Centre for Distance and Online Education
Punjabi University, Patiala

Class : B.A. I (Math) **Semester : 1**

Paper : III (Linear Algebra) **Unit–1 & 2**

Medium : English

Lesson No.

Department website : www.pbidde.org

B.A. / B.Sc. 1st Year (1st Semester)
MATHEMATICS

PAPER-III: LINEAR ALGEBRA

Maximum Marks: 50 **Pass Percentage: 35%**
Maximum Time: 3 Hrs

INSTRUCTIONS FOR THE PAPER SETTER

The question paper will consist of three sections A, B and C. Sections A and 8 will have four questions each from the respective sections of the syllabus and Section C will consist of one compulsory question having eight short answer type questions covering the entire syllabus uniformly. Each question in sections A and B will be of 7.5 marks and Section C will be of 20 marks.

INSTRUCTIONS FOR THE CANDIDATES

Candidates are required to attempt five questions in all selecting two questions from each of the Section A and B and compulsory question of Section C.

Section-A

Elementary operations on matrices, Inverse of a matrix using Gauss Jordan Method. Linear independence of row and column vectors, Row rank, Column rank and their equivalence. Eigen values, Eigen vectors, Characteristic equation of a matrix, Diagonalization, Cayley-Hamilton theorem and its use in finding inverse of a matrix, Consistency of a system of linear equations.

Section-B

Vector spaces, Examples, Linear dependence, Linear combinations, Bases and dimension. Subspaces. Linear transformation, Algebra of linear transformations, Matrices as linear transformations, Matrices and change of basis, Kernel and image, Rank and nullity theorem.

RECOMMENDED BOOKS:

1. Gibert Strang: Linear Algebra and its Applications, Cengage Learning Publishers (Fourth Edition)
2. P. B. Bhattacharya, S. K. Jain & S. R.Nagpaul: A first course in Linear Algebra, New Age International (P) Limited
3. Serge Lange: Introduction to Linear Algebra, Springer
4. Kenneth Hoffman, Kunze: Linear Algebra, PHI (Second Edition)
5. Charles W. Curtis: Linear Algebra: An Introductory Approach, Springer

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B.A. PART– I **MATHEMATICS : PAPER III**

(SEMESTER - I) **LINEAR ALGEBRA**

LESSON NO. 1.1 **AUTHOR : DR. CHANCHAL**

RANK OF A MATRIX

1.1.1 Objectives

The prime objective of this lesson is

- To understand the concept of rank of a matrix
- To study elementary transformations
- To determine rank and inverse of a matrix using elementary transformations

1.1.2 Introduction

To understand the concept of rank of a matrix, firstly we take a look at the various types of matricas the have studied in our presbes class.

(I) Transpose of a matrix : The matrix obtained from a given matrix A, by interchanging its rows and columns, is called the transpose of A and is generally denoted by A' or A^t or A^T.

Thus if $A = \left[a_{ij}\right]$ then (j, i)th element of A' is equal to (i, j) element of A

For example : If $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \end{bmatrix}$, then $A' = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 0 & 3 \end{bmatrix}$

Remarks :

(1) (A')' = A (2) (A + B)' = A' + B'

(3) (k A)' = k A', l being a complex number. (4) (AB)' = B'A'

(II) Symmetric and Skew-Symmetric Matrices

1. Any square matrix A = [a_{ij}] is said to be a symmetric matrix if a_{ij} = a_{ji} i.e., (i, j)th element of A is the same as the (j, i)th element of A. If we take the transpose of a symmetric matrix A, it is the same as A.

For example : $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 2 \\ -1 & 4 & 3 \\ 2 & 3 & 5 \end{bmatrix}$

2. Any square matrix $A = \left[a_{ij}\right]$ is said to be a skew-symmetric matrix if $a_{ij} = -a_{ji}$ i.e. (i, j)th element is the same as the negative of the (j, i) element.

∴ for a skew-symmetric matrix A, a_{ij} = – a_{ji}

If we put j = i, we get $a_{ij} = -a_{ji}$ or $a_{ii} = 0$ i.e, every diagonal element of A is zero.

Examples of skew-symmetric matrix are

$$\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}, \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$$

(III) Conjugate and Tranjugate of a Matrix

The Matrix obtained by replacing the elements by A by its complex conjugates, is called the conjugate of A and is generally denoted by A.

Thus, if $A = \left[a_{ij}\right], \bar{A} = \left[\bar{a}_{ij}\right]$ where denotes the conjugate of a_{ij}

For example : if $A = \begin{bmatrix} 2+3i & 7-5i & 6+i \\ 5 & 2+3i & 1-2i \\ -3-5i & 0 & 2-5i \end{bmatrix}$,

$$\text{then } \bar{A} = \begin{bmatrix} 2-3i & 7+5i & 6-i \\ 5 & 2-3i & 1+2i \\ -3-5i & 0 & 2+5i \end{bmatrix}$$

If all the element of A are real, then $\bar{A} = A$.

Note $(\bar{A}) = A$

Tranjugate of Matrix

The conjugate of the transpose of a matrix A is called tranjugate of A and is denoted by A^θ. Thus $A^\theta = \overline{(A')}$

Clearly $\overline{(A')} = (\bar{A})'$

$\therefore \quad A^\theta = (\bar{A})'$

For example : if

$$A = \begin{bmatrix} 2+3i & 6-i & 5+2i \\ 3 & 2 & -1+5i \\ 0 & 7-3i & -5+6i \end{bmatrix}$$

$$\text{then } A^\theta = \begin{bmatrix} 2-5i & 3 & 0 \\ 6+i & 2 & 7+3i \\ 5-2i & -1-5i & -5-6i \end{bmatrix}$$

Note : $(A^\theta)^\theta = A$.

(IV) Hermitian and Skew-Hermitian Matrices

(1) A square matrix $A = [a_{ij}]$ is said to be hermitian if $a_{ij} = \bar{a}_{ij}$ i.e., (i, j)th element is the conjugate of the (j, i)th element.

Now, d $a_{ij} = \bar{a}_{ij} \therefore a_{ii} = \bar{a}_{ii}$ i.e., the conjugate of any diagonal element is the same element.

$\therefore$ every diagonal element must be real.

For example :

$$\begin{bmatrix} 2 & 5-6i & 3-4i \\ 5+6i & 0 & 1-2i \\ 3+4i & 1+2i & 7 \end{bmatrix}, \begin{bmatrix} 0 & a+ib & c+id \\ a-ib & 1 & m+in \\ c-id & m-in & p \end{bmatrix}$$

(2) A square matrix $A=\left[a_{ij}\right]$ is said to be Skew-hermitian if $a_{ij}=-\bar{a}_{ji}$ i.e., (i, j)the element is the negative conjugate of (j, i) element.

Again as $a_{ij}=-\bar{a}_{ji} \therefore a_{ii}=-\bar{a}_{ii}$ i.e, $a_{ii}+\bar{a}_{ii}=0$.

$\therefore$ every diagonal element must be either zero or a purely imaginary number.

For example :

$$\begin{bmatrix} 4i & 4-3i & 6+5i \\ -4-3i & 0 & 2+7i \\ -6+5i & -2+7i & -9i \end{bmatrix}, \begin{bmatrix} 5i & 3-7i \\ -3-7i & 9i \end{bmatrix}$$

(v) Orthogonal Matrix

A square matrix P over the field of reals is said to be orthogonal if and only if P'P = I.

Now, if P is orthogonal, then P'P = I = PP'.

$$\Rightarrow |P'P| = |I| \quad \Rightarrow \quad |P'||P| = I$$

$$\Rightarrow |P||P| = I \quad \Rightarrow \quad |P|^2 = I$$

$$\Rightarrow |P| = \pm I \quad \Rightarrow \quad |P| \neq 0$$

$\Rightarrow$ P is invertible

$\therefore$ If P is orthogonal, then P is invertible.

Also P'P = I $\Rightarrow$ $P' = P^{-1}$

$\Rightarrow$ $PP' = PP^{-1}$ $\Rightarrow$ PP' = I

$\therefore$ P is orthgonal iff P'P = PP' = I i.e, iff $P' = P^{-1}$.

For example :

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

(vi) Unitary Matrix

A square matrix P over the fied of complex numbers is said to be unitary if and only if $P^{\theta}P\ I = PP^{\theta}$.

Now, if P is unitary, then $P^{\theta}P = I$

$$\Rightarrow |P^{\theta}P| = |I| \Rightarrow |P^{\theta}||P| = 1 \quad \Rightarrow |P||P| = 1 \quad \Rightarrow \left[\therefore |P^{\theta}| = |P|\right]$$

$$\Rightarrow |P|^2 = I \Rightarrow |P| \neq 0$$

$\Rightarrow$ P is invertible

$\therefore$ if P is unitary, then P is invertible.

Also $P^{\theta}P \Rightarrow I \Rightarrow P^{\theta} = P^{-1} \Rightarrow PP^{\theta} = PP^{-1} \Rightarrow PP^{\theta} = I$

$\therefore |P| = \pm 1$

$\Rightarrow$ absolute value of a determinant of a unitary matrix is 1.

(vii) Similar Matrices

Let A and B be square matrices of order n over a field. Then A is said to be similar to B over F if and only if there exists an n-rowed invertible matrix C over F such that

$AC = CB$ i.e. $B = C^{-1}AC$ or $A = CB^{-1}$.

1.1.3 Rank of a Matrix

Definition : A number r is said to be rank of a non-zero matrix A if

(i) there exists at least one minor of order r of A which does not vanish, and

(ii) every minor of order (r + 1), if any, vanishes

The rank of a matrix A is denoted by $\rho(A)$.

$\therefore$ We have $\rho(A) = r$.

In other words, the rank of a non-zero matrix is the largest order of any non-vanishing minor of the matrix.

Remarks : (i) the rank of a zero matrix is zero i.e., $\rho(O) = 0$ where O is a zero matrix.

(ii) the rank of a non-singular matrix of order n is n,

(iii) $\rho(A) \leq r$, if every minor of order (r + 1) vanishes,

(iv) $\rho(A) \geq r$, if there is a minor of order r whch does not vanish.

Some Important Results :-

Result 1: Prove that the rank of the transpose of a matrix A is the same as that of the original matrix A.

Proof . If $A = O$, then $A' = O$

$\therefore$ $\rho(A) = 0$ and $\rho(A') = 0$

$\Rightarrow$ $\rho(A') = \rho(A)$

$\therefore$ result is true in the case in which A is a zero matrix.

Now we discuss the case when $A \neq O$.

Let r be the rank of the matrix $A = [a_{ij}]$ where A is of type $m \times n$.

$\therefore$ there exists at least one square submatrix R or order r such that $|R| \neq 0$.
Now R' is also a square submatrix of A' of order r.

Also $|R'| = |R| \neq 0$

$\therefore \quad \rho(A') \geq r$

If possible, suppose $\rho(A') > r$

We take $\rho(A') = r+1 \Rightarrow \rho(A') \geq r+1$ [$\because$ (A')' = A]

$\Rightarrow \rho(A) \geq r+1$ [$\because$ (A')' = A]

which is impossible as ρ(A) = r

$\therefore$ our supposition is wrong

$\therefore \rho(A') \not> r$

from (1), we have.

$\rho(A') = r \Rightarrow \rho(A') = \rho(A)$.

Result 2 : Prove that $d\,\rho(\lambda A) = \rho(A)$ where λ is a non-zero scalar.

Proof : If A = O, then λA = O

$\therefore$ ρ(A) = 0 and ρ(λA) = 0

$\therefore$ ρ(λ A) = ρ(A)

$\therefore$ result is true in the case in which A is a zero matrix.

Now we discuss the case when A ≠ O.

Let r be the rank of the matrix $A = [a_{ij}]$ where A is of type m × n.

$\therefore$ there exists at least one square submatrix R of order r such that $|R| \neq 0$

Now λ R is a square submatrix of matrix λ A of order r.

$\therefore \quad |\lambda R| = \lambda^r |R| \neq 0$ as $\lambda \neq 0, |R| \neq 0$

$\therefore \rho(\lambda A) \geq r$ then $\rho(\lambda A) \geq r$ ...(1)

If possible, suppose $\rho(\lambda A) > r$

We take $\rho(\lambda A) > r+1$

$\therefore \rho\left(\frac{1}{\lambda}(\lambda A)\right) \geq r+1$ [$\because$ of (1)]

$\Rightarrow$ $\rho(A) \geq r+1$, which is impossible as $\rho(A)=r$

$\therefore$ Our supposition is wrong

$\because$ $\rho(\lambda A)=r$ or $\rho(\lambda A)=\rho(A)$

Result 3 : If A is an n-rowed non-singular matrix, then prove that $d\rho\left(A^{-1}\right)=\rho(A)$.

Hence deduce that $\rho(\text{adj}.A)=\rho(A)$.

Proof : Here A is an n-rowed non-singular matrix

$\therefore$ $|A| \neq 0$

$\Rightarrow$ $\rho|A|=n$

$\because$ A is non-singular

$\therefore$ A^{-1} exists and $AA^{-1} = 1$

$\Rightarrow \left|AA^{-1}\right|=|I| \Rightarrow |A|\left|A^{-1}\right|=1 \Rightarrow \left|A^{-1}\right| \neq 0$

$\therefore$ A^{-1} is an n-rowed non-singular matrix

$\Rightarrow$ $\rho\left(A^{-1}\right)=n \Rightarrow \rho\left(A^{-1}\right)=\rho(A)$

Deduction

$\rho(\text{adj}.A)=\rho\left(\frac{1}{|A|}\text{adj}.A\right)$ $[\because \rho(A)=\rho(\lambda A)]$

$=\rho\left(A^{-1}\right)=\rho(A) \Rightarrow \rho(\text{adj}.A)=\rho(A)$.

Problem 1 : Find the rank of the matrix $\begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$.

Solution : Let $A = \begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$

Since there does not exist any minor of order 4 or A

$\therefore \quad \rho(A) \le 3 \qquad \ldots(1)$

Now $\begin{vmatrix} 1 & -1 & 6 \\ 1 & 3 & -4 \\ 5 & 3 & 11 \end{vmatrix} = 1\begin{vmatrix} 3 & -4 \\ 3 & 11 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -4 \\ 5 & 11 \end{vmatrix} + 6\begin{vmatrix} 1 & 3 \\ 5 & 3 \end{vmatrix}$

$\therefore$ there exists a minor of order 3 of A which does not vanish.

$\therefore \rho(A) \ge 3$

From (1) and (2), we get,

$\rho(A) = 3$.

1.1.4 Elementary Operations (or Transformations)

We can also determive the rank of a matrix by using some other methods which are bared on the element ary transformations of a matrix, that includes :

(1) The interchange of any two parallel lines.

(2) The multiplication of all the elements of any line by any non-zero number.

(3) The addition to the elements of any line, the corresponding elements of any other line multiplied by any number.

Note : An elementary transformation is called a row transformation or a column transformation according as it applies to rows or columns. Therefore, there are three row transformation and three column transformations.

Symbols used for the transformations

(1) R_{ij} or $R_i \leftrightarrow R_j$ stands for the interchange of the ith and jth rows.

(2) $R_i^{(c)}$ or $R_i \to cR_i$ stands for the multiplication of the ith row by $c \neq 0$.

(3) $R_{ij}^{(k)}$ or $R_i \to R_i + kR_j$ stands for addition to the ith row, the product of the jth row by k. Similarly

(4) C_{ij} or $C_i \leftrightarrow C_j$ stands for the interchange of ith and jth columns.

(5) $C_i^{(c)}$ or $C_i \to cC$, stands for the multiplication of the elements of the ith column by $c \neq 0$.

(6) $C_{ij}^{(k)}$ or $C_i \to C_i + kC_j$ stands for addition to the ith column, the product of the jth column by k.

Definition of Elementary Matrix

A matrix, obtained from a unit matrix, by subjecting it to a single elementary transformation is called an elementary matrix.

Remarks :

1.1.5 Determination of Rank by Equivalent Matrix.

When an elementary transformation is applied to a matrix, it results into a matrix of the same order and same rank. The resulted matrix said to be equivalent to the given matrix and we use the symbol - to mean.

Let A be any given matrix, Reduce the matrix to equivalent matrix by using the following steps :

(i) Use row or column transformations, if necessary, to obtain a non-zero element (preferably 1) in the first row and the first column of the given matrix.

(ii) Divide the first row by this element, if it is not 1.

(iii) Subtract suitable multiples of the first row from the other rows so as to obtain zeros in the remainder of the first column.

(iv) Subtract suitable multiples of the first column from the other columns so as to get zeros in the remainder of the first row.

(v) Repeat the steps (i) – (iv) starting with the elements in the second-row and the second column.

(vi) Continue in this way down the "main diagonal" either until the end of the diagonal is reached or until all the remaining elements in the matrix are zero. The rank of this matrix, which is equivalent to the given matrix A, can be determined by inspection and consequently the rank of the given matrix A can be determined.

Problem 2 : Using elementary transformations, find the rank of the matrix

$$\begin{bmatrix} 1 & 3 & 2 \\ 4 & 6 & 5 \\ 3 & 5 & 4 \end{bmatrix}$$

Sol. Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 6 & 5 \\ 3 & 5 & 4 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -6 & -3 \\ 0 & -4 & -2 \end{bmatrix}, \text{ by } R_2 \to R_2 - 4R_1, R_3 \to R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \text{ by } R_2 \to -\frac{1}{3}R_2, R_3 \to -\frac{1}{2}R_3$$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ by } R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ by } C_2 \to C_2 - 3C_1, C_3 \to C_3 - 2C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ by } C_3 \to C_3 - \frac{1}{2}C_2$$

The rank of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is 2 as minor $\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}$ of order 2 does not vanish

$\therefore \rho(A) = 2$.

Problem 3 : Find the rank of the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 & 5 \\ 2 & 1 & -1 & -2 \\ 3 & -1 & -1 & 7 \end{bmatrix}$$

Sol. $A = \begin{bmatrix} 1 & -1 & 1 & 5 \\ 2 & 1 & -1 & -2 \\ 3 & -1 & -1 & 7 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 3 & -3 & -12 \\ 0 & 2 & -4 & -8 \end{bmatrix}, \text{by } R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 1 & 1 & -4 \\ 0 & 2 & -4 & -8 \end{bmatrix}, \text{by } R_2 \to R_2 - R_3$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & -6 & 0 \end{bmatrix}, \text{by } R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & -6 & 0 \end{bmatrix}, \text{by } C_2 \to C_2 + C_1, C_3 \to C_3 - C_1, C_4 \to C_4 - 5C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & 0 \end{bmatrix}, \text{by } C_3 \to C_3 - C_2, C_4 \to C_4 + 4C_2$$

The rank of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & 0 \end{bmatrix}$ is 3 as the minor $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -6 \end{vmatrix} = -6 \neq 0$ of order 3 does not vanish

$\therefore \quad \rho(A) = 3.$

Note : Normal form of a Matrix : The normal form of matrix A can be

$\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}, [I_r \quad O], \begin{bmatrix} I_r \\ O \end{bmatrix}$, where I_r is identity matrix of order 'r'.

