(-1)^m \int_0^\infty e^{-\lambda s} s^m \cos st\, ds = \frac{(-1)^m \lfloor m}{(\lambda^2 + t^2)^{\frac{m+1}{2}}} \cos\left((m+1)\tan^{-1}\frac{t}{\lambda}\right)$$

$$\left(\because \frac{d^n}{dx^n}\left(\frac{a}{x^2+a^2}\right) = \frac{(-1)^n \lfloor n}{(x^2+a^2)^{\frac{n+1}{2}}} \cos\left((n+1)\tan^{-1}\frac{a}{x}\right)\right)$$

$$\Rightarrow \quad (-1)^m \left(\frac{\pi}{2} f(t)\right) = \frac{(-1)^m \lfloor m}{(\lambda^2 + t^2)^{\frac{m+1}{2}}} \cos\left((m+1)\tan^{-1}\frac{t}{\lambda}\right) \qquad \text{(Using (iii))}$$

$$\Rightarrow \qquad f(t) = \frac{2}{\pi} \frac{2\lfloor m}{n\,(t^2+\lambda^2)^{\frac{m+1}{2}}} \cos\left((m+1)\tan^{-1}\frac{t}{\lambda}\right).$$

2.1.7 Self Check Exercise

1. Let $f(x) = \begin{cases} 2, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$. Find fourier integral of f(x).

2. Let $f(x) = \begin{cases} 2, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$. Find fourier sine integral of f(x) and evaluate

$$\int_0^\infty \frac{1-\cos s\pi}{s} \sin s\,x\,ds.$$

3. Find Fourier transform of

$$f(t) = \begin{cases} 1 + \dfrac{t}{2a}, & -2a < t < 0 \\ 1 - \dfrac{t}{2a}, & 0 < t < 2a \\ 0, & \text{otherwise} \end{cases}$$

4. Find the Fourier sine and cosine transform of f(t) where

$$f(t) = \begin{cases} t &, \ 0 < t < 1 \\ 2-t &, \ 1 < t < 2 \\ 0 &, \ t > 2 \end{cases}$$

5. If cosine Fourier transform of f(t) is

$$F_c(f(t)) = \begin{cases} \dfrac{2\lambda - s}{4\pi}, & s < 2\lambda \\ 0, & s > 2\lambda \end{cases}, \text{ Find } f(t)$$

LESSON NO. 2.2 **Author : Dr. Chanchal**

FOURIER TRANSFORMS-II

Structure :

2.2.1 Introduction

After introducing the concept of Fourier transform, we here, in this lesson, will go through the following properties of Fourier transforms :

1. F (af (t) + bg(t)) = aF(s) + bG(s), where F(s) and G(s) are Fourier transforms of f(t) and g(t). This is called **Linearity Property**.

2. $F(g(\lambda t)) = \frac{1}{\lambda} F\left(\frac{s}{\lambda}\right)$ where F(s) is Fourier transform of f(t). This is called change of scale property.

3. $F(f(t-\alpha)) = e^{is\alpha} F(s)$ or $F^{-1}(e^{-is\alpha} F(s))$ $f(t-\alpha)$ and $F(e^{i\alpha t} f(t)) = F(s+\alpha)$, where F(s) is Fourier transform of f(t). This is called shifting property.

4. **Modulation Theorem :** $F(\cos(\lambda t) f(t)) = \frac{F(s+\lambda) + F(s-\lambda)}{2}$, where F(s) is Fourier transform of f(t).

5. **Convolution Theorem :** F(f*g) = (F (f(t)) F(g(t)) where, f*g may be defined as The convolution of two functions f(t) and g(t) ($-\infty < t < \infty$) is denoted by f * g and defined as $f * g = \int_{-\infty}^{\infty} f(u)g(t-u)du$ or $\int_{-\infty}^{\infty} g(u)\, f(t-u)\, du$.

6. **Parsevel's Identity :** (i) $\int_{-\infty}^{\infty} f(t)\,\bar{g}(t)\,dt = \int_{-\infty}^{\infty} f(s)\,\bar{G}\,(s)\,ds$

$$\text{(ii)} \quad \int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_{-\infty}^{\infty} |F(s)|^2\, ds$$

Where F(s) and G(s) are respective complex fourier transform of f(t) and g(t). Further, $\bar{g}(t)$ and $\bar{G}(s)$ are the respective complex conjugates of g(t) and G(s).

Now, we prove all the above properties.

2.2.2 Linearity Property

Theorem 1 : If F(s) and G(s) are Fourier transforms of f(t) and g(t) respectively,

Then F (a f(t) + bg (t)) = a F(s) + b G (s) where a, b are constants.

Proof : We know $F(s) = \int_{-\infty}^{\infty} e^{ist} f(t)\,dt$ and $G(s) = \int_{-\infty}^{\infty} e^{ost} g(t)\,dt$ ... (1)

Now F (a f(t) + bg (t)) $= \int_{-\infty}^{\infty} e^{ist}(af(t) + bg(t)\,dt$

$$= \int_{-\infty}^{\infty} e^{ist} af(t)\,dt + \int_{-\infty}^{\infty} e^{-st} bg\,(t)\,dt$$

$$= a\int_{-\infty}^{\infty} e^{ist} f(t)\,dt + b\int_{-\infty}^{\infty} e^{-st} g\,(t)\,dt$$

= a F (s) + b G (s) [Using (1)]

Theorem 2 : If F_s (s) and G_s (s) are Fourier sine transforms and F_c (s) and G_c (s) are Fourier cosine transforms of f (t) and g (t) respectively,

Then F_s (a f (t) + bg (t)) = a F_s (s) + b G_s (s)

and F_c (af(t) + bg (t)) = a F_c (s) + b G_c (s)

Proof : We know $F_s(s) = \int_0^{\infty} f(t) \sin st\,dt,\ F_c(s) = \int_0^{\infty} f(t)\cos st\,dt$

and $G_s(s) = \int_0^{\infty} g(t) \sin st\,dt,\ G_c(s) = \int_0^{\infty} g(t)\cos st\,dt$