Remarks : 1. Every non-zero matrix of rank r can, by a sequence of elementary transformations, be reduced to the form $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$ where I, is a r-rowed unit matrix.

2. Let A be any non-zero matrix of rank r. Then there exist non-singular matrices P and Q such that $PAQ = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$.

3. A non-singular matrix can be reduced to a unit matrix by a series of elementary transformations.

4. Every non-singular matrix is a product of elementary matrices.

5. The rank of a matrix is not altered by pre-multiplication or post-multiplication of the matrix with any non-singular matrix.

6. The rank of a product of two matrices cannot exceed the rank of either matrix.

Problem 4 : Prove that the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ is equivalent to I_3.

Sol. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -6 \\ 0 & 1 & 2 \end{bmatrix}, \text{by } R_2 \to R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -6 \\ 0 & 1 & 2 \end{bmatrix}, \text{by } C_2 \to C_2 - 2C_1, C_3 \to C_3 - 3C_1$$

$$\sim\begin{bmatrix}1 & 0 & 0\\0 & 1 & 6\\0 & 1 & 2\end{bmatrix}, \text{by } R_2 \to -R_2$$

$$\sim\begin{bmatrix}1 & 0 & 0\\0 & 1 & 6\\0 & 0 & -4\end{bmatrix}, \text{by } R_3 \to R_3 - R_2$$

$$\sim\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & -4\end{bmatrix}, \text{by } C_3 \to C_3 - 6C_2$$

$$\sim\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}, \text{by } R_3 \to -\frac{1}{4}R_3$$

$\therefore A \sim I_3$

$\therefore$ given matrix is equivalent to I_3.

Problem 5 : If $A = \begin{bmatrix}1 & 1 & 1\\3 & 1 & 1\\1 & -1 & -1\end{bmatrix}$, then find the matrices P and Q such that PAQ is in the normal form. Hence find the rank of the matrix A.

Sol. Here $A = \begin{bmatrix}1 & 1 & 1\\3 & 1 & 1\\1 & -1 & -1\end{bmatrix}$,

We have A = I A I

$$\therefore \begin{bmatrix}1 & 1 & 1\\3 & 1 & 1\\1 & -1 & -1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{by } R_2 \leftrightarrow R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -3 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

by $R_2 \to R_2 - R_1, R_3 \to R_3 - 3R_1$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -3 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

by $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -3 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{by } R_2 \to -\frac{1}{2}R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -2 & 1 & -1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{by } R_3 - R_3 + 2R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -2 & 1 & -1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \text{by } C_3 \to C_3 - C_2$$

$$\Rightarrow \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix} = PAQ$$

where $P = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -2 & 1 & -1 \end{bmatrix}, Q = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & --1 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore$ PAQ is in the normal form and $\rho(A) = 2$.

Problem 6 : Reduce the matrix $\begin{bmatrix} 2 & -1 & 0 & 4 \\ 1 & 3 & 5 & -3 \\ 3 & -5 & -5 & 11 \\ 6 & 4 & 10 & 2 \end{bmatrix}$ to normal form. Hence find the rank of the matrix.

Solution : $A = \begin{bmatrix} 2 & -1 & 0 & 4 \\ 1 & 3 & 5 & -3 \\ 3 & -5 & -5 & 11 \\ 6 & 4 & 10 & 2 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 3 & 5 & -3 \\ 2 & -1 & 0 & 4 \\ 3 & -5 & -5 & 11 \\ 6 & 4 & 10 & 2 \end{bmatrix}, \text{by } R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 3 & 5 & -3 \\ 0 & -7 & 10 & 10 \\ 0 & -14 & -20 & 20 \\ 0 & -14 & -20 & 20 \end{bmatrix}, \text{by } R_1 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1, R_3 \to R_3 - 6R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 10 & 10 \\ 0 & -14 & -20 & 20 \\ 0 & -14 & -20 & 20 \end{bmatrix}, \text{by } C_2 \to C_2 - 3R_1, C_3 \to C_3 - 5C_1, C_4 \to C_4 + 3C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \end{bmatrix}, \text{by } C_2 \to -\frac{1}{7}C_2, C_3 \to -\frac{1}{10}C_3, C_4 \to \frac{1}{10}C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{by } R_3 \to R_3 \to 2R_2, R_4 \to R_4 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{by } C_3 \to C_3 - C_2, C_4 \to C_4 - C_2$$

$$\sim \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$$

$\therefore$ rank (A) = 2

1.1.6 Computation of Inverse using Elementary Transformations

We can understand this compoutation of finding invese with the help of following example:

If we are to find inverse of A, we write A = I A and go on performing row transformations on the product and the prefactor of A till we reach the result I = BA, then B is the inverse of A.

Problem 7 : Using elementary operations, find inverse of the matrix:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ -3 & 5 & 1 \\ 1 & 7 & 2 \end{bmatrix}$$

Solution : $A = \begin{bmatrix} 2 & 3 & 1 \\ -3 & 5 & 1 \\ 1 & 7 & 2 \end{bmatrix}$

Now A = I A

$$\therefore \begin{bmatrix} 2 & 3 & 1 \\ -3 & 5 & 1 \\ 1 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\therefore \begin{bmatrix} 1 & 7 & 2 \\ -3 & 5 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A, \text{by } R_2 \leftrightarrow R_3$$

$$\therefore \begin{bmatrix} 1 & 7 & 2 \\ 0 & 26 & 7 \\ 0 & -11 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & -2 \end{bmatrix} A, \text{by } R_2 \to R_2 + 3R_1, R_3 \to R_3 - 2R_1$$

$$\therefore \begin{bmatrix} 1 & 7 & 2 \\ 0 & 26 & 7 \\ 0 & -286 & -78 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 26 & 0 & -52 \end{bmatrix} A, \text{by } R_3 \to 26R_3$$

$$\therefore \begin{bmatrix} 1 & 7 & 2 \\ 0 & 26 & 7 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 26 & 11 & -19 \end{bmatrix} A, \text{by } R_3 \to R_3 + 11R_2$$

$$\therefore \begin{bmatrix} 1 & 7 & 0 \\ 0 & 26 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 52 & 22 & -37 \\ 182 & 78 & -130 \\ 26 & 11 & -19 \end{bmatrix} A, \text{by } R_1 \to R_1 + 2R_3, R_2 \to R_2 + 7R_3$$

$$\therefore \begin{bmatrix} 1 & 7 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 52 & 22 & -37 \\ 7 & 3 & -5 \\ -26 & -11 & 19 \end{bmatrix} A, \text{by } R_2 \to \frac{1}{26} R_2, R_3 \to -R_3$$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 \\ 7 & 3 & -5 \\ -26 & -11 & 19 \end{bmatrix} A, \text{by } R_1 \to R_1 - 7R_2$$

$\therefore \quad I = A^{-1} A$

$$\therefore \quad A^{-1} = \begin{bmatrix} 3 & 1 & -2 \\ 7 & 3 & -5 \\ -26 & -11 & 19 \end{bmatrix}.$$

1.1.7 Summary

In this lesson, we have gained knowledge about the rank of a matrix and learnt its evaluation using elementary transformations. We have also explained about the calculation of inverse of a matrix using elementary transformations. The concept is made more elaborative with the help of various suitable examples.

1.1.8 Key Concepts

Rank of a matrix, Equivalent matrices, Normal form.

1.1.9 Long Questions

1. Show that the matrix $A = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$ is unitary if and only if $a^2 + b^2 + c^2 + d^2 = 1$.

2. (i) If P, Q are unitary, prove that QP is also unitary.

(ii) If P, Q are orthogonal, prove that QP is also orthogonal.

3. If A is an orthogonal matrix, then A' and A^{-1} are also orthogonal.

4. Find the rank of each matrix $\begin{bmatrix} 0 & 6 & 6 & 1 \\ -8 & 7 & 2 & 3 \\ -2 & 3 & 0 & 1 \\ -3 & 2 & 1 & 1 \end{bmatrix}$.

5. Find the rank of the matrix $\begin{bmatrix} 1 & 2 & -3 & -1 \\ 3 & -4 & 1 & 2 \\ 5 & 2 & 1 & 3 \end{bmatrix}$.

6. Find the rank of the matrix $\begin{bmatrix} 3 & 4 & 1 & 2 \\ 3 & 2 & 1 & 4 \\ 7 & 6 & 2 & 5 \end{bmatrix}$, using equivalent matrix.

7. For the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$, find two non-singular matrices P and Q such that PAQ is in the normal form and hence find out rank of matrix A.

8. Reduce the matrix $\begin{bmatrix} 3 & -2 & 1 \\ 2 & -1 & 3 \\ 1 & -2 & 1 \end{bmatrix}$ to the form I_3 and find rank.

1.1.10 Short Questions

1. Find the rank of the matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

2. Find the rank of the matrix $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$, using equivalent matrix.

3. Use elementary transformation to find the inverse of

(i) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 1 \\ 5 & 2 & 3 \end{bmatrix}$

1.1.11 Suggested Readings

1. P. B. Bhattacharya, S. K. Jain & S. R. Nagpaul : A First Course in Linear Algebra, New Age International (P) Ltd.
2. Gilbert Strang : Linear Algebra and its Applications, Cengage Learning Publishers (Fourth Edition)

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(SEMESTER - I) LINEAR ALGEBRA

LESSON NO. 1.2 AUTHOR : DR. CHANCHAL

ROW RANK, COLUMN RANK AND THEIR EQUIVALENCE

1.2.1 Objectives

For this lesson, our prime objectives are

- To discuss about row rank and column rank of a matrix and their equivalence
- To discuss methods for checking linear dependence/independence of vectors

1.2.2 Row and Column Rank of a Matrix.

Firstly, we define the echelon form of a martrix:

A matrix A is said to be a row (column) equivalent to a matrix B if B can be obtained from A after a finite number of elementary row (column) operations, and we write $A \overset{R}{\sim} B$ or $A \overset{C}{\sim} B$.

Definition (Echelon Form):

A matrix A = $[a_{ij}]$ is said to be in the echelon form it

(i) The zero rows (columns) of A occur below all the non-zero rows (columns) of A

(ii)The number of zeros before the first non-zero element in a row (column) is less than the number of such zeros in the next row (column).

(iii) If $R_1, R_2, ...$ are non-zero rows (columns) of A, then first non-zero entry in these rows (columns) is 1. The Moreover Matrix is in (column) row reduced echelon form in addition to the above conditions, if a column (row) contains the first non-zero entry of any row(column), then every other entry in that column (row) is zero.

Row and column Rank of a Matrix

Let A be any matrix. Then Row rank of A, denoted by $\rho_R(A)$, is defined as the number of non-zero rows in a row echelon form of A.

Similarly,m column rank of A, denoted by $\rho_C(A)$, is defined as the number of non-zero column in a column echelon form of A.

Problem 1 : Find the row rank and column rank of $\begin{bmatrix} 1 & 3 & 2 & 4 \\ 5 & 2 & 0 & 1 \\ 3 & -4 & -4 & -7 \\ -7 & 5 & 6 & 10 \end{bmatrix}$.

Solution : Let $A = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 5 & 2 & 0 & 1 \\ 3 & -4 & -4 & -7 \\ -7 & 5 & 6 & 10 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & -13 & -10 & -19 \\ 0 & -13 & -10 & -19 \\ -0 & 26 & 20 & 38 \end{bmatrix}, \text{by } R_2 \to R_2 - 5R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 + 7R_1$$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & -13 & -10 & -19 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{by } R_3 \to R_3 - R_2, R_4 \to R_4 + 2R_2$$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 1 & \frac{10}{13} & \frac{19}{13} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{by } R_2 \to -\frac{1}{13} R_2$$

Which is in row-echelon form.

Since there are two non-zero rows in the row-echelon form

$\therefore$ row rank of A is 2

Now $A = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 5 & 2 & 0 & 1 \\ 3 & -4 & -4 & -7 \\ -7 & 5 & 6 & 10 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & -13 & -10 & -19 \\ 3 & -13 & -10 & -19 \\ -7 & 26 & 20 & 38 \end{bmatrix}, \text{by } C_2 \to C_2 - 3C_1, C_3 \to C_3 - 2C_1, C_4 \to C_4 - 4C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & -10 & -19 \\ 3 & 1 & -10 & -19 \\ -7 & -2 & 20 & 38 \end{bmatrix}, \text{by } C_2 \to C_2 - \frac{1}{13}C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -7 & -2 & 0 & 0 \end{bmatrix}, \text{by } C_3 \to C_3 + 10C_2, C_4 \to C_4 + 15C_2$$

which is column echelon form having two non-zero column.

$\therefore$ column rank = 2.

In order to understand the concept of row rank and column rank more deeply, we must have the knowledge of law vectors and column vectors.

1.2.3 Linear Dependence | Independence of Vectors

Definition (n-vector) : An ordered tuple of n numbers is called n-vector.

For example : $\{x_1, x_2, \ldots\ldots, x_n\}$ is an n-vector.

Linear Dependence of Vectors : A set V_1, V_2, ..., V_t of vectors is said to be linearly dependent set, if there exists t scalars p_1, p_2, ..., p_i, not all zero, such that

$p_1V_1 + p_2V_2 + ... + p_tV_t = O$, where O is a n-vector with all components zero.

Any set of vectors, which is not linearly dependent, is called linearly independent.

i.e. a set $V_1, V_2, ..., V_t$ of n-vectors is said to be linearly independent if every relation of the form $p_1V_1 + p_2V_2 + ... + p_tV_t = O$ implies $p_1 = p_2 = ... = p_t = 0$.

Linear Combination of Vectors :

A vector V is said to be a linear combination of the vectors

$V_1, V_2, ..., V_t$ if $V = p_1V_1 + p_2V_2 + ... + p_tV_t$, where $p_1, p_2, ..., p_t$ are scalars.

Result 1 : If a set of vectors is linearly dependent, show that at least one member of the set is a linear combination of the remaining members.

Proof : Let $V_1, V_2, ..., V_t$ be any linearly dependent set.

$\therefore$ the relation $p_tV_t + p_2V_2 + ... + p_tV_t = O$

implies that at least one of $p_1, p_2, ..., p_t$ is non-zero

Let p_t be non-zero

Now $p_1V_1 = -p_2V_2 - p_3V_3 - ... - p_tV_t$

$$\therefore V_t = \left(-\frac{p_2}{p_1}\right)V_2 + \left(-\frac{p_3}{p_1}\right)V_3 + ... + \left(-\frac{p_t}{p_1}\right)V_t$$

The relation (1) shows that V_1 is a linear combination of $V_2, V_3, ..., V_t$.

Result 2 : If η is a linear combination of the set $\{V_1, V_2, ..., V_r\}$, then the set $\{\eta, V_1, V_2, ..., V_r\}$ is linearly dependent.

Proof : Since η is a linear combination of $V_1, V_2, ..., V_r$

$\therefore \eta = k_1V_1 + k_2V_2 + ... + k_rV_r$

$\Rightarrow \eta - k_1V_1 - k_2V_2 - ... - k_rV_r = O$

Now at least one of the coefficients i.e. of η is non-zero.

$\therefore$ set $\eta, V_1, V_2, ..., V_r$ is L.D.

Result 3 : Prove that every super set of a linearly dependent set is linearly dependent.

Proof : Let $\{V_1, V_2, ..., V_p, V_{p+1}, ..., V_r\}$ be a super set of a linearly dependent set $\{V_1, V_2, ..., V_p\}$. Since $\{V_1, V_2, ..., V_p\}$ is linearly dependent set

$\therefore$ there exist scalars $k_1, k_2, ..., k_p$ (not all zero) such that

$$k_1V_1 + k_2V_2 + ... + k_pV_p = O$$

It can be re-written as

$k_1V_1 + k_2V_2 + ... + k_pV_p + ... + k_rV_r = O,$ where $k_1, k_2, ..., k_p, ..., k_r$ and not all zero.

$\therefore$ set $\{V_1, V_2, ..., V_p, ..., V_r\}$ is L.D.

Brief Outline of Vector Space :

In this lesson we briefly explain what a vector space is? The detailed studyof vector space will be done in lesson 0.5.

Definition : The n-vector Space : The set of all n-vectors over a field F, to be denoted by $V_n(F)$, is called the n-vector space over F.

Sub-space of n-vector Space V_n

Any non-zero empty set, S of vectors of $V_n(F)$ is called a subspace of $V_n(F)$, if when

(i) V_1, V_2 are any two members of S, then $V_1 + V_2$ is also a member of S and

(ii) If V is a member of S and k is a member of F, then kV is also a member of S.

Subspace Spanned by a Set of Vectors

Let V_1, V_2, …, V_t be a set of n-vectors.

The set of all linear combinations of the above set is called a subspace spanned by the set of vectors V_1, V_2, …, V_t.

Basis of a Subspace

A set of vectors is said to be the basis of a subspace, if

(i) the subspace is spanned by the set and

(ii) the set is linearly independent.

Dimension of a subspace

The number of vectors in any basis of a subspace is called the dimension of the subspace.

Another Method to check for the Linear Dependence of Vectors :

Let $V_1 = (b_{11}, b_{12}, ..., b_{1n}), V_2 = (b_{21}, b_{22}, ..., b_{2n})$

....

....

$v_n = (b_{n1}, b_{n2}, ..., b_{nn})$ be a vectors of the vector space.

By definition these vectors are L.D. vectors iff there exists scalars $\alpha_1, \alpha_2, ..., \alpha_n \in F$, not all zero such that $\alpha_1V_1 + \alpha_2V_2 + \alpha_nV_n = O$

$$\Rightarrow a_1(b_{11}, b_{12}, ..., b_{1n}) + \alpha_2(b_{21}, b_{22}, ..., b_{2n}) + ... + \alpha_n(b_{n1}, ..., b_{nn}) = O$$

$$\Rightarrow (\alpha_1 b_{11} + \alpha_2 b_{21} + \ldots + \alpha_n b_{n1}, \alpha_1 b_{12} + \alpha_2 b_{22} + \ldots\ldots$$

$$+\alpha_n b_{n2}, \ldots, \alpha_1 b_{1n} + \alpha_2 b_2 n + \ldots + \alpha_n b_{nn})$$

$$= (0, 0, \ldots, 0)$$

$\therefore$ $\alpha_1 b_{11} + \alpha_2 b_{21} + \ldots + \alpha_n b_{n1} = 0$

$\alpha_1 b_{12} + \alpha_2 b_{22} + \ldots + \alpha_n b_{n2} = 0$

....