Now $F_s(af(t)+bg(t)) = \int_0^\infty (af(t)+bg(t))\sin st\,dt$

$$= \int_0^\infty af(t)\sin st\,dt + \int_0^\infty bg(t)\sin st\,dt$$

$$= a\int_0^\infty f(t)\sin st\,dt + b\int_0^\infty g(t)\sin st\,dt$$

$$= a\,F_s(s) + b\,G_s(s).$$

And $F_c(a\,f(t) + bg(t)) = \int_0^\infty (a\,f(t)+bg(t)\cos st\,dt$

$$= \int_0^\infty af(t)\cos st\,dt + \int_0^\infty bg(t)\cos st\,dt$$

$$= a\int_0^\infty f(t)\cos st\,dt + b\int_0^\infty g(t)\cos st\,dt$$

$$= a\,F_c(s) + b\,G_c(s).$$

2.2.3 Change of Scale Property

Theorem 3 : I. If F (s) is complex Fourier transform of f(t)

Then $F(f(\lambda t)) = \frac{1}{\lambda}F\left(\frac{s}{\lambda}\right)$

II. If $F_s(s)$ is sine Fourier transform of f(t)

Then $F_s(f(\lambda t)) = \frac{1}{\lambda}F_s\left(\frac{s}{\lambda}\right)$

III. If $F_c(s)$ cosine Fourier transform of f(t)

Then $F_c(f(\lambda t)) = \frac{1}{\lambda}F_c\left(\frac{s}{\lambda}\right)$

Proof : (I) We know $F(s) = \int_{-\infty}^{\infty} e^{ist} f(t)\,dt$

Now $F(f(\lambda t)) = \int_{-\infty}^{\infty} e^{ist} f(t)\, dt$ [Put $\lambda t = u \Rightarrow dt = \frac{du}{\lambda}$]

$$= \int_{-\infty}^{\infty} e^{\frac{isu}{\lambda}} f(u) \frac{du}{\lambda}$$

$$= \frac{1}{\lambda} \int_{-\infty}^{\infty} e^{i\left(\frac{s}{\lambda}\right)u} f(u)\, du = \frac{1}{\lambda} F\left(\frac{s}{\lambda}\right).$$

(II) We know $F_s(s) = \int_0^{\infty} f(t) \sin st\, dt$

Now $F_s(f(\lambda t)) = \int_0^{\infty} f(\lambda t) \sin st\, dt$ [Put $\lambda t = u \Rightarrow dt = \frac{du}{\lambda}$]

$$= \int_0^{\infty} f(u) \sin\left(\frac{su}{\lambda}\right) \frac{du}{\lambda}$$

$$= \frac{1}{\lambda} \int_0^{\infty} f(u) \sin\left(\frac{s}{\lambda} u\right) du$$

$$= \frac{1}{\lambda} F_s\left(\frac{s}{\lambda}\right).$$

Now, the reader can easily prove the III part.

2.2.4 Shifting Property

Theorem 4 : If F (s) Fourier transform of f(t)

Then (i) $F(f(t-\alpha)) = e^{is\alpha} F(s)$ or $F^{-1}(e^{is\alpha} F(s)) = f(t-\alpha)$

(ii) $F(e^{i\alpha t} f(t)) = F(s+\alpha)$.

Proof : (i) We know $F(s) = \int_{-\infty}^{\infty} e^{ist} f(t)\, dt$

Now $F(f(t-\alpha)) = \int_{-\infty}^{\infty} f(t-\alpha)\, e^{ist}\, dt$

$$= \int_{-\infty}^{\infty} f(u)\, e^{is(u+\alpha)}\, du \qquad [\text{Put } t - \alpha = u \Rightarrow dt = du]$$

$$= \int_{-\infty}^{\infty} f(u)\, e^{isu} e^{is\alpha}\, du = e^{is\alpha} \int_{-\infty}^{\infty} f(u)\, e^{isu} du$$

$$= e^{is\alpha} F(s)$$

(ii) $$F(e^{i\alpha t} f(t)) = \int_{-\infty}^{\infty} e^{ist} e^{i\alpha t} f(t)\, du = \int_{-\infty}^{\infty} e^{i(s+\alpha)t} f(t)\, dt$$

$= F(s + \alpha)$.

2.2.5 Modulation Theorem

Theorem 5 : If F (s) is complex Fourier transform of f (t).

Then $F(\cos(\lambda t) f(t)) = \dfrac{F(s+\lambda) + F(s-\lambda)}{2}$

Proof : We know $F(s) = \int_{-\infty}^{\infty} e^{ist} f(t)\, dt$

Now $F(\cos \lambda t f(t)) = \int_{-\infty}^{\infty} e^{ist} \cos \lambda t\, f(t)\, dt$

$$= \int_{-\infty}^{\infty} e^{ist} \frac{(e^{i\lambda t} + e^{-i\lambda t})}{2} f(t)\, dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} (e^{i(s+\lambda)t} + e^{i(s-\lambda)t}) f(t)\, dt$$

$$= \frac{1}{2} \left(\int_{-\infty}^{\infty} e^{i(s+\lambda)t} f(t)\, dt + \int_{-\infty}^{\infty} e^{i(s-\lambda)t} f(t)\, dt \right)$$

$$= \frac{1}{2} (F(s+\lambda) + F(s\,'-'\,\lambda)) = \frac{F(s+\lambda) + F(s-\lambda)}{2}.$$

Theorem 6 : If $F_c(s)$ is cosine Fourier transform and $F_s(s)$ is sine Fourier transform of f (t).

Then

(i) $F_s(\cos(\lambda t) f(t)) = \dfrac{F_s(s+\lambda) + F_s(s-\lambda)}{2}$

(ii) $F_c(\cos(\lambda t) f(t)) = \dfrac{F_c(s+\lambda) + F_c(s-\lambda)}{2}$

(iii) $F_s(\sin(\lambda t) f(t)) = \dfrac{F_c(s-\lambda) - F_c(s+\lambda)}{2}$

(iv) $F_c(\sin(\lambda t) f(t)) = \dfrac{F_s(s+\lambda) - F_s(s-\lambda)}{2}$

Proof : (i) $F_s(\cos(\lambda t) f(t)) = \int_0^\infty \cos\lambda t\, f(t) \sin st\, dt$

$$= \frac{1}{2}\int_0^\infty 2 \sin st \cos \lambda t\, f(t)\, dt$$

$$= \frac{1}{2}\int_0^\infty (\sin(s+\lambda) t + \sin(s-\lambda) t\, f(t)\, dt$$

$$= \frac{1}{2}\int_0^\infty f(t) \sin(s+\lambda) t\, dt + \int_0^\infty f(t) \sin(s-\lambda) t\, dt$$

$$= \frac{1}{2}(F_s(s+\lambda) + F_s(s-\lambda))$$

(ii) $F_c(\cos(\lambda t) f(t)) = \int_0^\infty \cos\lambda t\, f(t) \cos st\, dt$

$$= \frac{1}{2}\int_0^\infty (2 \cos st \cos \lambda t) f(t)\, dt$$

$$= \frac{1}{2}\int_0^\infty (\cos(s+\lambda) t + \cos(s-\lambda) t\, f(t)\, dt$$

$$= \frac{1}{2} \int_0^\infty f(t) \cos (s + \lambda)\, t\, dt + \int_0^\infty f(t) \cos (s - \lambda)\, t\, dt$$

$$= \frac{1}{2} (F_c (s + \lambda) + F_c (s - \lambda))$$

Now, the reader can easily prove (iii) and (iv).

2.2.6 Convolution Theorem

Theorem 7 : Prove the Fourier transform of convolution of f(t) and g(t) is the product of their Fourier transforms i.e. F (f * g) = (F (f(t)) (F (g (t)))

Proof : We have $f * g = \int_{-\infty}^{\infty} f(u)\, g(t-u)\, du$

$$\Rightarrow \quad F (f * g) = F\left(\int_{-\infty}^{\infty} f(u)\, g(t-u)\, du \right)$$

$$= \int_{-\infty}^{\infty} e^{ist} \left\{ \int_{-\infty}^{\infty} f(u)\, g(t-u)\, du \right\} dt$$

$$= \int_{-\infty}^{\infty} f(u) \left\{ \int_{-\infty}^{\infty} e^{ist}\, g(t-u)\, dt \right\} du$$

$$= \int_{-\infty}^{\infty} f(u) \left\{ \int_{-\infty}^{\infty} e^{is(x+u)}\, g(x)\, dx \right\} du \quad [\text{Put } t - u = x \Rightarrow dt = dx]$$

[By change of order of integration]

$$= \int_{-\infty}^{\infty} f(u) \left\{ e^{isu} \int_{-\infty}^{\infty} e^{isx}\, g(x)\, dx \right\} du$$

$$= \int_{-\infty}^{\infty} e^{isu} f(u)\, F(g(x))\, du$$

$$= (F\, (g(x))) \left(\int_{-\infty}^{\infty} e^{isu} f(u)\, du \right)$$

= F (g (t)) F (f (t)) (F (g(x)) = F (g (t))

= F (f (t)) F (g (t))

Note : (i) We also denote F (s) and G (s) as Fourier transform of f and g respectively.

$\therefore$ F (f * g) = F (s) G (s)

(ii) By inversion formula

$F^{-1}(F(s)\ G(s)) = f * g = F^{-1}(F(s)) * F^{-1}(G(s))$.

2.2.7 Parseval's Identity

Theorem 8 : If the complex fourier transform of f (t) and g (t) be F (s) and G (s) respectively.

Prove : (i) $\int_{-\infty}^{\infty} f(t)\,\bar{g}(t)\,dt = \int_{-\infty}^{\infty} F(s)\,\bar{G}(s)\,ds$

where $\bar{g}(t)$ is complex conjugate of g (t) and $\bar{G}(s)$ is complex conjugate of G(s).

(ii) $\int_{-\infty}^{\infty} |f(t)|^2\,dt = \int_{-\infty}^{\infty} |F(s)|^2 ds$

Proof : (i) By inversion formula (for Fourier transform), we get

$$g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(s)\,e^{-ist} ds$$

Taking complex conjugate on both sides of (i)

We get $\overline{g(t)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{G(s)}\,e^{ist}\,ds$

$$\text{Now } \int_{-\infty}^{\infty} f(t)\,\overline{g(t)}\,dt = \int_{-\infty}^{\infty} f(t)\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{G(s)}\,e^{ist}\,ds\right)dt$$

$$= \int_{-\infty}^{\infty} \overline{G(s)}\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\,e^{ist}\,dt\right)ds$$

(By change of order of integration)

$$= \int_{-\infty}^{\infty} F(s)\,\overline{G(s)}\,ds$$

(ii) g (t) = f (t) in (i). Then G (s) = F (s)

$$\therefore \quad \int_{-\infty}^{\infty} f(t)\,\bar{f}(t)\,dt = \int_{-\infty}^{\infty} F(s)\,\overline{F(s)}\,ds \Rightarrow \int_{-\infty}^{\infty} |f(t)|^2\,dt = \int_{-\infty}^{\infty} |f(s)|^2\,ds \cdot$$

Another Form : (i) $\int_{-\infty}^{\infty} f(t)\,\overline{g(t)}\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\,\overline{G(s)}\,ds$

(ii) $\int_{-\infty}^{\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(s)|^2\,ds$.

Theorem 9 : If $F_c(s)$, $G_c(s)$ are Fourier cosine transforms and $F_s(s)$, $G_s(s)$ are Fourier sine transforms of f (t) and g (t) respectively.

Then prove (i) $\int_0^{\infty} f(t)\,g(t)\,dt = \frac{2}{\pi}\int_0^{\infty} F_s(s)\,G_s(s)\,ds$

(ii) $\int_0^{\infty} f(t)\,g(t)\,dt = \frac{2}{\pi}\int_0^{\infty} F_c(s)\,G_c(s)\,ds$

(iii) $\int_0^{\infty} (f(t))^2\,dt = \frac{2}{\pi}\int_0^{\infty} (F_s(s))^2\,ds$

(iv) $\int_0^{\infty} (f(t))^2\,dt = \frac{2}{\pi}\int_0^{\infty} (F_c(s))^2\,ds$

Proof : [(i) and (iii)]By inversion formula (for Fourier sine transform)

We get $g(t) = \frac{2}{\pi}\int_0^{\infty} G_s(s)\sin st\,ds$

Now $\int_0^{\infty} f(t)\,g(t)\,dt = \int_0^{\infty} f(t)\left(\frac{2}{\pi}\int_0^{\infty} G_s \sin st\,ds\right)dt$

$$= \frac{2}{\pi}\int_0^{\infty} G_s(s)\left(\int_0^{\infty} f(t)\sin st\,dt\right)ds$$

(By change of order of integral)

$$= \frac{2}{\pi}\int_0^{\infty} G_s(s)\,F_s(s)\,ds$$

$$= \frac{2}{\pi}\int_0^{\infty} F_s(s)\,G_s(s)\,ds$$

which proves (i)

Put g (t) = f(t) Then $G_s(s) = F_s(s)$

$$\therefore \quad \int_0^\infty (f(t))^2 dt = \frac{2}{\pi}\int_0^\infty (F_s(s))^2 ds \qquad \text{which proves (iii)}$$

Now, the reader caneasily prove (ii) and (iv) part.