$\alpha_1 b_{1n} + \alpha_2 b_{2n} + \ldots + \alpha_n b_{nn} = 0$

These homogenous equations must have a non-trivial (α_i's not all zero) solution. Moreover the above equations will have a non-trivial solution iff the determinant of its coefficient matrix is zero

i.e., iff $\begin{vmatrix} b_{11} & b_{21} & \cdots & b_{n1} \\ b_{12} & b_{22} & \cdots & b_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ b_{1n} & b_{2n} & \cdots & b_{nn} \end{vmatrix} = 0$

Hence vectors $V_1, V_2, \ldots, V_n$ are L.D.

iff $\begin{vmatrix} b_{11} & b_{21} & \cdots & b_{n1} \\ b_{12} & b_{22} & \cdots & b_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ b_{1n} & b_{2n} & \cdots & b_{nn} \end{vmatrix} = 0$

or iff $\begin{vmatrix} b_{11} & b_{21} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix} = 0$

[$\because$ value of det. remains unchanged if rows and columns are interchanged] and vectors are L.I. iff this determinant $\neq 0$.

Problem 2 : Examine whether (1, –3, 5) belongs to the linear space generated by S, where $S = \{(1,2,1), (1,1,-1), (4,5,-2)\}$ or not?

Sol. If possible, let (1, –3, 5) belong to the linear space generated by S

$\Rightarrow \exists$ scalars α_1, α_2 and α_3 such that

$$(1,-3,5) = \alpha_1(1,2,1) + \alpha_2(1,1,-1) + \alpha_3(4,5,-2)$$

$$= (\alpha_1, 2\alpha_1, \alpha_1) + (\alpha_2, \alpha_2 - \alpha_2) + (4\alpha_3, 5\alpha_3, -2\alpha_3)$$

$$= (\alpha_1 + \alpha_2 + 4\alpha_3, 2\alpha_1 + \alpha_2 + 5\alpha_3, \alpha_1 - \alpha_2 - 2\alpha_3)$$

By equality of vectors, we must have

$$\alpha_1 + \alpha_2 + 4\alpha_3 = 1$$

$$2\alpha_1 + \alpha_2 + 5\alpha_3 = -3$$

$$\alpha_1 - \alpha_2 - 2\alpha_3 = 5$$

Adding (1) and (3), we get,

$$2\alpha_1 + 2\alpha_3 = 6$$

$$\Rightarrow \alpha_1 + \alpha_3 = 3$$

and adding (2) and (3), we get,

$$3\alpha_1 + 3\alpha_3 = 2$$

$$\Rightarrow \alpha_1 + \alpha_3 = \frac{2}{3}$$

From (4) and (5), it is clear that we cannot found α_1 and α_2 and so α_3.

$\therefore$ our supposition is wrong.

Hence (1, –3, 5) does not belong to the Linear Space of S.

Problem 3 : Is the system of vectors [–1, 1, 2], [2, –3, 1], [10, –1, 0] linearly dependent ?

Sol. Given vectors are

$$V_1 = [-1,1,2], V_2 = [2,-3,1], V_3 = [10,-1,0]$$

Consider the relation

$$k_1V_1 + k_2V_2 + k_3V_3 = O$$

or $$k_1[-1,1,2] + k_2[2,-3,1] + k_3[10,-1,0] = [0,0,0]$$

$\therefore$ $$-k_1 + 2k_2 + 10k_3 = 0 \quad \ldots (1)$$

$$k_1 - 3k_2 - k_3 = 0 \quad \ldots (2)$$

$$2k_1 + k_2 = 0 \quad \ldots (3)$$

From (3), $k_2 = -2k_1$

$\therefore$ from (2), $k_1 + 6k_1 - k_3 = 0 \Rightarrow k_3 = 7k_1$

$\therefore$ from (1), $-k_1 - 4k_1 + 70k_1 = 0 \Rightarrow 65k_1 = 0 \Rightarrow k_1 = 0$

$\therefore$ $k_2 = 0, k_3 = 0$

$\therefore$ $k_1 = k_2 = k_3 = 0$

$\therefore$ $k_1V_1 + k_2V_2 + k_3V_3 = O \Rightarrow k_1 = k_2 = k_3 = 0$

$\therefore$ given set of vectors is L.I.

Problem 4 : Find the value of k so that the vectors

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \text{ and } \begin{bmatrix} k \\ 0 \\ 1 \end{bmatrix} \text{ are L.D.}$$

Sol. Let a, b, c be scalars, not all zero, such that

$$a\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + c\begin{bmatrix} k \\ 0 \\ 1 \end{bmatrix} = O$$

where O is 3 × 1 zero matrix

$$\Rightarrow \begin{bmatrix} a \\ -a \\ 3a \end{bmatrix} + \begin{bmatrix} b \\ 2b \\ -2b \end{bmatrix} + \begin{bmatrix} ck \\ 0 \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a + b + ck \\ -a + 2b \\ 3a - 2b + c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ $a + b + ck = 0$

$-a + 2b = 0$

$3a - 2b + c = 0$

From (2), we get a = 2b

$\therefore 3 \Rightarrow 3a - a + c = 0 \Rightarrow c = -2a$

Put the values of c and b in (1), we have

$$a + \frac{a}{2} - 2ak = 0 \Rightarrow \frac{3a}{2} - 2ak = 0 \Rightarrow 2a\left(\frac{3}{4} - k\right) = 0$$

But $a \neq 0$

[as if a = 0 then b = 0 and c = 0 which implies the given vectors are L.I.)

$$\Rightarrow \frac{3}{4} - k = 0 \Rightarrow k = \frac{3}{4}.$$

1.2.4 Equality of Row Rank and column Rank Raw Rank and column Rank:

If A is any m × n matrix, then

(i) the space spanned by the set of m rows is called row space of A and the number of independent row vectors is called the row rank of A.

(ii) the space spanned by the set of n columns is called Column Space of A and the number of independent column vectors is called the column rank of A.

In other words, Column rank of any matrix A is the maximum number of linearly independent columns of A.

Result 4 : Prove that pre-multiplication by a non-singular matrix does not alter the row rank of a matrix.

Proof : Let $A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}$ be m × n matrix and

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & P_{22} & \cdots & p_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}$$ be m × n non-singular matrix

Let $$B = PA = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & P_{22} & \cdots & p_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}$$

$$= \begin{bmatrix} p_{11}R_1 + p_{12}R_2 + ... + p_{1m}R_m \\ p_{21}R_1 + p_{22}R_2 + ... + p_{2m}R_m \\ .... \\ p_{m1}R_1 + p_{m2}R_2 + ... + p_{mm}R_m \end{bmatrix}$$

$\therefore$ each row of B is a linear combination of the rows R_1, R_2, ...R_m of matrix A.
$\therefore$ every member of the row space of B is as will a member of the row space of the matrix A.
Again $\quad A = P^{-1} B$
Proceeding as above, we find that every member of the row space of A is a member of the row space of B.
$\therefore$ row spaces of A and B are the same
$\therefore$ pre-multiplication with a non-singular matrix does not alter the row rank.

Note : We can prove that post-multiplication with a non-singular matrix does not alter the column rank of a matrix.

Result 5 : If s, be the row rank of an m × n matrix A, then there exists a non-singular matrix R of type m × m, such that

$$RA = \begin{bmatrix} K \\ O \end{bmatrix}$$

where K is an s × n matrix consisting of a set of a linearly independent rows of A.

Proof : Here A an s × n matrix with s as row rank, so out of m rows of A, s rows are L.I and remaining m-s are L.D. By row transformations, being s L.I. rows in first s rows. Now each of the last (m – s) rows, which are L.D., is a linear combination of first s rows.
Now subtracting suitable multiples of first s rows from the last (m – s) rows, we get a matrix in which each of last (m – s) rows is a zero row. Therefore the resulting matrix is of the type $\begin{bmatrix} K \\ O \end{bmatrix}$.

We know that every row transformation can be effected by pre-multiplying with a non-singular matrix. Let R be the product of all non-singular matrices corresponding to all row transformations.

$$\therefore \; RA = \begin{bmatrix} K \\ O \end{bmatrix}, \text{ where K is s × n matrix.}$$

Note : Similarly we can prove that $AS = \begin{bmatrix} L & O \end{bmatrix}$, where L is an m × s matrix, s being the column rank of A.

Result 6 : Prove that the row rank of a matrix is the same as its rank.

Proof : Let r be the rank and s be the row rank of m × n matrix A.

Since s is row rank of A

$\therefore$ there exists a non-singular matrix R such that

$RA = \begin{bmatrix} K \\ O \end{bmatrix}$, where K is s × n matrix

since each minor of order (s + 1) of the matrix RA involves at least are row of zeros

$\therefore$ $\rho(RA) \le s$

$\therefore$ $r \le s$ ...(1)

Since r is rank of A

$\therefore$ there exists a non-singular matrix P such that

$PA = \begin{bmatrix} G \\ O \end{bmatrix}$, where G is r × n matrix.

The row rank of PA, being the same as that A, is s. Also PA has only r non-zero rows.

$\therefore$ the row rank of PA can, at the most be r

$\therefore$ $s \le r$ ..(2)

From (1) and (2)

r = s

i.e. rank of A = row rank of A.

Corellary : Prove that the column rank of a matrix is the same as its rank.

Proof : We know that columns of A are the rows of A'.

$\therefore$ column rank of A = row rank of A'

= rank of A'

= rank of A

Hence the result

Remarks :

1. Rank of A = row rank of A = column rank of A.

2. The rank of a matrix is equal to the maximum number of its linearly independent rows and also to the maximum number of its linearly independent columns.

3. If A an n-rowed non-singular matrix, then its rows as well as columns form L.I. sets.

4. If A, B be two matrices of the same type, then $\rho(A+B) \leq \rho(A)+\rho(B)$.

5. If A, B are two n-rowed square matrices, then

$$\rho(AB) \geq \rho(A)+\rho(B)-n.$$

Problem 5 : Examine the linear independent or dependence of the rows of the matrix $A=\begin{bmatrix} 3 & 2 & 4 \\ 1 & 0 & 2 \\ 1 & -1 & -1 \end{bmatrix}$, hence find its rank.

Solution : $A=\begin{bmatrix} 3 & 2 & 4 \\ 1 & 0 & 2 \\ 1 & -1 & -1 \end{bmatrix}$

$$\therefore |A| = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 0 & 2 \\ 1 & -1 & -1 \end{bmatrix} = 3\begin{vmatrix} 0 & 2 \\ -1 & -1 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} + 4\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix}$$

$= 3(0 + 2) - 2 (-1 -2) + 4(-1-0)$

$= 3(2) - 2 (-3) + 4 (-1) = 6 + 6 - 4 = 8 \neq 0$

$\therefore$ A is non-singular matrix.

$\therefore$ three rows of A and L.I.

$\therefore$ rows of matrix A form of L.I. set

$\therefore$ $\rho(A)=3$.

Problem 6 : Find the value of k so that the vectors

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \text{and} \begin{bmatrix} k \\ 0 \\ 1 \end{bmatrix} \text{are L.D.}$$

Solution : Let a, b, c be scalars, not all zero, such that

$$a\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + c\begin{bmatrix} k \\ 0 \\ 1 \end{bmatrix} = O$$

where O is 3 × 1 zero matrix

$$\Rightarrow \begin{bmatrix} a \\ a \\ 3a \end{bmatrix} + \begin{bmatrix} b \\ 2b \\ -2b \end{bmatrix} + \begin{bmatrix} ck \\ 0 \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a+b+ck \\ -a+2b \\ 3a-2b+c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ a + b + ck = 0

–a + 2b = 0

3a – 2b + c = 0

From (2), we get a = 2b

$\therefore$ (3) ⇒ 3a – a + c = 0 ⇒ c = –2a

Put the values of c and b in (1) , we have

$$a + \frac{a}{2} - 2ak = 0 \Rightarrow \frac{3a}{2} - 2ak = 0 \Rightarrow 2a\left(\frac{3}{4} - k\right) = 0$$

But $a \neq 0$

[as if a = 0 then b = 0 and c = 0 which implies the given vectors are L.I.]

$$\Rightarrow \frac{3}{4} - k = 0 \Rightarrow k = \frac{3}{4}.$$

1.2.5 Summary

This lessons helps us to understand about the concept of linear combination of vectors and the vectors generating a subspace i.e. the vectors which form basis of a subspace. For this, we discussed about the linear dependence\independence of vectors. The concept is made more elaborative with the help of various suitable examples.

1.2.6 Key Concepts

Row rank, Column rank, Linear combination, Linear dependence/independence.

1.2.7 Long Questions

1. Reduce to row reduced echelon form the matrix

$$A = \begin{bmatrix} 0 & 1 & 3 & -1 & 4 \\ 2 & 0 & -4 & 1 & 2 \\ 1 & 4 & 2 & 0 & -1 \\ 3 & 4 & -2 & 1 & -1 \\ 6 & 9 & -1 & 1 & 6 \end{bmatrix} \text{ and find } \rho_R(A).$$

2. Find the row rank of the matrix $A=\begin{bmatrix}1 & 2 & -1 & 3\\ 4 & 1 & 2 & 1\\ 3 & -1 & 1 & 2\\ 1 & 2 & 0 & 1\end{bmatrix}$.

3. Show that the vectors $V_1=(1,2,3), V_2(0,1,2)$ and $V_3=(0,0,1)$ generate V_3(R).

4. Examine for linear dependence the vectors $[1,\ 2,\ 4,],[2,\ -1,\ 3],[0,\ 1,\ 2],[-3,\ 7,\ 2]$ and find the relation if it exists.

5. Prove that the vectors $x=(1,0,0), y=(0,1,0); z=(0,0,1)$ and $w=(1,1,1)$ form a linearly dependent set, but any three of them are linearly independent.

6. Show that the row vectors of the matrix $\begin{bmatrix}6 & 2 & 3 & 4\\ 0 & 5 & -3 & 1\\ 0 & 0 & 7 & -2\end{bmatrix}$ are linearly independent.

7. Determine whether the following matrices have same column space or not ?

$$A=\begin{bmatrix}1 & 3 & 5\\ 1 & 4 & 3\\ 1 & 1 & 9\end{bmatrix}, B=\begin{bmatrix}1 & 2 & 3\\ -2 & -3 & -4\\ 7 & 12 & 15\end{bmatrix}.$$

1.2.8 Short Questions

1. Define row rank of a matrix.

2. Define column rank of a matrix.

3. Define linear combination of vectors. 4. Define basis of a subspace.

4. Show that the vectors $[1\ \ 2\ \ 3],[3\ \ -2\ \ -1],[1\ \ -6\ \ -5]$ form a L.I. system.

1.2.9 Suggested Readings

1. P. B. Bhattacharya, S. K. Jain & S. R. Nagpaul : A First Course in Linear Algebra, New Age International (P) Ltd.
2. Gilbert Strang : Linear Algebra and its Applications, Cengage Learning Publishers (Fourth Edition)

B.A. PART– I **MATHEMATICS : PAPER III**

(SEMESTER - I) **LINEAR ALGEBRA**

LESSON NO. 1.3 **AUTHOR : DR. CHANCHAL**

EIGEN VALUES AND EIGEN VECTORS

1.3.1 Objectives

With the help of this lesson, the students would be able to get knowledge about

- Eigen values and Eigen vectors of a matrix
- Diagonalizable matrix and its corresponding diagonal matrix
- Cayley-Hamilton theorem
- The concept of minimal polynomial and minimal equation

1.3.2 Introduction

An expression of the form d $A_0x^m + A_1x^{m-1} + A_2x^{m-2} + ... + A_{m-1}x + A_m$. where $A_0, A_1, A_2, ..., A_m$ are all square matrices of the some order n and m is a positive integer, is called a n-rowed matrix polynomial of degree m.

Note : Two matrix polynomials are said to be equal iff the coefficients of the like powers of x are the same.

1. Eigen values are also known as proper values, characteristic values, latent roots or spectral values. Similarly eigen vectors are also called proper vectors, characteristic vectors, latest vectors or spectral vectors.

2. The set of characteristic roots of a matrix A is called the spectrum of the matrix A.

1.3.3 Some Important Results

Result 1 : Prove that λ is an eigen value of n-rowed square matrix A over a field F it and only if $|A-\lambda I|=0$.

Proof : (i) Assume that λ is an eigen value of A over F.

$\therefore$ there exists a non-zero column matrix X of type n × 1 such tat

$$AX=\lambda X$$

$$\Rightarrow AX-\lambda X=O$$

$$\Rightarrow AX-\lambda IX=O$$

$$\Rightarrow (A-\lambda I)X=O$$

$$\Rightarrow |A-\lambda I|=0 \qquad [\because X\neq O]$$

$[\because AX=O$ has a non-trivial solution iff $|A| = 0]$

(ii) Assume that $|A-\lambda I|=0$

$\therefore$ $|A-\lambda I|X=O$ has a non-trivial solution

$\therefore$ $AX-\lambda IX=O$

or $AX-\lambda X=O$

or $AX=\lambda X$

Where X is a non-zero matrix

$\therefore$ λ is an eigen value of A over F.

Note. λ is an eigen value of A over F iff $A-\lambda I$ is a singular matrix.

Result 2 : If X is a characteristic vector of a matrix corresponding to the characteristic value λ, then kX is also a characteristic vector of A corresponding to the same charactristic value $\lambda(k\neq 0)$.

Proof : Since X is a characteristic vector of A corresponding to the characteristic value λ.

$\therefore$ $X\neq O$ and

$AX = \lambda X$

Now $A(k X) = k(A X) = k(\lambda X) = \lambda(k X)$

Now kX is a non-zero vector such that $A(kX) = \lambda(kX)$

$\therefore$ kX is a characteristic vector of A corresponding to the characteristic value λ.

Note : Corresponding to a characteristic value λ , there may corresponds more than one characteristic vectors.

Result 3 : If X be an eigen vector of the n-rowed square matrix A over a field F, thn X cannot correspond to two distinct eigen values.

Proof : Since X is an eigen vector of A over F.

$\therefore$ X is a non-zero column matrix of order $n \times 1$.