2.2.8 Some Important Examples

Example 1 : Find Fourier sine and cosine transform of $f(t) = \begin{cases} e^{2t} - e^{-2t}, & 1 \le t < 2 \\ 0, & \text{otherwise} \end{cases}$.

Sol. $F_s(f(t)) = \int_0^\infty f(t) \sin st \, dt$

$$= \int_0^1 f(t) \sin st \, dt + \int_1^2 f(t) \sin st \, dt + \int_2^\infty f(t) \sin st \, dt$$

$$= \int_0^1 0 \sin st \, dt + \int_1^2 (e^{2t} - e^{-2t}) \sin st \, dt + \int_2^\infty 0 \sin st \, dt$$

(Using linear property)

$$= 0 + \int_1^2 e^{2t} \sin st \, dt - \int_1^2 e^{2t} \sin st \, dt + 0$$

$$= \left(\frac{e^{2t}}{s^2+4}(2 \sin st - s \cos st\right)_1^2 - \left(\frac{e^{-2t}}{s^2+4}(-2 \sin st - s \cos st)\right)_1^2$$

$$= \left(\frac{e^4}{s^2+4}(2 \sin 2s - s \cos 2s)\right) - \left(\frac{e^2}{s^2+4}(2 \sin s - s \cos s)\right)$$

$$+ \left(\frac{e^{-4}}{s^2+4}(2 \sin 2s + s \cos 2s)\right) - \left(\frac{e^{-2}}{s^2+4}(2 \sin s + s \cos s)\right)$$

$$= \frac{1}{s^2+4} \left(2 (e^4 + e^{-4}) \sin 2s - (e^4 - e^{-4}) s \cos 2s - 2 (e^2 + e^{-2}) \sin s + (e^2 - e^{-2}) s \cos\right)$$

$$= \frac{2}{s^2+4}(2 \cosh 4 \sin 2s - s \cos 2s \sinh 4 - 2 \cosh 2 \sin s + s \cos s \sinh 2)$$

Now, the reader can easily solve for the Fourier cosine transform.

Example 2 : Find Fourier cosine transform of $f(t)=\begin{cases}2\cos 3t, & 0\le t\le a\\ 0, & t<a\end{cases}$ using modulation theorem.

Sol. Here $f(t)=\begin{cases}2\cos 3t, & 0\le t\le a\\ 0, & t<a\end{cases}$

$$\Rightarrow \qquad f(t)=(\cos 3t)\left(\begin{cases}2, 0\le t\le a\\ 0, t>a\end{cases}\right)$$

= (cost 3 t) g (t) say

Firstly find Fourier cosine transform of $g(t)=\begin{cases}2, 0\le t\le a\\ 0, t>a\end{cases}$

$$F_c(g(t))=\int_0^{\infty} f(t)\cos st\,dt$$

$$=\int_0^{a} 2\cos st\,dt+\int_0^{\infty} 0\,.\cos st\,dt$$

$$=2\left.\frac{\sin st}{s}\right)_0^a+0=\frac{2}{s}\sin as=F_c(s)$$

By Modulation Theorem

$$F_c(\cos 3tg\ (t))=\frac{F_c(s+3)+F_c(s-3)}{2}$$

$$=\frac{1}{2}\left(\frac{2\sin(a(s+3))}{s+3}+\frac{2\sin(a(s-3))}{s-3}\right)$$

$$=\frac{\sin a\ (s+3)}{s+3}+\frac{\sin a\ (s-3)}{s-3}$$

$$=\frac{(s-3)\,(\sin(a(s+3)))+(s-3)\,(\sin(a\ (s-3)))}{s^2-9}$$

$$=\frac{s\ (\sin(a(s+3)))+\sin(a(s-3))-3\ (\sin(a(s+3)))-\sin(a\ (s-3))}{s^2-9}$$

$$= \frac{2s \sin a\, s \cos 3a - 6 \cos as \sin 3a}{s^2 - 9}.$$

Example 3 : Use Parsevel's identity to prove $\int_0^\infty \frac{\sin ax}{x\,(a^2 + x^2)}\, dx = \frac{\pi\,(1 - e^{-a^2})}{2a^2}$

Sol. Let $f(t) = e^{-at}$

Then $F_c(f(t)) = \int_0^\infty e^{-at} \cos st\, dt = \frac{a}{s^2 + a^2}$

Further take $g(t) = \begin{cases} 1, 0 < t < a \\ 0, t > a \end{cases}$

Then $F_c\,(g(t)) = \int_0^\infty g(t) \cos st\, dt$

$$= \int_0^a 1 . \cos st\, dt + \int_a^\infty 0 . \cos st\, dt$$

$$= \left.\frac{\sin st}{s}\right)_0^a + 0$$

$$= \frac{\sin sa}{s}$$

By Parsevel's identity for Fourier cosine transform, we get

$$\int_0^\infty f(t)\, g(t)\, dt = \frac{2}{\pi} \int_0^\infty F_c(f\,(t))\, F_c\,(g(t))\, ds$$

$$\Rightarrow \int_0^a f(t)\, g(t)\, dt + \int_0^\infty f(t)\, g(t)\, dt = \frac{2}{\pi} \int_0^\infty \left(\frac{a}{s^2 + a^2}\right)\left(\frac{\sin sa}{s}\right) ds$$

$$\Rightarrow \int_0^a e^{-at} . 1\, dt + \int_a^\infty e^{-at} . 0\, dt = \frac{2}{\pi} \int_0^\infty \frac{a \sin as}{s\,(s^2 + a^2)}\, ds$$

$$\Rightarrow \left(\frac{e^{-at}}{-a}\right)_0^a + 0 = \frac{2}{\pi} \int_0^\infty \frac{a \sin as}{s\,(s^2 + a^2)}\, ds$$

$$\Rightarrow \qquad \frac{1}{-a}(e^{-a^2}-1) = \frac{2}{\pi}\int_0^\infty \frac{a \sin as}{s\,(s^2+a^2)}\,ds$$

$$\Rightarrow \qquad \int_0^\infty \frac{a \sin as}{s\,(s^2+a^2)}\,ds = \frac{\pi}{2}\left(\frac{1-e^{-a^2}}{a}\right)$$

$$\Rightarrow \qquad \int_0^\infty \frac{\sin as}{x\,(x^2+a^2)}\,dx = \frac{\pi\,(1-e^{-a^2})}{2a^2}.$$

Example 4 : Use convolution to find $F^{-1}\left(\frac{1}{12-s^2+7\text{ is}}\right)$

Sol. We have $12 - s^2 + 7\text{ is} = 12 + 4\text{ is} + 3\text{ is} + i^2 s^2$

$= 4\,(3 + is) + is\,(3 + is)$

$= (4 + is)\,(3 + is)$

$$\therefore \qquad F^{-1}\left(\frac{1}{12-s^2+7\text{ is}}\right) = F^{-1}\left(\frac{1}{(4+is)\,(3+is)}\right)$$

$$= F^{-1}\left(\left(\frac{1}{3+is}\right).\left(\frac{1}{4+is}\right)\right)$$

Already we have done in example 4, that

$$F^{-1}\left(\frac{1}{3+is}\right) = e^{-3t}H\,(t)$$ where H (t) is unit step function

Similarly $$F^{-1}\left(\frac{1}{4+is}\right) = e^{-4t}H(t)$$

So that $$F^{-1}\left(\frac{1}{12-s^2+7\text{ is}}\right) = (e^{-3t}H(t)) * (e^{-4t}H(t))$$

$$= \int_{-\infty}^{\infty} e^{-3u}H(u)\,e^{-4(t-u)}H(t-u)\,du$$

$$= e^{-4t} \int_{-\infty}^{\infty} e^{u} H(u)\, H(t-u)\, du$$

Here H (u) H (t – u) $= \begin{cases} 1 \text{ for } 0 < u < t \\ 0 \text{ for } u < 0 \text{ or } u > t \end{cases}$

So that $F^{-1}\left(\dfrac{1}{12 - s^2 + 7\,is}\right) = e^{-rt}\left(\int_{-\infty}^{0} e^{u}.0\,du + \int_{0}^{1} e^{u}\,.\,1du + \int_{1}^{\infty} e^{u}.\,0\,du\right)$

$$= e^{-4t}(e^{u})_0^t = e^{-4t}\,(e^{t} - 1) = e^{-3t} - e^{-4t}.$$

2.2.9 Self Check Exercise

1. Using Parsevel's identity, prove $\displaystyle\int_0^{\infty} \frac{dt}{(t^2+1)^2} = \frac{\pi}{4}$

2. Prove if F (f (t) = F (s) then F (t^n f (t)) = $(-i)^n \dfrac{d^n F(s)}{ds^n}$.

3. Use convolution to find $F^{-1}\left(\dfrac{1}{6 + 5\,is - s^2}\right)$

4. Solve for f (t) if $\displaystyle\int_0^{\infty} f(t)\cos st\, ds = \begin{cases} 1, 0 \le t < 1 \\ 2, 1 \le t < 2 \\ 0, t \ge 2 \end{cases}$

5. Find Fourier sine transform of $f(t) = \begin{cases} 2t\cos 3t, & 0 \le t \le a \\ 0, & t > a \end{cases}$.

LESSON NO. 2.3 | Author : Dr. Chanchal

FOURIER TRANSFORMS-III

Structure :

2.3.1 Introduction

Definition 1. Finite Fourier Sine Transform

Let f (t) be a function defined on (0 l) and satisfying Dirichlet's conditions on (0 l).

The **finite fourier sine transform** of f(t), 0 < t < l is defined as

$$F_s(f(t)) = \int_0^l f(t) \sin \frac{\pi st}{l} dt; s \in N$$

Definition 2. Inverse Finite Fourier Sine Transform :

The above function f (t) is known as **inverse finite fourier sine transform of $F_s(f(t))$** and is defined as

$$F_s^{-1}(F_s (f(t))) = f(t) = \frac{2}{l} \sum_{s=1}^{\infty} F_s(f(t)) \frac{\sin \pi st}{l}$$

Definition 3. Finite Fourier Cosine Transform

Let f(t) be a function defined on (0, l) and satisfying Dirichlet's conditions on (0, l).

Then **finite fourier cosine transform** of f(t), 0 < t < l is defined as

$$F_c(f(t)) = \int_0^l f(t) \frac{\cos \pi st}{l} dt; s \in N$$

Definition 4. Inverse Finite Fourier Cosine Transform :

The above function f (t) is known as **inverse finite fourier cosine transform**

of F_c(f (t)) or F_c(s) and is defined as

$$\left(\text{This formula is got from Fourier cosine series } f(t)=\frac{a_0}{2}+\sum_{s=1}^{\infty} a_n \frac{\cos n\pi t}{l}\right).$$

2.3.2 Inversion Formulae

The inversion formuale for finite Fourier sine transform is given by equation (1) given in the introduction. It can be proved as :

Proof : Given function f(t) is defined in (0, l) satisfying Dirichlet's conditions in (0 l).

Now define f(t) on ($-l$ 0) such that f (–t) = –f (t) so that the function f(t) becomes an odd function on ($-l$ l), satisfying Dirichlet's condition on ($-l$ l).

∴ By result from Fourier series,

$$f(t)=\sum_{s=1}^{\infty} b_s \frac{\sin \pi st}{l}$$

where $b_s=\dfrac{2}{l}\displaystyle\int_0^l f(t)\frac{\sin \pi st}{l}\,dt,\ s\in N$

$$=\frac{2}{l}F_s\,(f(t)) \qquad \text{[Using (i)]}$$

Hence $f(t)=\displaystyle\sum_{s=1}^{\infty}\frac{2}{l}F_s\,(f(t))\frac{\sin \pi st}{l}$

$$=\frac{2}{l}\sum_{s=1}^{\infty}F_s(f(t))\frac{\sin \pi st}{l}.$$

Now, the inversion formulae for finite Fourier cosine transform, given by equation (2), can be proved as :

Proof : Given function f(t) is defined in (0, l) satisfying Dirichlet's conditions in (0 l)

Now define f(t) on ($-l$ 0) such that f (–t) = f(t)

∴ By result from Fourier series,

$$f(t)=\frac{a_0}{2}+\sum_{s=1}^{\infty}a_s\cos\left(\frac{s\pi t}{l}\right)$$

where $a_s=\dfrac{2}{l}\displaystyle\int_0^l f(t)\cos\left(\frac{s\pi t}{l}\right)dt,\ s\in N$

$$= \frac{2}{l} F_c\ (f(t)) \qquad \text{[Using (i)]}$$

And $\quad a_0 = \frac{2}{l} \int_0^l f(t)\, dt = \frac{2}{l} F_c(0)$

so that $f(t) = \frac{1}{l} F_c(0) + \frac{2}{l} \sum_{s=1}^{\infty} F_c(f(t)) \cos\left(\frac{s\pi\, t}{l}\right)$

$$= \frac{1}{l} F_c(0) + \frac{2}{l} \sum_{s=1}^{\infty} F_c(s) \cos\left(\frac{s\pi\, t}{l}\right).$$