Suppose eigen vector X corresponds to two eigen values λ_1, λ_2 of A.

$$\therefore \quad AX = \lambda_1 X \text{ and } AX = \lambda_2 X$$

$$\Rightarrow \lambda_1 X = \lambda_2 X$$

$$\Rightarrow (\lambda_1 - \lambda_2) X = 0$$

$$\Rightarrow \lambda_1 - \lambda_2 = 0$$

Hence the result.

Result 4 : Prove that any system of eigen vectors X_1, X_2, ...X_m corresponding respectively to a system of distinct eigen values $\lambda_1, \lambda_2, \lambda_m$ of a matrix A is linearly independent.

Proof : Try Yourself.

Result 5 : Prove that the characteristic roots of a hermitian matrix are real.

Proof : Let λ be a characteristic roots of a hermitian matrix A.

$\therefore$ there exists a non-zero $n \times 1$ column matrix X such that

$AX = \lambda X$

$$\Rightarrow X^\theta (AX) = X^\theta (\lambda X)$$

$$\Rightarrow X^\theta AX = \lambda X^\theta X$$

$$\Rightarrow \left(X^\theta AX\right) = \left(\lambda X^\theta X\right)^\theta$$

$$\Rightarrow X^\theta A^\theta \left(X^\theta\right)^\theta = \bar{\lambda} X^\theta \left(X^\theta\right)^\theta$$

$\Rightarrow X^{\theta}AX = \bar{\lambda}\, X^{\theta}X$ [$\because A^{\theta} = A$ as A is hermitian and $\left(X^{\theta}\right)^{\theta} = X$]

$\Rightarrow X^{\theta}\lambda\, X = \bar{\lambda}\, X^{\theta}X$ $[\because AX = \lambda X]$

$\Rightarrow \lambda X^{\theta}X = \bar{\lambda}\, X^{\theta}X$

$\Rightarrow \left(\lambda - \bar{\lambda}\right) X^{\theta}X = O$

$\Rightarrow \lambda - \bar{\lambda} = 0$ $\left[\because X^{\theta} \neq O \text{ as } X \neq O\right]$

$\Rightarrow \bar{\lambda} = \lambda$

$\Rightarrow \lambda$ is real

Hence the result.

Result 6 : Prove that any two characteristic vectors corresponding to two distinct characteristics roots of a hermitian matrix are orthogonal.

Proof : Let X_1, X_2 be the characteristic vectors corresponding to characteristic roots λ_1, λ_2 of the hermitian matrix A.

$\therefore$ $AX_1 = \lambda_1 x_1$

$AX_2 = \lambda_2 X_2$

From (1), $X_2{}^{\theta}AX_1 = X_2{}^{\theta}\lambda_1 X_1$

From (2), $X_1{}^{\theta}AX_2 = X_1{}^{\theta}\lambda_2 X_2$

Now $\left(X_2{}^{\theta}AX_1\right)^{\theta} = X_1{}^{\theta}A^{\theta}\left(X_2{}^{\theta}\right)^{\theta} = X_1{}^{\theta}AX_2,$

since $A^{\theta} = A$ as A is hermitian

$\therefore$ $\left(X_2{}^{\theta}\lambda_1 X_1\right)^{\theta} = X_1{}^{\theta}\lambda_2 X_2$

$\Rightarrow$ $\lambda_1 X_1{}^{\theta}\left(X_2{}^{\theta}\right)^{\theta} = \lambda_2 X_1{}^{\theta}X_2$ $[\because \lambda_1, \lambda_2 \text{ are real}]$

$\Rightarrow$ $\lambda_1 X_1{}^{\theta}X_2 = \lambda_2 X_1{}^{\theta}X_2$

$\Rightarrow$ $(\lambda_1 - \lambda_2)\, X_1{}^{\theta}X_2 = O$

But $\lambda_1 - \lambda_2 \neq 0$

$\therefore \quad X_1^\theta X_2 = 0$

$\Rightarrow \quad X_1, X_2$ are orhogonal.

Result 7 : Prove that characteristic roots of a unitary matrix are of unit modulus.

Proof : Let A be given unitary matrix.

$$\therefore \quad A^\theta A = 1 \qquad \ldots (1)$$

Let λ be a characteristic root of A.

$\therefore$ there exists a non-zero vector X such that

$$AX = \lambda X \qquad \ldots (2)$$

$$\Rightarrow \quad (AX)^\theta = (\lambda X)^\theta$$

$$\Rightarrow \quad X^\theta A^\theta = \bar{\lambda} X^\theta \qquad \ldots (3)$$

From (2) and (3), we get,

$$\left(X^\theta A^\theta\right)(AX) = \left(\bar{\lambda} X^\theta\right)(\lambda X)$$

$$\Rightarrow \quad X^\theta \left(A^\theta A\right) X = \lambda\bar{\lambda}\, X^\theta X$$

$$\Rightarrow \quad X^\theta I X = \lambda\bar{\lambda}\, X^\theta X \qquad [\because \text{ of } (1)]$$

$$\Rightarrow \quad X^\theta X = \lambda\bar{\lambda}\, X^\theta X$$

$$\Rightarrow \quad \left(\lambda\,\bar{\lambda} - 1\right) X^\theta X = O$$

$$\Rightarrow \quad \lambda\bar{\lambda} - 1 = O \qquad [\because X^\theta X \neq O \text{ as } X \neq O]$$

$$\Rightarrow \quad |\lambda|^2 - 1 = O$$

$$\Rightarrow \quad |\lambda|^2 = 1$$

$$\Rightarrow \quad |\lambda| = 1$$

Hence the result.

Result 8 : Prove that any two characteristic vectors corresponding to two distinct characteristic roots of a unitary matrix are orthogonal.

Proof : Try Yourself.

1.3.4 Characterirtic Equation of a Matrix

If A be any n-rowed square matrix over a field F and λ an indeterminate, then the

matrix $A-\lambda I$ is called the characteristic matrix of A.
The determinant $|A - \lambda I|$, an algebraic polynomial in λ of degree n, is called the characteristic polynomial of A.
The equation $|A - \lambda I| = 0$ is called characteristic equation of A.

Remark : An eigen value λ of matrix A is always a root of its characteristic equation and every root of the characteristic equation of A is an eigen value of A.
$\therefore$ in order to find eigen values of A, we should find roots of the characteristic equation of A.

1.3.5 Diagonalizable Matrix

An $n \times n$ matrix A is called diagonalizable if there exists an invertible $n \times n$ matrix P such that $P^{-1} AP$ is a diagonal matrix.

Method to find Diagonal Matrix for a Diagonalizable matrix.

Step I : Find eigen values $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ of A.

Step II : Find corresponding eigen vectors $X_1, X_2, \ldots, X_n$. If number of eigen vectors < n, A is not diagonalizable.

Step III : Find $P = \{X_1 X_2 X_3 \ldots X_n\}$ and P^{-1}.

Step IV : $p^{-1}AP = \text{Diag.}(\lambda_1, \lambda_2, \ldots\ldots, \lambda_n)$

is required diagonal matrix.

Note : A is diagonalizable if and only if A has n L.I. eigen vectors.

1.3.6 Cayley Havilton Theorem

Statement : Every square matrix satisfies its characteristic equation.

Proof : Let A be any square matrix of order n, and its characteristic equation be

$$p_0 + p_1\lambda + p_2\lambda^2 + \ldots + p_n\lambda^n = 0$$

We have to prove that A satisfies this equation

i.e., $p_1 1 + p_1 A + p_2 \lambda^2 + \ldots + p_n A^n = O$

For proving this, we proceed as follows :

We know that $(A - \lambda I)\,\text{adj.}(A - \lambda I) = |A - \lambda| I$ $\quad [\because A\,\text{adj.}\,A = |A|I]$

Let adj. $(A - \lambda I) = B_0 + B_1\lambda + B_2\lambda^2 + \ldots + B_{n-1}\lambda^{n-1}$

$\therefore$ we have, $(A - \lambda I)\left(B_0 + B_1\lambda + B_2\lambda^2 + \ldots + B_{n-1}\lambda^{n-1}\right)$

$$= \left(p_0 + p_1\lambda + p_2\lambda^2 + \ldots + p_n\lambda^n\right)I$$

Equating the coefficients of like powers of λ, we get,

$$AB_0 = p_0 I$$

$$AB_1 - B_0 = p_1 I$$

$$AB_2 - B_1 = p_2 I$$

$$\ldots \ldots \ldots \ldots \ldots \ldots$$

$$AB_{n-1} - B_{n-2} = p_{n-1} I$$

$$-B_{n-1} = p_n I$$

Pre-multiplying above equations by I, A, A^2, ..., A^n respectively and adding, we get,

$O = p_0 I + p_1 A + p_2 A^2 + \ldots + p_n A^n$, which is same as (1).

Hence the theorem.

1.3.7 Minimal Polynomial and Minimal Equation

If m(x) be a scalar polynomial of the lowest degree with leading coefficient unity, such that m(x) = 0 is satisfied by A i.e. m(A) = O, then the polynomial m(x) is called the minimal polynomial of A and m(x) = 0 is called the minimum equation of A.

Note. The degree of the minimal equation of an n-rowed matrix is less than or equal to that of its characteristic equation which is n.

Derogatory and Non-derogatory Matrices

An n-rowed matrix is said to be derogatory or non-derogatory, according as the degree of its minimal equation is less than or equal to n.

1.3.8 Problems

Problem 1 : Prove that a square matrix A and its transpose A have the same set of eigen values.

Sol. Characteristic polynomial of A^t

$$= |A^t - \lambda I| = |A - \lambda I^t| = |(A - \lambda I)^t|$$

$$= |A - \lambda I| \qquad \left[\because |A^t| = |A|\right]$$

= characteristic polynomial of A

∴ A and A^t have same characteristic polynomial and hence the same set of eigen values.

Problem 2 : If α is an eigen value of a non-singular matrix A, then prove that $\frac{|A|}{\alpha}$ is an eigen value of adj. A.

Sol. Since α is an eigen of a non-singular matrix A

$\therefore$ $\alpha \neq 0$ and there exists a non-zero column vector X such that

$$AX = \alpha X$$

$$\Rightarrow \quad (\text{adj. } A)(AX) = (\text{adj. } A)(\alpha X)$$

$$\Rightarrow \quad [(\text{adj. } A)(A)]X = \alpha[(\text{adj. } A)X]$$

$$\Rightarrow \quad (|A|I)X = (\text{adj. } A)X$$

$$\Rightarrow \quad (\text{adj. } A)X = \frac{|A|}{\alpha}X$$

$\Rightarrow$ $\frac{|A|}{\alpha}$ is an eigen value of adj. A.

Problem 3 : The characteristic roots of $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & k \end{bmatrix}$ are 0, 3 and 15. Find the value of k.

Sol. Let $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & k \end{bmatrix}$

$$\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\therefore \quad A - \lambda I = \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & k-\lambda \end{bmatrix}$$

$\therefore$ characteristics equation of matrix A is $|A - \lambda I| = 0$

or $$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & k-\lambda \end{vmatrix} = 0$$

Since $\lambda = 0$ is a root of it

$\therefore$ $$\begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & k \end{vmatrix} = 0$$

$\therefore$ $$8\begin{vmatrix} 7 & -4 \\ -4 & k \end{vmatrix} - (-6)\begin{vmatrix} -6 & -4 \\ 2 & k \end{vmatrix} + 2\begin{vmatrix} -6 & 7 \\ 2 & -4 \end{vmatrix} = 0$$

or $$8(7k-16)+6(-6k+8)+2(24-14)=0$$

or $$56\,k-128-36k+48+20=0$$

$\therefore$ $$20k = 60 \;\Rightarrow k = 3.$$

Problem 4 : Define similar matrices and prove that similar matrices have same characteristic polynomial and hence same eigen values.

Sol. Let A and B be square matrices of order n over a field F. Then A is said to be similar to B over F if and only if there exists an n-rowed invertible matrix P over F such that

$$AP = PB \text{ i.e. } B = P^{-1}\,AP \;\text{ or }\; A = PB\,P^{-1}$$

Let A and B be two similar matrices

$\therefore$ $$B = P^{-1}AP$$

$\therefore$ $$B-\lambda I = P^{-1}AP-\lambda I = P^{-1}\,AP-\lambda P^{-1}P \qquad \left[\because P^{-1}P = I\right]$$

$$= P^{-1}AP - P^{-1}(\lambda\, I)\,P = P^{-1}\,(A-\lambda I)\,P$$

$\therefore$ $$|\,B-\lambda I\,| = |\,P^{-1}\,(A-\lambda I)\,P\,|$$

$$= |\,P^{-1}\,|\,|\,A-\lambda I\,|\,P\,|$$

$$= |\,A-\lambda I\,|\,|\,P^{-1}\,|\,|\,P\,| = |\,A-\lambda I\,|\,|\,P^{-1}P\,|$$

$$= |\,A-\lambda I\,|\,|\,I\,|$$

$\therefore \quad |B - \lambda I| = |A - \lambda I| \qquad [\because |I| = 1]$

$\therefore$ matrices A and B = $P^{-1}AP$ have the same characteristic polynomial and hence the same set of eigen values.

Problem 5 : Determine the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

Is it diagonalisable ? Justify.

Sol. $A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{bmatrix}$$

$\therefore$ the characteristics equation of A is $|A - \lambda I| = 0$

or $$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0$$

or $$\begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0, \text{ by } R_1 \to R_1 + R_2 + R_3$$

or $$(6-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0 \text{ or } (6-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 2 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

or $(6-\lambda)\left[(1)(2-\lambda)(2-\lambda)\right]=0$

or $(6-\lambda)(2-\lambda)^2=0$

$\therefore$ $\lambda = 2, 2, 6$

which are the eigen values of A.

The eigen vector $X=\begin{bmatrix}x\\y\\z\end{bmatrix}\neq O$ corresponding to the eigen value $\lambda=6$ is given by

$$AX=\lambda X \text{ or } (A-6I)X=O$$

or $\begin{bmatrix}-3 & 1 & 1\\ 2 & -2 & 2\\ 1 & 1 & -3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ or $\begin{bmatrix}1 & 1 & -3\\ 2 & -2 & 2\\ -3 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

or $\begin{bmatrix}1 & 1 & -3\\ 0 & -4 & 8\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$, by $R_3 \to R_3 + R_2$

Now the coefficient matrix of these equations is of rank 2. Therefore these equations have only 3 – 2 = 1 L.I. solution. Thus there is only one L.I. eigen vector corresponding to the value 6. These equations can be written as

$x + y - 3z = 0$

$-4y+8z=0 \qquad \Rightarrow y=2z$

$\therefore\ x+2z-3z=0 \qquad \Rightarrow x=z$

Take $z = 1,$ $\quad \therefore x=1, y=2$

$X=\begin{bmatrix}1\\2\\1\end{bmatrix}$ is an eigen vector of A.

The eigen vector $X=\begin{bmatrix}x\\y\\z\end{bmatrix}\neq O$ corresponding to the eigen value $\lambda=2$ is given by

$$AX = 2X \text{ or } (A = 2I)X = O$$

or
$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{by } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

The coefficient matrix of these equations is of rank 1. Therefore these equations have 3 – 1 = 2 L.I. solutions. These equations can be written as

$x + y + z = 0$ or $x = -y - z$

Take $y = 1, z = 0$; $y = 0, z = 1$

Therefore we find two L.I. eigen vectors of A as $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

$$\therefore \quad P = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$|P| = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = 1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= 1(1-0) + 1(2-0) - 1(0-1)$$

$$= 1(1) + 1(2) - 1(-1) = 1 + 2 + 1$$

$$= 4$$

Co-factors of the elements of first row of $|P|$ are

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, -\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \text{ i.e. } 1, 2, 1 \text{ respectively}$$

Co-factors of the elements of second row of $|P|$ are

$$= \begin{vmatrix} -1 & -1 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} \text{ i.e. } 1, 2, -1 \text{ respectively}$$

Co-factors of the elements of third row of $|P|$ are

$$\begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix}, -\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \text{ i.e. } 1, -2, 3 \text{ respectively}$$

$$\therefore \qquad \text{adj}\,P = \begin{bmatrix} 1 & -2 & -1 \\ 1 & 2 & -1 \\ 1 & -2 & 3 \end{bmatrix}' = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix}$$

$$\therefore \qquad p^{-1} = \frac{\text{adj}\,P}{|P|} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix}$$

$$p^{-1}AP = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 24 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

which is a diagonal matrix.

Problem 6 : Verify Cayley Hamilton Theorem for the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & -1 & 0 \end{bmatrix}$.

Hence find A^{-1} .

Solution : $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & -1 & 0 \end{bmatrix}$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \lambda L = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\therefore \quad A - \lambda I = \begin{bmatrix} -\lambda & 0 & 1 \\ 1 & 2-\lambda & 0 \\ 2 & -1 & -\lambda \end{bmatrix} = -\lambda \begin{vmatrix} 2-\lambda & 0 \\ -1 & -\lambda \end{vmatrix} + 1 \begin{vmatrix} 1 & 2-\lambda \\ 2 & -1 \end{vmatrix}$$

$$= -\lambda\left(-2\lambda + \lambda^2\right) + 1(-1 - 4 + 2\lambda) = -\lambda^3 + 2\lambda^2 - 5 + 2\lambda$$

$$\therefore \quad |A - \lambda I| = -\lambda^3 + 2\lambda^2 + 2\lambda - 5$$

The characteristic equation of A is $|A - \lambda I| = 0$

or $\quad -\lambda^3 + 2\lambda^2 + 2\lambda - 5 = 0$ or $\quad \lambda^3 - 2\lambda^2 - 2\lambda - 5 = 0$

We are to prove that A satisfies thiws equation i.e. $A^3 - 2A^2 - 2A + 5I = O$

Now $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & -1 & 0- \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 4 & 1 \\ -1 & -2 & 2 \end{bmatrix}$

$$A^3 = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 4 & 1 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & -1 & 0- \end{bmatrix} = \begin{bmatrix} -1 & -2 & 0 \\ 6 & 7 & 2 \\ 2 & -6 & -1 \end{bmatrix}$$

Consider $A^3 - 2A^2 - 2A + 5I$

$$= \begin{bmatrix} -1 & -2 & 2 \\ 6 & 7 & 2 \\ 2 & -6 & -1 \end{bmatrix} - 2 \begin{bmatrix} 2 & -1 & 0 \\ 2 & 4 & 1 \\ -1 & -2 & 2 \end{bmatrix} - 2 \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & -1 & 0 \end{bmatrix} + 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \quad A^3 - 2A^2 - 2A + 5I = O$$

Re-multiplying both sides by A^{-1}, we get,

$$A^2 - 2A - 2I + 5I^{-1} = 2I$$

$$= -\begin{bmatrix} 2 & -1 & 0 \\ 2 & 4 & 1 \\ -1 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & -1 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$5A^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & -1 \\ 5 & 0 & 0 \end{bmatrix}$$

$$\therefore \qquad A^{-1} = \frac{1}{5}\begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & -1 \\ 5 & 0 & 0 \end{bmatrix}$$

1.3.9 Summary

In this lesson, we have studied about characteristic equation and the terms related to it. An important theorem based upon it i.e. Cayley-Hamilton theorem has been discussed. Moreover, the concept of diagonalizable matrix and to find out a diagonal matric for it, has been also elaborated. The concepts are made more clear with the help of various suitable examples.

1.3.10 Key Concepts

Characteristic roots, Characteristic equation, Diagonalizable matrix, Cayley-Hamilton theorem, Minimal polynomial, Minimal equation.

1.3.11 Long Questions

1. If λ is an eigen value of a square matrix A, then prove that $\bar{\lambda}$ is an eigen value of A^{θ} and conversely.

2. Show that the necessary and sufficient condition for a 2 × 2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to have zero as an eigen value is that a d – b c = 0.

5. Diagonalize, if possibel, the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & -1 & 4 \end{bmatrix}$.

6. Verify Cayley-Hamilton Theorem for the matrix $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$. Hence find A^{-1}.

7. Using Cayley Hamilton theorem, find A^8, if $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

1.3.12 Short Questions

1. Define diagonalizable matrix.

2. Discuss the concept of minimal polynomial and minimal equation.

3. Determine eigen values of the matrix $\begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$.

4. Find the characteristic roots and the spectrum of the matrix $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

1.3.13 Suggested Readings

1. P. B. Bhattacharya, S. K. Jain & S. R. Nagpaul : A First Course in Linear Algebra, New Age International (P) Ltd.

2. Gilbert Strang : Linear Algebra and its Applications, Cengage Learning Publishers (Fourth Edition)

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B.A. PART– I
(SEMESTER - I)

MATHEMATICS : PAPER III
LINEAR ALGEBRA

LESSON NO. 1.4

AUTHOR : DR. CHANCHAL

SYSTEM OF LINEAR EQUATIONS AND ITS CONSISTENCY

1.4.1 Objectives

With the help of this lesson, the students would be able to get knowledge about

- Linearly independent solution for the system of homogeneous linear equations.
- Consistency of a system of non-homogeneous linear equations

1.4.2 Homogeneous and Non-Homogeneous Linear Equations (An Introduction)

Let
$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_2 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0 \end{array}\right]$$

be a set of m linear equations in n unknowns x_1, x_2, ..., x_n.

The above set of linear equations can be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = O$$

i.e., $AX = O$

$$\text{where } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, O = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The matrix A is called the coefficient matrix.

Remarks : Any set of values $x_1, x_2, \ldots\ldots, x_n$ which satisfy simultaneously the m equtions in (1), is called a solution of the system.

A system of equations, which has a solution, is called consistent or compatible. If the system does not has any solution, it is called inconsistent.

1.4.3 Linearly Independent Solutions of AX = O

Conditions under which a set of homogeneous equations possess a (i) trivial solution of (ii) non-trivial solution.

Let there be m equations in n unknowns. So the coefficient matrix A is of type $m \times n$. Let r be rank of A.

Now either $r < n$ or $r = n$

(i) If $r = n$, then the equation $AX = O$ has $n - n = 0$

i.e. no linearly independent solution. Therefore, the equation $AX = O$ has trivial solution.

(ii) If $r < n$, then the equation $AX = 0$ has $n - r$ linearly independent solutions. Any linear combination of these $n - r$ solutions will also be a solution of $AX = O$. So, there are infinite number of non-trivial solutions.

Article 1 : Let A be an $m \times n$ matrix of rank r. Then the equation $AX = O$ has $(n-r)$ linearly independent solutions.

Proof : The given equation is $AX = O$ ... (1)

We have to prove two results :

(i) $AX = O$ has $(n - r)$ solutions

(ii) $(n - r)$ solutions form a linearly independent set.

For proving first part, we proceed as following :

$\because$ rank of A = r

$\therefore$ column rank of A = r

$\therefore$ A has r linearly independent columns. We assume that first r columns are linearly independent and last (n–r) columns are linearly dependent.

Let $A = \begin{bmatrix} C_1 & C_2 \ldots C_r & C_{r+1} \ldots C_n \end{bmatrix}$

$\therefore$ equation (1) can be written as

$$\begin{bmatrix} C_1 & C_2 \ldots C_r & C_{r+1} \ldots C_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = O$$

i.e., $C_1 x_1 + C_2 x_2 + \ldots + C_r x_r + C_{r+1} x_{r+1} + \ldots + C_n x_n = O$

$\because$ $C_{r+1}, C_{r+1}, \ldots . C_n$ are linearly dependent columns.

$\therefore$ each is a linear combination of $C_1, C_2, \ldots \ldots, C_r$

Let $C_{r+1} = p_{11}\, C_1 + p_{12}\, C_2 + \ldots + p_{1r}\, C_r$

$C_{r+2} = p_{21}\, C_1 + p_{22}\, C_2 + \ldots + p_{2r}\, C_r$

....

$C_n = p_{t1} C_1 + p_{t2}\, C_2 + \ldots + p_{tr} C_r$ where t = n – r $\quad [\because n = r + (n - r)]$

The above equation can be written as

$$\left.\begin{array}{l} p_{11}C_1 + p_{12}C_2 + \ldots . + p_{1r}C_r + (-1)C_{r+1} + 0.C_{r+2} + \ldots . + 0.C_n = O \\ p_{21}C_1 + p_{22}C_2 + \ldots . + p_{2r}C_r + 0.C_{r+1} + (-1)C_{r+2} + \ldots . + 0.C_n = O \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ p_{11}C_1 \;\; p_{12}C_2 \ldots . + p_{1r}C_r + 0.C_{r+1} + 0.C_{r+2} + \ldots . + (-1)\, C_n = O \end{array}\right]$$

Comparing one by one the equation in (3) with equation (2), we get,

$$X_1 = \begin{bmatrix} p_{11} \\ p_{12} \\ : \\ p_{1r} \\ -1 \\ 0 \\ 0 \\ : \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} p_{21} \\ p_{22} \\ : \\ p_{2r} \\ 0 \\ -1 \\ 0 \\ : \\ 0 \end{bmatrix}, \ldots, X_t = \begin{bmatrix} p_{t1} \\ p_{t2} \\ : \\ p_{tr} \\ 0 \\ 0 \\ 0 \\ : \\ -1 \end{bmatrix}$$

as t = n – r solutions of the equation

Now we have to show that these n – r solutions $X_1, X_2, \ldots, X_t$ are linearly independent vectors.

For this we consider the relation

$$p_1X_1 + p_2X_2 + \ldots + p_tX_t = O$$

i.e.

$$p_1 \begin{bmatrix} p_{11} \\ p_{12} \\ : \\ p_{1r} \\ -1 \\ 0 \\ 0 \\ : \\ 0 \end{bmatrix} + p_2 \begin{bmatrix} p_{21} \\ p_{22} \\ : \\ p_{2r} \\ 0 \\ -1 \\ 0 \\ : \\ 0 \end{bmatrix} + \ldots + p_t \begin{bmatrix} p_{t1} \\ p_{t2} \\ : \\ p_{tr} \\ 0 \\ 0 \\ 0 \\ : \\ -1 \end{bmatrix} = O$$

Comparing (r + 1)th, (r + 2)th ….. nth elements, we get,

$$-p_1 = 0, -p_2 = 0, \ldots, -p_t = 0$$

$\therefore \quad p_1 = p_2 = \ldots = p_t = 0$

$\therefore \quad p_1X_1 + p_2X_2 + \ldots.. + p_tX_t = 0$

$\Rightarrow \quad p_1 = p_2 = \ldots.. = p_t = 0$

$\therefore \quad X_1, X_2, \ldots\ldots, X_t$ are L.I. vectors

$\therefore$ A X = O has n – r L.I. solution.

Article 2 : The equation AX = O has a non-zero (i.e., non-trivial solution) iff A is singular.

Proof : Assume that AX = O has a non-zero solution.

$\therefore$ n – r > 0 where r is the rank of n-rowed matrix A implies n > r.

i.e., rank of A is less than the order of the matrix.

$\therefore$ A is a singular matrix.

Again, assume that A is a singular matrix

$\therefore$ |A| = 0

$\Rightarrow$ rank of A < order of A

$\Rightarrow$ r < n

$\Rightarrow$ n – r > 0

$\Rightarrow$ equation AX = 0 has a non-zero solution.

1.4.4 Consistency of AX = B

Conditions under which a system of non-homogeneous equations will have :

(i) no solution (ii) a unique solution (iii) infinity of solutions.

Let AX = B be a system of non-homogeneous equations.

(i) The equation AX = B has no soluiton if A and [A B] do not have the same rank.

(ii) The equation AX = B, has a solution if the rank of A is the same as that of [A B]. If in addition, A is non-singular, then equation has a unique solution.

(iii) The equation AX = B will have infinite solutions if A and [A B] have the same rank and A is singular.

Article 2 : The necessary and sufficient condition that the system of equations AX = B is consistent (i.e., has a solution), is tha the matrices A and [A B] are of the same rank.

Proof : Let $\rho(A) = r$ where A is m × n matrix.

$\therefore$ column rank of A is also r.

$\therefore$ r columns of A are linearly independent and the remaining (n – r) are linearly dependent.

Let $C_1, C_2, \ldots, C_r$ be linearly independent and $C_{r+1}, \ldots, C_n$ be linearly dependent where A = $[C_1 \quad C_2 \ldots C_n]$.

The given equation is Ax = B where $X = \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_n \end{bmatrix}$

$$\text{i.e., } \begin{bmatrix} C_1 & C_2 \ldots C_r & C_{r+1} \ldots C_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_r \\ x_{r+1} \\ : \\ x_n \end{bmatrix} = B$$

$$\text{i.e., } x_1C_1 + x_2C_2 + \ldots + x_rC_r + x_{r+1}C_{r+1} + \ldots + x_nC_n = B$$

Condition is necessary.

Asume that the equation AX = B has a solution $X = \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_n \end{bmatrix}$

$$\therefore \text{ we have } \begin{bmatrix} C_1 & C_2 .. C_r & C_{r+1} \ldots C_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_r \\ x_{r+1} \\ : \\ x_n \end{bmatrix} = B$$

$\therefore$ $x_1C_1 + x_2C_2 + ... + x_rC_r + x_{r+1}C_{r+1} + ... + x_nC_n = B$ ... (2)

$\because$ $C_{r+1}\ C_{r+2}, ..., C_n$ are linearly dependent and

$C_1, C_2, .., C_r$ are linearly independent.

$\therefore$ $C_{r+1}, ..., C_n$ are linear combination of $C_1, C_2, ..., C_r$ and consequently from (2), B is also a linear combination of $C_1, C_2, ..., C_r$.

$\therefore$ number of linearly independent columns of [A B] is also r.

$\therefore$ if the equation AX = B has a solution, then rank of A is the same as that of [A B].

Condition is sufficient.

Assume rank of A as well as of [A B] is r.

$\because$ rank of [A B] is r.

$\therefore$ number of independent columns of [A B]

i.e., $\left[C_1\ \ C_2...C_r\ \ C_{r+1}...C_n\ \ B\right]$ is r.

But C_1, C_2, ..., C_r are already linearly independent.

$\therefore$ B is linearly dependent column

$\because$ B is a linear combination of C_1, C_2, ..., C_r

$\therefore$ there exists r scalars p_1, p_2, ..., p_r such that

$$B = p_1C_1 + p_2C_2 + ... + p_rC_r$$

The above equation can be written as

$$p_1C_1 + p_2C_2 + ... + p_rC_r + 0 . C_{r+1} + ... + 0.C_n = B \quad ... (3)$$

Comparing (1) and (3), we get,

$$x_1 = p_1, x_2 = p_2, ..., x_r = p_r, x_{r+1} = ... = x_n = 0.$$

$\therefore$ $X = \begin{bmatrix} p_1 \\ p_2 \\ : \\ p_r \\ 0 \\ : \\ 0 \end{bmatrix}$ is a solution of AX = B.

$\therefore$ if ranks of A and [A B] are same, the equation AX = B has a solution.

Article 3 : The equation AX = B has a unique solution if A is non-singular.

Proof : (i) Assume that A is non-singular i.e., A^{-1} exists.

$\therefore$ from the equation AX = B, we have,

$$A^{-1}(AX) = A^{-1}B \text{ i.e., } X = A^{-1}B$$ which is a solution of AX = B.

(ii) We prove that the solution is unique.

If possible, let X_1, X_2 be two different solutions of AX = B

$\therefore$ $AX_1 = B$ and $AX_2 = B$

Consequently $AX_1 = AX_2$

$\Rightarrow$ $A^{-1}(AX_1) = A^{-1}(AX_2)$

$\Rightarrow$ $X_1 = X_2$

which is not possible as X_1 , X_2 are distinct.

$\therefore$ our supposition is wrong.

$\therefore$ Ax = B has a unique solution.

1.4.5 Problem

Problem 1 : Find the value of k so that the equation

x – 2y + z = 0, 3x – ty + 2z = 0, y + kx = 0 have

(i) unique solution

(ii) infinitely many solution. Also find solutions for these values of k.

Solution : The given equations ar

x – 2y + z = 0

3x – y + 2 = 0

0x + y + kz = 0

which can be written as

$$\begin{bmatrix} 1 & -2 & 1 \\ 3 & -1 & 2 \\ 0 & 1 & k \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ AX = O

Where $$A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -1 & 2 \\ 0 & 1 & k \end{bmatrix}$$

$$|A| = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -1 & 2 \\ 0 & 1 & k \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & -1 \\ 0 & 1 & k \end{bmatrix}, \text{by } R_2 \to R_2 - 3R_1$$

$$= 1\begin{vmatrix} 5 & -1 \\ 1 & k \end{vmatrix} = 1(5k+1) = 5k+1$$

(i) Equations have a unique solution

if $|A| \neq 0$

i.e. if $5k + 1 \neq 0$

i.e. if $k \neq -\frac{1}{5}$

(ii)System has infinitely many solutions

if $|A| = 0$

i.e. if $5k + 1 = 0$

i.e. if $k = -\frac{1}{5}$

When $k = -\frac{1}{5}$, we have

$$\begin{bmatrix} 1 & -2 & 1 \\ 3 & -1 & 2 \\ 0 & 1 & -\frac{1}{5} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{by } R_2 \to R_2 - 3R_1$$

$$\therefore \begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & -1 \\ 0 & 5 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{by } R_3 \to 5R_3$$

$$\therefore \begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{by } R_3 \to R_3 - R_2$$

$\therefore$ $x - 2y + z = 0$

$$5y - z = 0 \Rightarrow 5y = z \Rightarrow y = \frac{1}{5}z$$

$$\therefore \quad x - \frac{2}{5}z + z = 0 \Rightarrow x + \frac{3}{5}z = 0 \Rightarrow x = -\frac{3}{5}z$$

Put z = k

$\therefore$ solutions are $x = -\frac{3}{5}k, y = \frac{1}{5}k, z = k,$ where k is a parameter.

Problem 2 : Find non-trivial solution of the system of equations

$$x - 2y - 3z = 0$$

$$-2x + 3y + 5z = 0$$

$$3x + y - 2z = 0, \text{ if possible.}$$

Sol. The given equations are

$$x - 2y - 3z = 0$$

$$-2x + 3y + 5z = 0$$

$$3x + y - 2z = 0$$

which can be written as

$$\begin{bmatrix} 1 & -2 & -3 \\ -2 & 3 & 5 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 1 & -2 & -3 \\ 0 & -1 & -1 \\ 0 & 7 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_2 \to R_2 + 2R_1, R_3 \to R_3 - 3R_1$$

$$\therefore \quad \begin{bmatrix} 1 & -2 & -3 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{by } R_3 \to R_3 + 7R_2$$

$$\therefore \quad x - 2y - 3z = 0$$

$$-y - z = 0 \quad \Rightarrow y = -z$$

$$\therefore \quad x + 2z - 3z = 0 \;\Rightarrow x = z$$

Put z = k

$\therefore$ x = k, y = –k, z = k, where k is a parameter.

Problem 3 : Show that the system of equations

$$x + y + z = 4,\; 2x + 5y - 2z = 3,\; x + 7y - 7z = -6$$

is consistent and solve it.

Sol. The given equations are

$$x + y + z = 4$$

$$2x + 5y - 2z = 3$$

$$x + 7y - 7z = -6$$

which can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ -6 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -4 \\ 0 & 6 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ -10 \end{bmatrix}, \text{by } R_2 \to R_2 - 2R_1,\; R_3 \to R_3 - R_1$$

$$\therefore \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 0 \end{bmatrix}, \text{by } R_3 \to R_3 - 2R_2$$

Now rank of $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix}$ as well as of $\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is 2 .

$\therefore$ given equations are consistent and solutions are given by

$$x + y + z = 4.$$

$$3y - 4z = -5 \Rightarrow 3y = 4z - 5 \Rightarrow y = \frac{4}{3}z - \frac{5}{3}$$

$$\therefore \quad x + \frac{4}{3}x - \frac{5}{3} + z = 4 \Rightarrow x + \frac{7}{3}z = \frac{17}{3} \Rightarrow x = -\frac{7}{3}z + \frac{17}{3}$$

Put z = k

$\therefore$ solutions are $x = -\frac{7}{3}k + \frac{17}{3}, y = \frac{4}{3}k - \frac{5}{3}, z = k$

where k is a parameter.

Problem 4 : Investigate for what values of a, b the following equations

$$x + y + 5z = 6$$

$$x + 2y + 3az = b$$

$$x + 3y + ax = 1$$

have

1. no solution
2. unique solution
3. an infinite number of solutions.