2.3.3 Some Important Examples

Example 1 : Find finite Fourier sine and cosine transform of

$$f(t) = \begin{cases} t,\ 0 \le t \le \dfrac{\pi}{2} \\ \pi - t,\ \dfrac{\pi}{2} \le t \le \pi \end{cases}.$$

Sol. (i) Here $F_s\ (f(t)) = \int_0^{\pi} f(t) \sin st\, dt$

$$= \int_0^{\frac{\pi}{r}} f(t) \sin st\, dt + \int_{\frac{\pi}{2}}^{\pi} f(t) \sin st\, dt = \int_0^{\frac{\pi}{r}} t \sin st\, dt + \int_{\frac{\pi}{2}}^{\pi} (\pi - t) \sin st\, dt$$

$$= \left(r\left(\frac{-\cos st}{s}\right)\right)_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 1 . \left(\frac{-\cos st}{s}\right) dt + \left((\pi - t)\left(\frac{-\cos st}{s}\right)\right)_{\frac{\pi}{s}}^{\pi} - \int_{\frac{\pi}{2}}^{\pi} (-1)\left(\frac{-\cos st}{s}\right) dt$$

$$= -\frac{1}{s}\left(\frac{\pi}{2} \frac{\cos s\,\pi}{2} - 0\right) + \frac{1}{s}\left(\frac{\sin st}{s}\right)_0^{\frac{\pi}{2}} + \left(0 + \frac{1}{s} \frac{\pi}{2} \cos \frac{s\pi}{2}\right) - \frac{1}{s}\left(\frac{\sin st}{s}\right)_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{1}{s^2}\left(\frac{\sin s\pi}{2} - 0\right) - \frac{1}{s^2}\left(\sin s\pi - \frac{\sin \pi s}{2}\right)$$

$$= \frac{2}{s^2}\frac{\sin \pi s}{2} \qquad (\because \sin s\,\pi = 0)$$

$$= \begin{cases} \frac{2}{s^2}(0) \text{ if s even integer} \\ \frac{2}{s^2}(1) \text{ if s odd integer i.e. s=1, 5,0,....} \\ \frac{2}{s^2}(-1) \text{ if s odd integer i.e. s=3, 7, 11,....} \end{cases}$$

$$= \begin{cases} 0 \text{ if s even integer} \\ \frac{2}{s^2} \text{ if s odd integer i.e. s=1, 5,9,....} \\ \frac{2}{s^2} \text{ if s odd integer i.e. s=3, 7, 11,....} \end{cases}$$

(ii) Here $F_c f(t) = \int_0^{\pi} f(t) \cos st\, dt$

$$= \int_0^{\frac{\pi}{2}} f(t) \cos st\, dt + \int_{\frac{\pi}{2}}^{\pi} f(t) \cos st\, dt$$

$$= \int_0^{\frac{\pi}{2}} (t) \cos st\, dt + \int_{\frac{\pi}{2}}^{\pi} (\pi - t) \cos st\, dt$$

$$= \left(t\,\frac{\sin st}{s}\right)_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 1\,.\,\frac{\sin st}{s}\, dt + \left((\pi - t)\,\frac{\sin st}{s}\right)_{\frac{\pi}{2}}^{\pi} - \int_{\frac{\pi}{2}}^{\pi} (-1)\,\frac{\sin st}{s}\, dt$$

$$= \frac{\pi}{2s}\sin\left(\frac{s\pi}{2}\right) - 0 - \frac{1}{s}\left(\frac{-\cos st}{s}\right)_0^{\frac{\pi}{2}} + \left(\frac{0\sin\pi s}{s} - \frac{\pi}{2s}\sin\left(\frac{s\pi}{2}\right)\right) + \frac{1}{s}\left(\frac{-\cos st}{s}\right)_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{1}{s^2}\left(\cos\left(\frac{s\pi}{2}\right) - 1\right) - \frac{1}{s^2}\left(\cos s\pi - \cos\left(\frac{s\pi}{2}\right)\right)$$

$$= \frac{2}{s^2}\cos\left(\frac{s\pi}{2}\right) - \frac{\cos s\pi}{s^2} - \frac{1}{s^2}$$

$$= \begin{cases} \frac{2}{s^2}(0) - \frac{(-1)}{s^2} - \frac{1}{s^2} \text{ if s odd integer} \\ \frac{2}{s^2}(-1) - \frac{1}{s^2} - \frac{1}{s^2} \text{ if s even integer i.e. s=2,6,10,....} \\ \frac{2}{s^2}(1) - \frac{1}{s^2} - \frac{1}{s^2} \text{ if s even integer i.e. s=4,8,12,....} \end{cases}$$

$$= \begin{cases} 0 \text{ if s odd integer} \\ \frac{-4}{s^2} \text{ if s even integer i.e. s=2,6,10,....} \\ 0 \text{ if s even integer i.e. s=4,8,12,....} \end{cases}.$$

Example 2 : Find finite Fourier sine and cosine transform of $f(t) = \sin \alpha t$, $t \in (0\ \pi)$.
Sol. Case (i) when $\alpha \neq \pm s$

$$F_s(\sin \alpha t) = \int_0^{\pi} \sin \alpha t \sin st\, dt$$

$$= \frac{1}{2}\int_0^{\pi} (\cos(\alpha - s)t - \cos(\alpha + s)t)\, dt$$

$$= \frac{1}{2}\left(\frac{\sin(\alpha - s)t}{\alpha - s} - \frac{\sin(\alpha + s)t}{\alpha + s}\right)_0^{\pi}$$

$$= \frac{1}{2}\left(\frac{\sin(\alpha - s)\,\pi}{\alpha - s} - \frac{\sin(\alpha + s)}{\alpha + s} - 0\right)$$

$$= \frac{1}{2}\left(\frac{(\alpha + s)\,(\sin(\alpha - s)\,\pi) - (\alpha - s)\,(\sin(\alpha + s)\,\pi)}{\alpha^2 - s^2}\right)$$

$$= \frac{1}{2}\left(\frac{\alpha\,(\sin(\alpha - s)\,\pi - \sin(\alpha + s) + s\,(\sin(\alpha - s)\,\pi - \sin(\alpha + s)\,\pi)}{\alpha^2 - s^2}\right)$$