Sol. The given equations are

$$x + y + 5z = 6$$

$$x + 2y + 3az = b$$

$$x + 3y + ax = 1$$

which can be written as

$$\begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 3a \\ 1 & 3 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ b \\ 1 \end{bmatrix}$$

i.e., $AX = B$ where $A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 3a \\ 1 & 3 & a \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 6 \\ b \\ 1 \end{bmatrix}$

The given equations will have a unique solution.

if $\begin{vmatrix} 1 & 1 & 5 \\ 1 & 2 & 3a \\ 1 & 3 & a \end{vmatrix} \neq 0$

i.e., if $\begin{vmatrix} 1 & 1 & 5 \\ 0 & 1 & 3a-5 \\ 0 & 2 & a-5 \end{vmatrix} \neq 0$, by $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

i.e., if $\begin{vmatrix} 1 & 1 & 5 \\ 0 & 1 & 3a-5 \\ 0 & 0 & -5a+5 \end{vmatrix} \neq 0$, by $R_3 \to R_3 - 2R_2$

i.e., if $-5a + 5 \neq 0$ i.e., if $a \neq 1$

$\therefore$ given equations will have a unique solution when $a \neq 1$ and b has any value When $a = 1$,

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & -2 \\ 0 & 2 & -4 \end{bmatrix}, \text{ by } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}, \text{ by } R_3 \to R_3 - 2R_2$$

$\therefore \rho(A) = 2$

$$[A\ B] = \begin{bmatrix} 1 & 1 & 5 & 6 \\ 1 & 2 & 3 & b \\ 1 & 3 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 5 & 6 \\ 0 & 1 & -2 & b-6 \\ 0 & 2 & -4 & -5 \end{bmatrix}, \text{by } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 5 & 6 \\ 0 & 1 & -2 & b-6 \\ 0 & 0 & 0 & -2b+7 \end{bmatrix}, \text{by } R_3 \to R_3 - 2R_2$$

Rank of [A B] is 3 if $b \neq \frac{7}{2}$

$\therefore$ rank of A and [A B] are not equal if $b \neq \frac{7}{2}$

$\therefore$ if a = 1, $b \neq \frac{7}{2}$, the given set of equations does not have any solution. If $a = 1, b = \frac{7}{2}$, then the ranks of A and [A B] are equal and A is singular.

$\therefore$ the given system of equations has an infinite number of solutions.

1.4.6 Summary

In this lesson, we have studied about the homogeneous AX = 0 and non-homogeneous system of linear equations AX = B and their solutions. We have studied various conditions under which a system can possess different types of solutions such as unique solution, no solutions and infinite many solutions. Moreover, more clarity of concept has been developed by using some simple examples.

1.4.7 Key Concepts

System of homogeneous linear equation, System of non-homogeneous linear equations, Linearly independent solutions, Consistency, Unique solution, No solution, Infinite many solutions,

1.4.8 Long Questions

1. Determine the value of λ so that the equations

$$2x + y + 2z = 0$$

$$x + y + 3z = 0$$

$$4x + 3y + \lambda z = 0$$

have non-zero solution.

2. Solve the following equations :

$$x + y + z = 0$$

$$x + 2y + 3z = 0$$

$$x + 3y + 4z = 0$$

3. Show that the equations

$$x + y + z + 3 = 0$$

$$3x + y - 2z + 2 = 0$$

$$2x + 4y + 7z - 7 = 0$$

are inconsistent.

1.4.9 Short Questions

1. For what value of λ, does the system $\begin{bmatrix} -1 & 2 & 1 \\ 3 & -1 & 2 \\ 0 & 1 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$ has (i) a unique solution (ii) more than one solution.

2. Solve the equations

$$x - y + z = 5$$

$$2x + y - z = -2$$

$$3x - y - z = -7$$

3. Examine the consistency of the following equations and if consistent, find the complete solution

$$4x - 2y + 6z = 8$$

$$x + y - 3z = -1$$

$$15x - 3y + 9z = 21$$

1.4.10 Suggested Readings

1. P. B. Bhattacharya, S. K. Jain & S. R. Nagpaul : A First Course in Linear Algebra, New Age International (P) Ltd.
2. Gilbert Strang : Linear Algebra and its Applications, Cengage Learning Publishers (Fourth Edition)

B.A. PART– I **MATHEMATICS : PAPER III**
(SEMESTER - I) **LINEAR ALGEBRA**

LESSON NO. 2.1 **AUTHOR : DR. CHANCHAL**

VECTOR SPACES-I

2.1.1 Objectives

With the help of this lesson, the students would be able to get knowledge about

- Vector space and its important results
- Subspace and sum of subspaces
- A special subspace called linear span

2.1.2 Vector Space (An Introduction)

Definition : Let $< V, + >$ be an abelian group and $<F, +, . >$ be a field. Define a function (called scalar multiplication) from $F \times V \to V$, s.t., for all $\alpha \in F$, $v \in V$, $\alpha . v \in V$. Then V is said to form a vector space over F if for all $x, y \in V$, $\alpha, \beta \in F$, the following hold

(i) $(\alpha + \beta)\, x = \alpha x + \beta x$

(ii) $\alpha\, (x + y) = \alpha x + \alpha y$

(iii) $(\alpha\beta)\, x = \alpha\, (\beta x)$

(iv) $1 . x = x$, 1 being unity of F.

Also then, members of F are called scalars and those of V are called vectors.

Remark : We have used the same symbol + for the two different binary compositions of V and F, for convenience. Similarly same symbol, is used for scalar multiplication and product of the field F.

Since < V, + > is group, its identity element is denoted by 0. Similarly the field F would also have zero element which will also be represented by 0.

Theorem 1 : In any vector space V(F) the following results hold

(i) $0.x = 0$

(ii) $\alpha.0 = 0$

(iii) $(-\alpha)\, x = -\,(\alpha x) = \alpha(-x)$

(iv) $(\alpha - \beta)x = \alpha x - \beta x,\ \alpha, \beta \in F,\ x \in V$

Proof : (i) $0.x = (0 + 0)\,.\,x = 0.x + 0.x$

$\Rightarrow$ $0 + 0.x = 0.x + 0.x$

$\Rightarrow$ $0 = 0.x$ (cancellation in V)

(ii) $\alpha.0 = \alpha.(0 + 0) = \alpha.0 + \alpha.0 \Rightarrow \alpha.0 = 0$

(iii) $(-\alpha)x + \alpha x = [(-\alpha) + \alpha]\, x = 0.x = 0$

$\Rightarrow$ $(-\alpha x) = -\alpha x$

(iv) follows from above.

Example 1 : Let < F, +, . > be a field

Let $V = \{(\alpha_1, \alpha_2) \mid \alpha_1, \alpha_2 \in F\}$

Define + and . (scalar multiplication) by

$(\alpha_1, \alpha_2) + (\beta_1, \beta_2) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2)$

$\alpha\, (\alpha_1\, .\, \alpha_2) = (\alpha\alpha_1, \alpha\alpha_2)$

One can check that all conditions in the definition are satisfied. Here $V = F \times F = F^2$. One can extend this to F^3 and so on. In general we can take n-tuple $(\alpha_1, \alpha_2,\ldots,\alpha_n)$; $\alpha_i \in F$ and define F^n or $F^{(n)} = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) \mid \alpha_i \in F\}$ as a Vector space over F.

Example 2 : Let V = set of all real valued continuous functions defined on [0, 1]. Then V forms a vector space over the field R of reals under addition and scalar multiplication defined by

$(f + g)\, x = f(x) + g(x)\ f, g \in V$

$(\alpha\, f)\, x = \alpha f(x) \quad \alpha \in R$

for all $x \in [0, 1]$

It may be recalled here that sum of two continuous functions is continuous and scalar multiple of a continuous function is continuous.

Example 3 : The set F [x] of all polynomials over a field F in an indeterminate x forms a vector space over F w.r.t, the usual addition of polynomials and the scalar multiplication defined by:

For $f(x) = a_0 + a_1x + \ldots + a_nx^n \in F[x],\ \alpha \in F$

$\alpha\, .\, (f\,(x)) = \alpha a_0 + \alpha a_1 x + \ldots + \alpha a_n x^n.$

Example 4 : $M_{m\times n}(F)$, the set of all m × n matrices with entries from a field F forms a vector space under addition and scalar multiplication of matrices.

We use the notation $M_n(F)$ for $M_{n\times n}(F)$.

Example 5 : Let F be a field and X a non empty set.

Let $F^X = \{f | f: X \to F\}$, the set of all mappings from X to F. Then F^X forms a vector space over F under addition and scalar multiplication defined as follows:

For $f, g \in F^X; \alpha \in F$

Define $f + g : X \to F$, $\alpha F : X \to, F$ such that

$$(f + g)(x) = f(x) + g(x)$$

$$(\alpha f)(x) = \alpha f(x) \qquad \forall x \in X$$

Example 6 : Let V be the set of all vectors in three dimensional space. Addition in V is taken as the usual addition of vectors in geometry and scalar multiplication is defined as :

$\alpha \in R$, $\vec{v} \in V \Rightarrow \alpha\vec{v}$ is a vector in V with magnitude $|\alpha|$ times that of V. Then V forms a vector space over R.

2.1.3 Subspaces

Definition : A non empty subset W of a vector space V(F) is said to form a subspace of V if W forms a vector space under the operations of V.

Theorem 2 : A necessary and sufficient condition for a non empty subset W of a vector space V(F) to be a subspace is that W is closed under addition and scalar multiplication.

Proof : If W is a subspace, the result follows by definition.

Conversely, let W be closed under addition and scalar multiplication

Let $x, y, \in W$ since $| \in F, -1 \in F$

$\therefore -1, y \in W \Rightarrow -y \in W$

$x, -y \in W \Rightarrow x - y \in W$

$\Rightarrow < W, + >$ forms a subgroup of $< V, + >$.

Rest of the conditions in the definition follow trivially.

Theorem 3 : A non empty subset W of a vector space V(F) is a subspace of V if $\alpha x + \beta y \in W$ for $\alpha, \beta \in F$, $x, y \in W$.

Proof : If W is a subspace, result follows by definition.

Conversely, let given condition hold in W.

Let $x, y \in W$ be any elements. Since $I \in F$

$I . x + I. y = x + y \in W$

$\Rightarrow$ W is closed under addition.

Again, $x \in W$, $\alpha \in F$ then

$\alpha x = \alpha x + 0.y$ for any $y \in W$, $0 \in F$

which is in W. (Note here 0 may not be in W)

Hence W is closed under scalar multiplication.

The result thus follows by previous theorem.

Remark : V and {0} will be trivial subspaces of any vector space V(F).

Example 1 : Consider the vector space $R^2(R)$

then $W_1 = \{(a, 0) \mid a \in R\}$

$W_2 = \{(0, b) \mid b \in R\}$

are subspaces of R^2

As for any $\alpha, \beta \in R$, $(a_1, 0), (a_2, 0) \in W_1$, we find

$$\alpha (a_1, 0) + \beta (a_2, 0) = (\alpha a_1, 0) + (\beta a_2, 0)$$
$$= (\alpha a_1 + \beta a_2, 0) \in W_1.$$

Hence W_1 is a subspace. Similarly we can show W_2 is a subspace of R^2.

Problem 1 : Show that union of two subspaces may not be a subspace.

Solution : Consider the previous example.

$W_1 \cup W_2$ will be the set containing all pairs of the type (a, 0), (0, b)

In particular $(1, 0), (0, 1) \in W_1 \cup W_2$

But $(1, 0) + (0, 1) = (1, 1) \notin W_1 \cup W_2$.

Hence $W_1 \cup W_2$ is not a subspace.

We take up few more examples of subspaces.

Example 2 : Let V = R [x] and suppose $W = \{f(x) \in V \mid f(x) = f(1 - x)\}$

Then W is a subspace of V as

$W \neq \phi$ since $0 \in W$ as $f(x) = 0 = f(1 - x)$

Again, if $f(x), g(x) \in W$, then $f(x) = f(1 - x)$, $g(x) = g(1 - x)$

Let $f(x) + g(x) = h(x)$

Then $h(1 - x) = f(1 - x) + g(1 - x)$

$= f(x) + g(x) = h(x)$

$\Rightarrow$ $h(x) \in W$ or that $f(x) + g(x) \in W$

Again, for $\alpha \in R$, let $\alpha f(x) = r(x)$

Then $r(1 - x) = \alpha f(1 - x) = \alpha f(x) = r(x)$

$\Rightarrow$ $r(x) \in W \Rightarrow \alpha f(x) \in W$

Hence W is a subspace.

Example 3 : Let $V = F^X$ (see example 7) and suppose $Y \subseteq X$

Then $W = \{f \in V \mid f(y) = 0 \ \forall\, y \in y\}$ is a subspace of V

Clearly $0 \in W$ and for $f, g \in W$, $f(y) = 0 = g(y) \quad \forall\, y \in Y$

So $(f + g)(y) = f(y) + g(y) = 0 \quad \forall\, y \in Y$

$\Rightarrow$ $f + g \in W$

Again, if $\alpha \in F$, then $(\alpha f)\, y = \alpha(f(y)) = 0 \quad \forall\, y \in Y$

$\Rightarrow$ $\alpha f \in W$.

Example 4 : If $V = R^n$, then

$W = \{x_1, x_2, ..., x_n) \mid x_1 + x_2 + ... + x_n = 1\}$ will not be subspace of V.

Notice, $(1, 0, 0, ..., 0) + (0, 1, 0, ..., 0) = (1, 1, 0, ..., 0) \notin W$.

Example 5 : Let $V = M_{2\times 1}(F)$. Let A be a 2×2 matrix over F.

Then $W = \left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in V \mid A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0\right\}$ forms a subspace of V

$$W \neq \phi \text{ as } \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in W$$

For $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ in W, we have

$$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 = A\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\Rightarrow A\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = 0$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in W$$

Also $A\left(\alpha\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \alpha A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \alpha\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in W$

Hence W is a subspace of V.

2.1.4 Sum of Subspaces

If W_1 and W_2 be two subspaces of a vector space V(F) then we define

$$W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_1, w_2 \in W_2\}$$

$$W_1 + W_2 \neq \varphi \text{ as } 0 = 0 + 0 \in W_1 + W_2$$

Again, $x, y \in W_1 + W_2$, $\alpha, \beta \in F$ implies

$$x = w_1 + w_2$$

$$y = w_1' + w_2' \qquad w_1, w_1' \in W_1, w_2, w_2' \in W_2$$

$$\alpha x + \beta y = \alpha(w_1 + w_2) + \beta(w_1' + w_2')$$

$$= \left(\alpha\omega_1 + \beta w_1'\right) + \left(\alpha w_2 + \beta w_2'\right) \in W_1 + W_2$$

Showing thereby that sum of two subspaces is a subspace.
One can extend the definition, similarly, to the sum of n subspaces W_1, W_2,...,W_n, which would also be a subspace and we write $W_1 + W_2 + \ldots + W_n = \sum_{i=1}^{n} W_i$.

Definition : Let W_1, W_2,..., W_n be subspaces of V then $W_1 + W_2 + \ldots + W_n$ is called the direct sum if each $x \in W_1 + W_2 + \ldots + W_n$ can be expressed uniquely as $x = w_1 + w_2 + \ldots + w_n, w_i \in W_i$ and in that case we write

$$W_1 + W_2 + \ldots + W_n = W_1 \oplus W_2 \oplus \ldots \oplus W_n$$

We say, a vector space V is the direct sum of its subspaces W_1, W_2,..., W_n if $V = W_1 \oplus W_2 \oplus \ldots \oplus W_n$, i.e., if

$$V = W_1 + W_2 + \ldots + W_n$$

and each $v \in V$ can be expressed uniquely as $v = w_1 + w_2 + \ldots + w_n$, $w_i \in W_i$.

Theorem 4 : $V = W_1 \oplus W_2 \Leftrightarrow V = W_1 + W_2 W_1 \cap W_2 = (0)$.

Proof : Let $V = W_1 \oplus W_2$

We need to prove $W_1 \cap W_2 = (0)$
Let $x \in W_1 \cap W_2$, then $x \in W_1$ and $x \in W_2$

$\Rightarrow \quad x = 0 + x \in W_1 + W_2 = V$

$\Rightarrow \quad x = x + 0 \in W_1 + W_2 = V$

Since x has been expressed as x = x + 0 and 0 + x and the representation has to be unique, we get x = 0

$\Rightarrow \quad W_1 \cap W_2 = (0)$.

Conversely, let $v \in V$ be any element and suppose

$$v = w_1 + w_2$$

$$v = w_1' + w_2'$$

are two representation of v

then $\quad w_1 + w_2 = w_1' + w_2' (= v)$

$$\Rightarrow w_1 - w_1' = w_2' - w_2$$

Now L.H.S. is in W_1 and R.H.S. belongs to W_2
i.e., each belongs to $W_1 \cap W_2 = (0)$

$$\Rightarrow \quad w_1 = w_1' = w_2' - w_2 = 0$$

$$\Rightarrow \quad w_1 = w_1', w_2 = w_2'.$$

Hence the result.

Remark: The above theorem can also be stated as

$$W_1 + W_2 = W_1 \oplus W_2 \Leftrightarrow W_1 \cap W_2 = \{0\}.$$

If W be a subspace of a vector space V(F) then since <W, +> forms an abelian group of < V, + >, we can talk of cosets of W in V. Let $\frac{V}{W}$ be the set of all cosets W + v, v $\in$ V, then we show that $\frac{V}{W}$ also forms a vector space over F, under the operations defined by

$$(W + x) + (W + y) = W + (x + y) \qquad x, y \in V$$
$$\alpha (W + x) = W + \alpha x \qquad \alpha \in F$$

Addition is well defined, since,

$$W + x = W + x'$$
$$W + y = W + y'$$
$$\Rightarrow \quad x - x' \in W, y - y' \in W$$
$$\Rightarrow \quad (x - x') + (y - y') \in W$$
$$\Rightarrow \quad (x + y) - (x' + y') \in W$$
$$\Rightarrow \quad W + (x + y) = W + (x' + y')$$

Again, $W + x = W + x'$

$$\Rightarrow \quad x - x' \in W$$
$$\Rightarrow \quad \alpha(x - x') \in W \qquad \alpha \in F$$
$$\Rightarrow \quad \alpha x - \alpha x' \in W$$
$$\Rightarrow \quad W + \alpha x = W + \alpha x'$$
$$\Rightarrow \quad \alpha(W + x) = \alpha(W + x')$$

Thus, scalar multiplication is also well defined. It should now be a routine exercise to check that all conditions in the definition of a vector space are satisfied.

$W + 0$ will be zero of $\frac{V}{W}$

W – x will be inverse of W + x

2.1.5 Linear Span

Definition : Let V(F) be a vector space, $v_i \in V, \alpha_i \in F$ be elements of V and F respectively. Then elements of the type $\sum_{i=1}^{n} \alpha_i v_i$ are called linear combinations of $v_1, v_2, \ldots, v_n$ over F.

Let S be a non empty subset of V, then the set

$$L(S) = \left\{ \sum_{i=1}^{n} \alpha_i v_i \mid \alpha_i \in F, v_i \in S, \text{n finite} \right\}$$

i.e., the set of all linear combinations of finite sets of elements of S is called linear span of S. It is also denoted by <S>. If S = φ, define L(S) = {0}.

Problem 1 : Let S = {(1, 4), (0, 3)} be a subset of $R^2(R)$. Show that (2, 3) belongs to L(S).

Solution: (2, 3) ∈ L(S) if it can be put as a linear combination of (1, 4) and (0, 3).

Now $(2, 3) = \alpha(1,4) + \beta(0,3)$

$$\Rightarrow (2,3) = (\alpha + 0, 4\alpha + 3\beta)$$

$$\Rightarrow 2 = \alpha,\ 4\alpha + 3\beta = 3$$

$$\Rightarrow \alpha = 2,\ \beta = -\frac{5}{3}$$

Hence $(2,3) = 2\,(1,4) - \frac{5}{3}(0,3)$

Showing that $(2,3) \in L(S)$.

Problem 2 : Let V = R^4(R0 and let S = {(2, 0, 0, 1), (–1, 0, 1, 0)}. Find L (S).

Solution : Any element $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in L(S)$ is a linear combination of members of S.

Let $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \alpha(2,0,01) + \beta(-1,0,1,0),\ \alpha, \beta \in R$

then $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (2\alpha - \beta, 0, \beta, \alpha)$

i.e., $L(S) = \{(2\alpha - \beta, 0, \beta, \alpha) \mid \alpha, \beta \in R\}$

Problem 3 : Show that the vector F[x] is not finite dimensional.

Solution : Let V = F[x] and suppose it is finite dimensional.

Then $\exists S \subseteq V$, s.t., V = L(S0 and S is finite.

Suppose S = $\{p_1, p_2, \ldots, p_k\}$. We can assume $p_1 \neq 0 \quad \forall i$

Let deg $p_i = r_i$ and let $t = \text{Max}\{r_1, r_2, \ldots, r_k\}$

Now $x^{r+1} \in V$ and since V = L(S),

$$x^{r+1} = \alpha_1 p_1 + \alpha_2 p_2 + \ldots + \alpha_k p_k, \alpha_i \in F$$

So $$0 = (-1)\, x^{t+1} + \alpha_1 p_1 + \ldots + \alpha_k p_k$$

Since x^{r+1} does not appear in $p_1, p_2, \ldots, p_k$

we get –1 = 0, a contradiction. Hence V is not FDVS over F.

Note if S = $\{1, x, \ldots, x^n, \ldots\}$ then V = L(S).

Theorem 5 : L(S) is the smallest subspace of V, containing S.

Proof : L(S) $\neq \varphi$ as $v \in S \Rightarrow v = 1 \,.\, v,\ 1 \in F$

$$\Rightarrow v \in L\,(S)$$

thus, in fact, $S \subseteq L(S)$.

Let $x, y \in L(S), \alpha, \beta \in F$ be any elements

then $x = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$

$$y = \beta_1 v_1' + \beta_2 v_2' + \ldots + \beta_m v_m' \quad v_i, v_j' \in S, \alpha_i, \beta_j \in F$$

Thus $\alpha x + \beta y = \alpha\alpha_1 v_1 + \alpha\alpha_2 v_2 + \ldots + \alpha\alpha_n v_n + \beta\beta_1 v_1' + \ldots + \beta\beta_m . v_m'$.

R.H.S. being a linear combination belongs to L(S)

Hence L(S) is a subspace of V, containing S.

Let now W be any subspace of V, containing S

We show $L(S) \subseteq W$

$x \in L(S) \Rightarrow x' = \sum \alpha_i v_i \quad v_i \in S, \alpha_j \in F$

$v_i \in S \subseteq W$ for all i and W is a subspace

$\Rightarrow \sum \alpha_i v_i \in W \Rightarrow x \in W$

$\Rightarrow L(S) \subseteq W$

Hence the result follows.

Theorem 6 : If W is a subspace of V, then L(W) = W and conversely.

Proof : $W \subseteq L(W)$ by definition and sine L(W) is the smallest subspace of V containing W and W is itself a subspace

$L(W) \subseteq W$

Henc $L(W) = W$.

Conversely, let L(W) = W

Let $x, y \in W, \alpha, \beta \in F$

Then $x, y \in L(W)$

$\Rightarrow$ x, y are linear combinations of members of W.

$\Rightarrow$ $\alpha x + \beta y$ is a linear combination of members of W

$\Rightarrow$ $\alpha x + \beta y \in L(W)$

$\Rightarrow$ $\alpha x + \beta y \in W$

$\Rightarrow$ W is a subspace.

Definition : If V = L(S), we say S spans (or generates) V. The vector space V is said to be finite-dimensional (over F) if there exists a finite subset S of V such that V = L (S). We use notation F.D.V.S. for a finite dimensional vector space.

It now follows from the results we've proved that

If S_1 and S_2 are two subspaces of V, then $S_1 + S_2$ is the subspace spanned by $S_1 \cup S_2$

Indeed, $L(S_1 \cup S_2) = L(S_1) + L(S_2) = S_1 + S_2$.

2.1.6 Summary

In this lesson, we have studied about an important mathematical structure known as vector space and the definition is made more clear using simple examples. Further, the idea of subspace of a vector space, sum of subspaces, direct sum of subspaces, union of subspaces has been elaborated alongwith various results. An interesting subspace called linear span is also introduced and given proofs of its concerning theorems.

2.1.7 Key Concepts

Group, Field, Scalar multiplication, Vector space, Subspace, Direct sum, Linear combination, Linear Span, Finite dimensional vector space.

2.1.8 Long Questions

1. Let $\alpha_1 = (1, 1, -2, 1)$, $\alpha_2 = (3, 0, 4, -1)$, $\alpha_3 = (-1, 2, 5, 2)$. Show that the vector (4, –5, 9, –7) is spanned by $\alpha_1, \alpha_2, \alpha_3$.

2. If S spans V then show that every super set of S spans V.

2.1.9 Short Questions

1. Define vector space with suitable example.

2. Discuss the concept of subspace. Give an example to show that union of two subspaces may be a subspace.

3. What is the difference between sum and direct sum of subspaces.

4. Define Linear span.

2.1.10 Suggested Readings

1. P. B. Bhattacharya, S. K. Jain & S. R. Nagpaul : A First Course in Linear Algebra, New Age International (P) Ltd.

2. Gilbert Strang : Linear Algebra and its Applications, Cengage Learning Publishers (Fourth Edition)

B.A. PART– I **MATHEMATICS : PAPER III**
(SEMESTER - I) **LINEAR ALGEBRA**

LESSON NO. 2.2 **AUTHOR : DR. CHANCHAL**

VECTOR SPACES-II

2.2.1 Objectives

With the help of this lesson, the students would be able to get knowledge about

- Linear dependence / independence of vectors
- Basis of a vector space
- Dimension of a vector space

2.2.2 Linear Dependence and Independence

Let V(F) be a vector space. Elements $v_1, v_2, \ldots, v_n$ in V are said to be linearly dependent (over F) if $\exists$ scalars $\alpha_1, \alpha_2, \ldots \alpha_n \in F$, (not all zero) such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n = 0$$

($v_1, v_2, \ldots, v_n$ are finite in number, not essentially distinct).

Thus for linear dependence $\sum \alpha_i v_i = 0$ and at least one $\alpha_i \neq 0$.

If $v_1, v_2 \ldots v_n$ are not linearly dependent (L.D.) these are called linearly independent (L.I.).

In other words, $v_1, v_2, \ldots, v_n$ are L.I. if

$$\sum \alpha_i v_i = 0 \Rightarrow \alpha_i = 0 \text{ for all i}$$

A finite set $X = \{x_1, x_2 \ldots, x_n\}$ is said to be L.D. or L.I. according as its n members are L.D. or L.I.

In general anysubset Y of V (F) is called L.I. if every finite non empty subset of Y is L.I. otherwise it is called L.D.

So, if some subsets are L.I. and some are L.D. then Y is called L.D.

Observations : (i) A non zero vector is always L.I. as $v \neq 0$, $\alpha v = 0$ would mean $\alpha = 0$.

(ii) Zero vector is always L.D.

$$1.\,0 = 0 \qquad 1 \neq 0,\ 1 \in F$$

Thus any collection of vectors to which zero belongs is always L.D.

In other words, if $v_1, v_2, \ldots, v_n$ are L.I. then none of these can be zero.

(iii) v is L.I. iff $v \neq 0$.

(iv) Any subset of a L.I. set is L.I.

(v) Any super set of a L.D. set is L.D.

(vi) Empty set φ is L.I. since it has no non empty finite subset and consequently it satisfies the condition for linear independence.

(vii) A set of vector is L.I. if and only if every finite subset of it is L.I.

Example 1 : Consider $R^2(R)$, R = reals.

$$v_1 = (1,0),\ v_2 = (0,1) \in R^2 \text{ are L.I.}$$

as

$$\alpha_1 v_1 + \alpha_2 v_2 = 0 \text{ for } \alpha_1, \alpha_2 \in R$$

$$\Rightarrow \alpha_1 (1, 0) + \alpha_2 (0, 1) = (0, 0)$$

$$\Rightarrow (\alpha_1, \alpha_2) = (0, 0) \Rightarrow \alpha_1 = \alpha_2 = 0.$$

Example 2 : Consider the subset

$$S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (2, 3, 4)\}$$

in the vector space $R^3(R)$

Since $2\,(1, 0, 0) + 3\,(0, 1, 0) + 4\,(0, 0, 1) - 1\,(2, 3, 4) = (0, 0, 0)$

we find S is L.D.

Example 3 : In the vector space F[x] of polynomials the vectors $f(x) = 1 - x$, $g(x) = x - x^2$, $h(x) = 1 - x^2$ are L.D. since $f(x) + g(x) - h(x) = 0$.

Problem 1 : Show that the vectors $v_1 = (0, 1, -2)$ $v_2 = (1, -1, 1)$, $v_3 = (1, 2, 1)$ are L.I. in $R^3(R)$.

Solution : Let $\sum \alpha_i v_i = 0$ for $\alpha_i \in R$

Then

$$\alpha_1 (0,1,-2) + \alpha_2 (1,-1,1) + \alpha_3 (1,2,1) = (0,0,0)$$

$$\Rightarrow (0, \alpha_1, -2\alpha_1) + (\alpha_2, -\alpha_2, \alpha_2) + (\alpha_3, 2\alpha_3, \alpha_3) = (0,0,0)$$

$$\Rightarrow 0 + \alpha_2 + \alpha_3 = 0$$

$$\alpha_1 - \alpha_2 + 2\alpha_3 = 0$$

$$-2\alpha_1 + \alpha_2 + \alpha_3 = 0$$

Since the coefficient determinant $\begin{vmatrix} 0 & 1 & 1 \\ 1 & -1 & 2 \\ -2 & 1 & 1 \end{vmatrix}$ is $-6 \neq 0$ the above equations have only the zero common solution

$$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0 \Rightarrow v_1, v_2, v_3 \text{ are L.I.}$$

Problem 2 : Show that {f(x), g(x), h(x)} is L.I. in F[x], whenever. deg f(x), deg g(x), deg h(x) are distinct.

Solution : Let $f(x) = a_0 + a_1x + ... + a_mx^m, a_m \neq 0$

$g(x) = b_0 + b_1x + ... + b_nx^n, b_n \neq 0$

$h(x) = c_0 + c_1x + ... + c_rx^r, c_1 \neq 0$

Let $\alpha f(x) + \beta g(x) + \gamma h(x) = 0, \alpha, \beta, \gamma \in F$

Let m < n < t (without any loss of generality)

then $\gamma C_t = 0 \Rightarrow \gamma = 0$ as $c_t \neq 0$

$\therefore$ $\alpha f(x) + \beta g(x) = 0$

and so $\beta b_n = 0 \Rightarrow \beta = 0$ as $b_n \neq 0$

$\Rightarrow \alpha f(x) = 0 \Rightarrow \alpha\, a_m = 0 \Rightarrow \alpha = 0$ as $a_m \neq 0$

Hence {f(x), g(x), h(x)} is L.I. in F[x] over F.

Problem 3 : Show that the vectors

v_1 = (1, 1, 2, 4), v_2 = (2, –1, –5, 2), v_3 = (1, –1, –4, 0) and v_4 = (2, 1, 1, 6) are L.D. in R^4 (R).

Solution : Suppose $av_1 + bv_2 + cv_3 + dv_4 = 0, a, b, c, d \in R$

then a(1, 1, 2, 4) + b (2, –1, –5, 2)
+ c (1, –1, –4, 0) + d (2, 1, 1, 6) = (0, 0, 0, 0)

or (a, a, 2a, 4a) + (2b, –b, –5b, 2b) + (c, –c, –4c, 0)
+ (2d, d, d, 6d) = (0, 0, 0, 0)

$\Rightarrow$ a + 2b + c + 2d = 0
a – b – c + d = 0
2a – 5b – 4c + d = 0
4a + 2b + 0c + 6d = 0

$$\Rightarrow \quad \begin{vmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{vmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1, R_4 \to R_4 - 4R_1$$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & -3 & -2 & -1 \\ 0 & -3 & -2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_4 \to \frac{1}{2}R_4, R_3 \to \frac{1}{3}R_3$$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & -1 & -2/3 & -1/3 \\ 0 & -3/4 & -1 & -1/2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_4 \to R_4 - R_2, R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow a + 2b + c + 2d = 0$

$-3b - 2c + d = 0$

$3b + 2c + d = 0$

$a = -1$, $b = -1$, $c = 1$, $d = 1$ satisfy the equations.

Since coefficients are non zero, the given vectors are L.D.

Problem 4 : Show that

(i) $\{1, \sqrt{2}\}$ is L.I in R over Q.

(ii) $\{1, \sqrt{2}, \sqrt{3}\}$ is L.I in R over Q.

(iii) $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$ is L.I in R over Q.

Solution : (i) Suppose $a + b\sqrt{2} = 0, \quad a, b \in Q$

Suppose $b \neq 0$, then $\sqrt{2} = -\frac{a}{b} \in Q$, a contradiction

Hence b = 0 and so a = 0. Thus $\{1, \sqrt{2}\}$ is L.I. in R over Q.

(ii) Let $a + b\sqrt{2} + c\sqrt{3} = 0$, a, b, c $\in$ Q

Let c $\neq$ 0, then

$$\sqrt{3} = -\frac{a}{c} - \frac{b}{c}\sqrt{2} = \alpha + \beta\sqrt{2}, \alpha\beta \in Q$$

$$\Rightarrow 3 = \alpha^2 + \alpha\beta^2 + 2\alpha\beta\sqrt{2}$$

$$\Rightarrow \alpha\beta\sqrt{2} \in Q \Rightarrow \alpha\beta = 0$$

Let α = 0 then $\beta = \sqrt{\frac{3}{2}}$, a contradiction

So, c = 0 giving $a + b\sqrt{3} = 0 \Rightarrow a = b = 0$ by (i)

Hence the result follows.

(iii) Let $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} = 0,$ a, b, c, d $\in$ Q

Then $\left(a + b\sqrt{2}\right) + \sqrt{3}\left(c + d\sqrt{2}\right) = 0$

Let $c + d\sqrt{2} \neq 0$

Then $\sqrt{3} = \frac{-\left(a + b\sqrt{2}\right)}{\left(c + d\sqrt{2}\right)} = \frac{-\left(a + b\sqrt{2}\right)\left(c - d\sqrt{2}\right)}{c^2 - 2d^2}$

$$= \alpha + \beta\sqrt{2}, \qquad \alpha, \beta \in Q$$

$\Rightarrow \alpha.1+\beta\sqrt{2}+(-1)\sqrt{3}=0$

$\Rightarrow -1 = 0$ by (ii), a contradiction

$\therefore c+d\sqrt{2}=0 \Rightarrow c=d=0 \Rightarrow a+b\sqrt{2}=0$

$\Rightarrow a = b = 0$

Hence the result follows.

Problem 5 : If two vectors are L.D. then one of them is scalar multiple of the other.

Solution : Suppose v_1, v_2 are L.D. then $\exists\ \alpha_i \in F$, s.t.,

$\alpha_1 v_1 + \alpha_2 v_2 = 0$ for some $\alpha_i \neq 0$

without loss of generality we can take $\alpha_1 \neq 0$, then α_1^{-1} exists and $\alpha_1 v_1 = (-\alpha_2 v_2)$

$\Rightarrow v_1 = \left(-\alpha_1^{-1}\alpha_2\right)v_2 = \beta v_2$

which proves the result.

Probelm 6 : If x, y, z are L.I. over the find C of complex.nos. then so are x + y + z and z + x over C.

Solution : Suppose $\alpha_1(x+y)+\alpha_2(y+z)+\alpha_2(z+x)=0, \alpha_i \in C$

Then $(\alpha_1+\alpha_3)x+(\alpha_1+\alpha_2)y+(\alpha_2+\alpha_3)z=0$

$\alpha_1+\alpha_3=\alpha_1+\alpha_2=\alpha_2+\alpha_3=0$ as x, y, x are L.I.

Solving we find

$\alpha_1=\alpha_2=\alpha_3=0$

Hence the result.

2.2.3 Basis and Dimensions

Note : We have already showed that (1, 0) and (0, 1) are L.I. in $R^2(R)$. If $v=(a,b)\in R^2$ be any element then since (a, b) = a(1, 0), a, b $\in$ R

We find any element of R^2 can be written as a linear combination of {(1, 0), (0, 1)} = S

i.e., $v \in R^2 \Rightarrow v \in L(S)$

$\Rightarrow R^2 \subseteq L(S)$

But $L(S) \subseteq R^2$

i.e., $R^2 \subseteq L(S)$

or the S spans R^2.

Definition : Let V(F) be a vector space. A subset S of V is called a basis of V if S consists of L.I. elements (i.e., any finite number of elements in S are L.I.) and V = L(X), i.e, S spans V.

Therefore, S = {(1, 0),(0, 1)} is a basis of $R^2(R)$. It is rather easy to see then that {(1, 0, 0).(0, 0, 1)} will form basis of $R^3(R)$, and one can trivially extends this to $R^{(n)}$.

Again {(1, 1, 0), (1, 0, 0),(0, 1, 1)} also forms a basis of $R^3(R)$. (Show !) Thus a vector space may have more than one basis.