$$= \frac{1}{2}\left(\frac{\alpha\,(2\cos\alpha\pi\sin s\pi) + s\,(2\sin\alpha\pi\cos s\pi)}{\alpha^2 - s^2}\right)$$

$$= \frac{1}{2}\left(\frac{0 + 2s\sin\alpha\pi\,(-1)^s}{\alpha^2 - s^2}\right)$$

$$= \frac{(-1)^s s\,(\sin\alpha\pi)}{\alpha^2 - s^2},\ a \neq \pm s.$$

Case (ii) : When $\alpha = \pm s$

$$F_s(\sin\alpha t) = \int_0^{\pi} \sin(\pm st)\sin st\,dt$$

$$= \int_0^{\pi} \pm\sin^2 st\,dt = \pm\int_0^{\pi} \frac{1 - \cos 2st}{s}\,dt$$

$$= \pm\frac{1}{2}\left(t - \frac{\sin 2st}{2s}\right)_0^{\pi} = \pm\frac{\pi}{2}.$$

Further to find fourier cosine transform.

Case (i) : $F_c(\sin\alpha t) = \int_0^{\pi} \sin\alpha t\cos st\,dt$

$$= \frac{1}{2}\int_0^{\pi} (\sin(\alpha + s)\,t + \sin(\alpha - s)\,t)\,dt$$

$$= \frac{1}{2}\left(\frac{-\cos(\alpha + s)t}{\alpha + s} - \frac{\cos(\alpha - s)t}{\alpha - s}\right)_0^{\pi}$$

$$= \frac{1}{2}\left(\frac{-\cos(\alpha+s)t}{\alpha+s} - \frac{\cos(\alpha-s)\,t}{\alpha-s}\right)_0^{\pi}$$

$$= -\frac{1}{2}\left(\frac{\cos(\alpha+s)\,\pi}{\alpha+s} + \frac{\cos(\alpha-s)\,\pi}{\alpha-s}\right) + \frac{1}{2}\left(\frac{1}{\alpha+s} + \frac{1}{\alpha-s}\right)$$

$$= -\frac{1}{2}\left(\frac{(\alpha-s)\cos(\alpha+s)\pi + (\alpha+s)\cos(\alpha-s)\pi}{\alpha^2-s^2}\right) + \frac{\alpha}{\alpha^2-s^2}$$

$$= -\frac{1}{2}\left\{\frac{a\,(\cos(\alpha+s)\pi + \cos(\alpha-s)\,\pi) - s(\cos(\alpha+s)\pi - \cos(\alpha-s)\pi}{\alpha^2-s^2}\right\} + \frac{\alpha}{\alpha^2-s^2}$$

$$= -\frac{1}{2}\left(\frac{\alpha(2\cos\alpha\pi\cos s\pi) + s(2\sin\alpha\pi\sin s\pi)}{\alpha^2-s^2}\right) + \frac{\alpha}{\alpha^2-s^2}$$

$$= -\frac{1}{2}\left(\frac{(2\alpha\cos\alpha\pi)\,(-1)^s + 0}{\alpha^2-s^2}\right) + \frac{\alpha}{\alpha^2-s^2}$$

$$= \frac{\alpha}{\alpha^2-s^2}(1-(-1)^s\cos\alpha\pi);\ \alpha \neq \pm s$$

Case (ii) : When $\alpha = \pm s$

$$F_c(\sin\alpha t) = \int_0^{\pi} \sin(\pm st)\cos st\,dt$$

$$= \pm\frac{1}{2}\int_0^{\pi} \sin 2st\,dt = \pm\frac{1}{2}\left(\frac{-\cos 2st}{2s}\right)_0^{\pi}$$

$$= \pm\frac{1}{2}(\cos 2s\pi - \cos 0) = \pm\frac{1}{2}(1-1) = 0.$$

2.3.4 Solution of Heat Conduction Problems by Fourier Fourier Transforms

Example 4 : Solve $\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}$ with following conditions

(i) $V_x(0, t) = 0$ for $t > 0$ (ii) $V(x,0) = \begin{cases} x, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$

(iii) V (x, t) is bounded

Sol. Given P.D. E is $\frac{\partial V}{\partial t} - \frac{\partial^2 V}{\partial x^2}$

As $V_x = 0$ is given at x = 0 … (i)

So take Fourier cosine transform on both sides of (i), we get

$$\int_0^\infty \frac{\partial V}{\partial t} \cos sx \, dx = \int_0^\infty \frac{\partial^2 V}{\partial x^2} \cos sx \, dx$$

$$\Rightarrow \quad \frac{d}{dt}\left(\int_0^\infty V \cos sx \, dx\right) = F_c\left(\frac{\partial^2 V}{\partial x^2}\right) = -\left(V_x\right)_{x=0} - s^2 \bar{V}_c$$

$$\Rightarrow \quad \frac{d}{dt}\bar{V}_c = -0 - s^2 \bar{V}_c \Rightarrow \frac{d\bar{V}_c}{\bar{V}_c} = -s^2 dt$$

Integrate,we get $\log \bar{V}_c = -s^2 t + c'$

$$\Rightarrow \quad \bar{V}_c = e^{-s^2 t + c'} = e^{c'} e^{-s^2 t}$$

$$\Rightarrow \quad \bar{V}_c = c\, e^{-s^2 t}. \text{ (say) where } e^{c'} = c$$

Also given $V(x,0) = \begin{cases} x, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$

Taking cosine fourier transform,

We get $\left(\bar{V}_c\right)_{t=0} = \int_0^\infty V(x,0) \cos sx \, dx$

$$= \int_0^1 x \cos sx \, dx + \int_1^\infty a \cos sx \, dx = \left(x \frac{\sin sx}{s}\right)_0^1 - \int_0^1 \frac{\sin sx}{x} dx + 0$$

$$= \frac{\sin s}{s} + \frac{1}{s}\left(+\frac{\cos sx}{s}\right)_0^1 = \frac{\sin s}{s} + \frac{1}{s^2}(\cos s - 1)$$

$$\Rightarrow \qquad c \,.\, e^0 = \left(\frac{\sin s}{s} + \frac{\cos s - 1}{s^2}\right)$$

$$\Rightarrow \qquad c = \left(\frac{\sin s}{s} + \frac{\cos s - 1}{s^2}\right)$$

Hence $\bar{V}_c = \left(\frac{\sin s}{s} + \frac{\cos s - 1}{s^2}\right) e^{-s^2 t}$.

Taking inverse cosine fourier transform,

We get, $V(x, t) = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin s}{s} + \frac{\cos s - 1}{s^2}\right) e^{-s^2 t} \cos sx \, dx$.

Example 5 : Solve $\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}$ using finite Fourier transform, if V (0, t) = 0 and V (4, t) = 0. And V (x, 0) = 4x, where $x \in (0\ 4)$; t > 0.

Sol. Given P.D.E. is $\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}$ … (i)

Subject to V (0, t) = 0, V (4,t) = 0 and V (x, 0) = 4x where $x \in (0, 4)$; t > 0

As V (0, t)and V (4, t) are given, so we should take finite Fourier sine transform on both sides of (i), so we get

(Here $l = 4$)

$$\int_0^4 \frac{\partial V}{\partial t} \sin \frac{s\pi x}{4} \, dx = \int_0^4 \frac{\partial^2 V}{\partial x^2} \sin \frac{s \pi x}{4} \, dx$$

$$\Rightarrow \qquad \frac{d}{dt}\left(\int_0^4 V \sin \frac{s\pi x}{4} dx\right) = \left(\frac{\partial V}{\partial x} \sin \frac{s\pi x}{4}\right)_0^4 - \int_0^4 \frac{\partial V}{\partial x}\left(\frac{s\pi}{4} \cos \frac{s\pi x}{4}\right) dx$$

$$\Rightarrow \qquad \frac{d\bar{V}_s}{dt} = 0 - \frac{s\pi}{4} \int_0^4 \frac{\partial V}{\partial x} \cos \frac{s\pi x}{4} \, dx$$

$$= -\frac{s\pi}{4}\left(\left(V \cos \frac{s\pi x}{4}\right)_0^4 - \int_0^4 V\left(-\frac{s\pi}{4} \sin \frac{s\pi x}{4}\right) ds\right)$$

$$= -\frac{s\pi}{4}\left[\{V(4,t)\cos s\pi - V(0,t)\cos 0) + \frac{s\pi}{4}\int_0^4 V \sin\frac{s\pi x}{4}\,dx\right)$$

$$= -\frac{s\pi}{4}\left(0 - 0 + \frac{s\pi}{4}\int_0^4 V \sin\frac{s\pi x}{4}\,dx\right) = -\frac{s^2\pi^2}{16}\bar{V}_s$$

$$\Rightarrow \quad \frac{d\bar{V}_s}{dt} = \frac{s^2\pi^2}{16}\bar{V}_s$$

$$\Rightarrow \quad \frac{d\bar{V}_s}{\bar{V}_s} = -\frac{s^2\pi^2}{16}\,dt$$

Integrating, $\log \bar{V}_s = -\frac{s^2\pi^2}{16}t + c'$

$$\Rightarrow \quad \bar{V}_s = e^{\frac{s^2\pi^2}{16}t + c'} = e^{c'}\, e^{\frac{s^2\pi^2}{16}6}$$

$$\Rightarrow \quad \bar{V}_s(s,t) = ce^{\frac{s^2\pi^2}{16}t} \text{ where } C = e^{c'} \text{ is arbitrary constant}$$

Further put t = 0, we get

$$\bar{V}_s(s,0) = ce^0 = c$$

$$\Rightarrow \quad C = \bar{V}_s(s,0) = \int_0^4 V(x,0)\sin\frac{s\pi x}{4}\,dx$$

$$= \int_0^4 4x \sin\frac{s\pi x}{4}\,dx$$

$$\Rightarrow \quad c = 4\left(x\left(-\frac{\cos\frac{s\pi x}{4}}{\frac{s\pi}{4}}\right)_0^4 - \int 1 \cdot \left(\frac{-\cos\frac{s\pi x}{4}}{\frac{s\pi}{4}}\right)dx\right)$$

$$\Rightarrow \qquad c = 4\left(-\frac{4\times 4}{s\pi}\cos s\pi + 0 + \frac{4}{s\pi}\left(\frac{\sin\frac{s\pi x}{4}}{\frac{s\pi}{4}}\right)_0^4\right)$$

$$\Rightarrow \qquad c = 4\left(\frac{-16\,(-1)^s}{s\;\pi} + \frac{16}{s^2\pi^2}\,(\sin s\pi - \sin\theta)\right)$$

$$\Rightarrow \qquad c = \frac{64\,(-1)^{s+1}}{s\pi} + 0$$

$$\therefore \qquad \bar{V}_s(s,t) = \frac{(-1)^{s+1}64}{s\pi}\,e^{\frac{-s^2\pi^2}{16}t}$$

Apply inversion theorem for finite Fourier sine transform,

We get $V\,(x,t) = \frac{2}{4}\sum_{s=1}^{\infty}\bar{V}_s(s,t)\,\sin\frac{s\pi x}{4}$

$$= \frac{1}{2}\sum_{s=1}^{\infty}(-1)^{s+1}\,\frac{64}{s\pi}\,e^{\frac{s^2\pi^2}{16}}\,\sin\frac{s\pi x}{4}$$

$$= \frac{32}{\pi}\sum_{s=1}^{\infty}\frac{(-1)^{s+1}}{s}\,e^{\frac{s^2\pi^2}{16}t}\,\sin\frac{s\pi x}{4}.$$

Note : The above problem can be interpreted as :

Find the temperature V (x, t) at any point x and at any time t of a solid bounded by planes x = 0 and x = 4, whose both ends are kept at temperature zero and V (x, 0) = 4x.

2.3.5 Self Check Exercise

1. Find finite fourier sine and cosine transform of $f(t) = \frac{\pi}{3} - t + \frac{t^2}{2\pi}, 0 < t < \pi$.
2. Find finite Fourier sine and cosine transform of $f(t) = e^{at}$, $0 < t < l$.
3. Find finite Fourier sine transform of f(t) = 2, $0 < t < \pi$. Apply inversion theorem to find Fourier series for f(t) = 2, $0 < t < \pi$ and then find

   $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots$

4. Find solution of $\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}$

Subject to V (0, t) = 1, V (π, t) = 3 and V (x, 0) = 1 for $x \in (0\pi)$, t > 0
Also interpret the above problem physically.

5. Using finite Fourier sine transform solve $\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}$

Subject to conditions V (0, t) = 09 = V (π, t) and V (x, 0) = 2x, $x \in (0\ \pi)$, t>0.

6. Find the temperature V (x, t) in a slab $0 < x < \pi$, with initial temperature = 1 and its faces are being kept at temperature zero and λ = diffusivity = 1.