If the elements in a basis are written in a certain specific order, we call it ordered basis. Also {(1, 0).(0, 1)}{(1, 0, 0),(0, 1, 0),(0, 0, 1)} etc. are called standard basis of R^2, R^3 etc. Also ϕ is a basis for V = {0}.

Problem 7 : Show that the set S= {(1, 2, 1),(2, 1, 0),(1, -1, 2)} form a basis of $R^3(R)$.

Solution : Let

$$\alpha_1(1,2,1)+\alpha_2(2,1,0)+\alpha_3(1,-1,2)=(0,0,0,)\,\alpha_i \in R$$

$$\Rightarrow (\alpha_1+2\alpha_2+\alpha_3, 2\alpha_1+\alpha_2-\alpha_3, \alpha_1+0+2\alpha_3)=(0,0,0)$$

$$\Rightarrow \alpha_1+2\alpha_2+\alpha_3=0$$

$$2\alpha_1+\alpha_2-\alpha_3=0$$

$$\alpha_1+0+2\alpha_3=0$$

In matrix form, we get $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e. AX = 0

where $|A| = -9 \neq 0$

So A is a non singular matrix and thus AX = 0 has the unique zero solution $\alpha_1=\alpha_2=\alpha_3=0$.

Hence S is L.I. set

Again, to show that L(S) = R^3, let (a, b, c) $\in R^3$ be any element. We want that (a, b, c) = β_1 (1, 2, 1) + β_2 (2, 1, 0) + β_3 (1, -1, -2) for same $\beta_1,\beta_2,\beta_3 \in R$

i.e., we want some $\beta_1 \in R$ s.t., the equations

$$\beta_1+2b_2+\beta_3=a$$

$$2\beta_1 + \beta_2 - \beta_3 = b$$

$$\beta_1 + 0\beta_2 - 2\beta_3 = c$$

are satisfied i.e, in matrix form

$$AX = B \text{ where } A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Since $|A| = -9 \neq 0$, AX = B has a unique solution i.e, $\exists$ some β_i s.t., above equation are satisfied or that it is possible to express any (a, b, c) $\in R^3$ as a linear combination members of S . i.e, L(S) = R^3
Hence S forms a basis of $R^3(R)$.

Theorem 1 : If $S = \{v_1, v_2, ...v_n\}$ is basis of V, thenf every element of V can be expressed uniquely as a linear combination of $v_1, v_2, ...v_n$.

Proof : Since, by definition of basis, V = L(S), each element $v \in V$ can be expressed as linear combination of $v_1, v_2, ...v_n$.

Suppose $v = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n, \alpha_i \in F$

$v = \beta_1 v_1 + \beta_2 v_2 + ... + \beta_n v_n, \beta_i \in F$

then $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = \beta_1 v_1 + \beta_2 v_2 + ... + \beta_n v_n$

$$\Rightarrow (\alpha_1 - \beta_1) v_1 + (\alpha_2 - \beta_2) v_2 + ... + (\alpha_n - \beta_n) v_n = 0$$

$$\Rightarrow \alpha_i - \beta_i = 0 \text{ for all } i (v_1, v_2, ...v_n \text{are L.I.})$$

$$\Rightarrow \alpha_i = \beta_i \text{ for all i.}$$

Theorme 2 : Suppose S is a finite subset of a vector space V such that V = L(S) [i.e., V is F.D.V.S] then there exists a subset of S which is a basis of V.

Proof : If S consists of L.I. elements then S itself forms basis of V and we've nothing to prove.

Let now T be a subset of S, such that T spans V and T is such minimal subset of S. (Existence of T is ensured as S is finite).

Suppose $T = \{v_2, v_2, ..., v_n\}$

we show T is L.I.

Let $\sum \alpha_i v_i = 0, \alpha_i \in F$

Suppose $\alpha_i \neq 0$ for some i. Without any loss of generality we can take $\alpha_1 \neq 0$.

Then α_1^{-1} exists.

Now $\alpha_1 v_1 + \alpha_2 v_2 + \ldots.. + \alpha_n v_n = 0$

$$\Rightarrow a_1^{-1}\left(\alpha_1 v_1 + \alpha_2 v_2 + \ldots.. + \alpha_n v_n\right) = 0$$

$$\Rightarrow v_1 = \left(-\alpha_1^{-1}\alpha_2\right)v_2 + \left(-\alpha_1^{-1}\alpha_3\right)v_3 + \ldots + \left(-\alpha_1^{-1}\alpha_n\right)v_n$$

$$= \beta_2 v_2 + \beta_3 v_3 + \ldots\ldots + \beta_n v_n \quad \beta_i \in F$$

If $v \in V$ be any element then

$$v = \gamma_1 v_1 + \gamma_2 v_2 + \ldots.\gamma_n v_n \qquad \gamma_i \in F \quad \text{as } V = L(T)$$

$$\Rightarrow v = \gamma_1\left(\beta_2 v_2 + \ldots.. + \beta_n v_n\right) + \gamma_2 v_2 + \ldots.\gamma_n v_n$$

i.e., any element of V is a linear combination of v_2, v_3, ….v_n

$\Rightarrow \{v_2, v_3, \ldots.. v_n\}$ spans V, which contradicts our choice of T(as T was such minimal)

Hence $\alpha_1 = 0$

or that $\alpha_i = 0$ for all i

$$\Rightarrow v_1, v_2, \ldots.. v_n \text{ are L.I.}$$

and thus T is a basis V.

Cor : A F.D.V.S has a basis.

In fact, one can prove this result for any vector spece.(i.e. any vector space has a basis).

Theorem 3 : If V is a F.D.V.S and $\{v_1, v_2, \ldots.. v_r\}$ is a L.I. subset of V, then it can be extended to form a basis of V.

Proof : If $\{v_1, v_2, \ldots.. v_r\}$ spans V, then it self forms a basis of V and there is nothing to prove.

Let $S = \{v_1, v_2, \ldots.. v_r, v_{r+1}, \ldots\ldots.. v_n\}$ be the maximal L.I. subset of V, containing $\{v_1, v_2, \ldots.. v_r\}$.

We show S is a basis of V, for which it is enough to prove that S spans V.

Let $v \in V$ be any element

then $T = \{v_1, v_2, ...v_n, v\}$ is L.D. by choice of S

$\Rightarrow \exists\ \alpha_1, \alpha_2, ...\alpha_n, \alpha \in F$ (not all zero) such that

$$\alpha_1 v_1 + ... + \alpha_n v_n + \alpha v = 0$$

We claim $\alpha \neq 0$. Suppose $\alpha = 0$

then $\alpha_1 v_1 + ... + \alpha_n v_n = 0$

$\Rightarrow \alpha_1 = 0$ for all i as $v_1, v_2, ...v_n$ are L.I.

$\therefore \alpha = \alpha_i = 0$ for all i which is not truie.

Hence $\alpha \neq 0$ and so α^{-1} exists.

Since $v = \left(\alpha^{-1}\alpha_1\right)v_1 + \left(\alpha^{-1}\alpha_2\right)v_2 + ... + \left(\alpha^{-1}\alpha_n\right)v_n$

v is a linear combination of $v_1, v_2, ...v_n$

which proves our assertion.

Theorem 4 : If dim V = n and S = $S = \{v_1, v_2, ..., v_n\}$ spans V then S is a basis of V.

Proof : Since dim V = n, any basis of V has n elements. But theorem 8, a subset of S will be a basis of V but as S contains n elements, it will itself form basis of V.

Theorem 5 : If dim V = n and $S = \{v_1, v_2, ..., v_n\}$ is L.I. subset of V then S is a basis of V.

Proof : Since $\{v_1, v_2, ..., v_n\}$ = S is L.I. it can be extended to form a basis of V, but dim V being n it will itself be a basis of V.

Aliter : Let $v \in V$, then

$v, v_1, v_2, ...v_n$ will be L.D.. Thus $\exists \alpha, \alpha_1, \alpha_2, ..., \alpha_n \in F$ s.t.,

$\alpha v + \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = 0$

where some α_i or α is not zero.

If α = 0, then

$$\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = 0$$

$$\Rightarrow \alpha_i = 0 \forall i \text{ as } v_1, v_2, ..., v_n \text{ are L.I.}$$

Thus $\alpha \neq 0$ and so

$$v = \left(-\alpha^{-1}\alpha_1\right)v_1 + \ldots\ldots + \left(-\alpha^{-1}\alpha_n\right)v_n \in L(S)$$

$$\Rightarrow V \subseteq L(S)$$

$\Rightarrow V = L(S)$ and as S is L.I. S is a basis of V.

Problem 8 : If $\{v_1, v_2, \ldots, v_n\}$ is a basis of F.D.V.S V of dim n and v = $\sum \alpha_i v_i, \alpha_r \neq 0$ then prove that $\{v_1, v_2, \ldots, v_{r-1}, v, v_{r+1} \ldots, v_n\}$ is also a basis of V.

Solution : We have

$$v = \alpha_1 v_1 + \ldots + \alpha_r v_r + \ldots + \alpha_n v_n \quad \alpha_r \neq 0 \therefore a_r^{-1} \text{exists}$$

$$\Rightarrow v_1 = \left(\alpha_r^{-1}\alpha_1\right)v_1 + \ldots + \left(-\alpha_r^{-1}\alpha_{r-1}\right)v_{r-1} + \alpha_r^{-1}v + \ldots + \left(-\alpha_r^{-1}\alpha_n\right)v_n$$

$$\Rightarrow \beta_1 v_1 + \ldots + \beta_{r-r}v_{r-1} + \beta_r v + \beta_{r+1} + \ldots + \beta_n v_n$$

If $x \in V$ be any element, then

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n \quad \alpha_i \in F$$

$$\Rightarrow x = \alpha_1 v_1 + \ldots + \alpha_{r-1}v_{r-1} + \alpha_r\left(\beta_1 v_1 + \ldots + \beta_n v_n\right) + \ldots + a_n v_n$$

or that x is a linear combination of

$$v_1, \ldots, v_{r-1}, v, v_{r+1}, \ldots, v_n$$

and x being any element, we find V is spanned by $\{v_1, \ldots, v_{r-1}, v, v_{r+1}, \ldots v_n\}$ and it forms a basis of V, using theorem done above.

Theorem 6 : Two finite dimensional vector spaces over F are isomorphic iff they have the same dimension.

Proof : Let V and W be two isomorphic vector spaces over F and let $\theta : V \to W$ be the isomorphism.

Let dim V = n and $\{v_1, v_2, \ldots, v_n\}$ be a basis of V.

We claim $\{\theta(v_1), \theta(v_2), \ldots, \theta(v_n)\}$ is basis of W.

Now $\sum_{i=1}^{n} \alpha_i \theta(v_i) = 0 \quad \alpha_i \in F$

$$\Rightarrow \sum \theta(\alpha_i v_i) = 0 = \theta(0)$$

$$\Rightarrow \sum \alpha_i v_i = 0 \quad (\theta \text{ is } 1-1)$$

$$\Rightarrow \alpha_i = 0 \text{ for all i as } v_1, v_2, ..., v_n \text{ are L.I.}$$

$$\Rightarrow \theta(v_1), \theta(v_2), ..., \theta(v_n) \text{ are L.I.}$$

Again, we $w \in W$ is any element, then as θ is onto, $\exists$ some $v \in V$ s.t., $\theta(v) = w$

Now $\quad v \in V \Rightarrow v = \sum_{i=1}^{n} \alpha_i v_i$ for some $\alpha_i \in F$

$$\Rightarrow w = \theta(V) = \theta\left(\sum \alpha_i v_i\right)$$

$$\Rightarrow 'w = \sum \theta(\alpha_i v_i) = \alpha_i \theta(v_1) + \alpha_1 \theta(v_2) + ... + \alpha_n \theta(v_n)$$

or that w is a linear combination of $\theta(v_1), \theta(v_2), ..., \theta(v_n)$

Hence $\theta(v_1), \theta(v_2), ..., \theta(v_n)$ span W and therefore, form a basis of W showing that dim W = n.

Connersely, let dim V = dim W = n and suppose, $\{v_1, v_2, ..., v_n\}$ and $\{w_1, w_2, ..., w_n\}$ are basis of V and W respectively.

Define a map $\theta : V \to W$ s.t.,

$$\theta(v) = \theta(\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n)$$

$$\alpha_1 w_1 + \alpha_2 w_2 + ... + \alpha_n w_n$$

then θ is easily seen to be well defined. (Indeed any $v \in V$ is unique linear combination of members of basis).

If $v, v' \in V$ be any elements then

$$v = \sum \alpha_i v_i, v' = \sum \beta_i v_i \qquad \alpha_i, \beta_i \in F$$

$$\theta(v + v') = \theta\left(\sum \alpha_i v_i + \sum \beta_i v_i\right)$$

$$= \theta\left(\sum (\alpha_i \beta_i) v_i\right)$$

$$= \sum (\alpha_i + \beta_i) w_i$$

$$= \sum \alpha_i w_i + \sum \beta_i w_i = \theta(v) + \theta(v')$$

Also $\quad \theta(\alpha v) = \theta\left(\alpha \sum \alpha_i v_i\right) = \theta\left(\sum \alpha \alpha_i v_i\right) = \sum (\alpha \alpha_i) w_i$

$$\alpha\sum \alpha_i w_i = \alpha\theta(v)$$

Thus θ is a homomorphism

Now if $v \in \text{Ker } \theta$

then $\theta(v) = 0$

$\Rightarrow \theta\left(\sum \alpha_i v_i\right) = 0$

$\Rightarrow \sum \alpha_i w_i = 0$

$\Rightarrow \alpha_i = 0$ for all i $w_1, w_2, ..., w_n$ being L.I.

$\Rightarrow v = 0$

$\Rightarrow \text{Ker } \theta = \{0\}$

$\Rightarrow \theta$ is one-one.

That θ is onto is obvious. Hence θ is an isomorphism.

Problem 9 : (1, 1, 1) is L.I. vector in R^3 (R) Extend it to form a basis of R^3.

Solution : (1, 1, 1) is non zero vector and it therefore L.I. in R^3.

Let $S = \{(1, 1, 1)\}$, then $L(S) = \{\alpha(1, 1, 1)\alpha \in R\}$

Now $(1, 0, 0) \in R^3$, but $(1, 0, 0) \notin L(S)$

thus by above proble $S_1 = \{(1,1,1),(1,0,0)\}$ is L.I.

Now $L(S_1) = \{\alpha(1,1,1) + \beta(1,0,0) \mid \alpha, \beta \in R\}$

$= \{(\alpha + \beta, \alpha, -\alpha) - \alpha, \beta \in R\}$

Again $(0, 1, 0) \notin L(S_1)$ and by above problem

$S_2 = \{(1, 1, 1),(1, 0, 0),(0, 1, 0)\}$ is L.I. subset of R^3.

Since dim $R^3 = 3$, we find S_2 will be a basis of R^3.

Theorem 7 : Let W be a subspace of a F.D.V.S. V, then W is finite dimensional and dim W $\leq$ dim V. In fact, dim V = dim W iff V = W.

Proof : Let dim V = n, then n is maximum number of L.I. elements in any subset of V. Since any subset of W will be a subset of V, n is the maximum number of L.I. elements in W.

Let $w_1, w_2, ...,w_m$ be the maximum number of L.I. elements in W hten $m \leq n$.

We show $\{w_1, w_2, ...,w_m\}$ is basis of W. These are already L.I. If $w \in W$ be any element then the set $\{w_1, w_2, ...,w_m, w\}$ is L.D.

$\Rightarrow \exists\ \alpha_1, \alpha_2, ...\alpha_m, \alpha$ in F (not all zero) s.t.,

$\alpha_1 w_1 + ... + \alpha_m w_m + \alpha w = 0$

If $\alpha = 0$ we get $\alpha_i = 0$ for all i as $w_1, ..., w_m$ are L.I. which is not true. Thus $\alpha \neq 0$ and so α^{-1} exists.

The above equation then gives us

$$w = \left(-\alpha^{-1}\alpha_1\right)w_1 + ... + \left(-\alpha^{-1}\alpha_m\right)w_m$$

Showing that $\{w_1, w_2, ..., w_m\}$ spans W (and thus W is finite dimensional)

$$\Rightarrow \{w_1, w_2, ..., w_m\} \text{ is a basis of W}$$

$$\Rightarrow \dim W = m \leq n = \dim V$$

Finally, if dim V = dim W = n

and $\{w_1, w_2, ..., w_m\}$ be a basis of W then as $\{w_1, w_2, ..., w_n\}$ is L.I. in W it will be L.I. in V. and as dim V = n, $\{w_1, w_2, ..., w_n\}$ is a basis of V.

Now if $v \in V$ be any element then

$$\Rightarrow v = \alpha_1 w_1 + \alpha_2 w_2 + ... + \alpha_n w_n \in W$$

$$\Rightarrow V \subseteq W \Rightarrow V = W$$

Conversely, of course, V = W $\Rightarrow$ dim V = dim W.

Theorem 8 : Let W be a subspace of a F.D.V.S. V. Then

$$\dim \frac{V}{W} = \dim V - \dim W.$$

Proof : Let dim W = m and let $\{w_1, w_2, ..., w_m\}$ be a basis of W.

$w_1, w_2, ..., w_m$ being L.I. in W will be L.I. in V and thus $\{w_1, w_2, ..., w_m\}$ can be extended to form a basis of V.

Let $\{w_1, w_2, ..., w_m, v_1, v_2, ..., v_n\}$ be this extended basis of V.

Then dim V = n + m

Consider the set $S = \{w + v_1, w + v_2, ..., w + v_n\}$, we show if forms a basis of $\frac{V}{W}$.

Let $\alpha_1 (W + v_1) + ... + \alpha_n (W + v_n) = W, \alpha_i \in F$

Then $W + (\alpha_1 v_1 + ... + \alpha_n v_n) = W$

$$\Rightarrow \alpha_1 v_1 + ... + \alpha_n v_n \in W$$

$\Rightarrow \alpha_1 v_1 + ... + \alpha_n v_n$ is a linear combination of $w_1, ..., w_m$

$\Rightarrow \alpha_1 v_1 + ... + \alpha_n v_n = \beta_1 w_1 + ... + \beta_m w_m \ \beta_j \in F$

$\Rightarrow \alpha_1 v_1 + ... + \alpha_n v_n - \beta_1 w_1 - ..... - \beta_m w_m = 0$

$\Rightarrow \alpha_i = \beta_j = 0$ for all i, j.

$\Rightarrow \{W + v_1, W + v_2, ..., W + v_n\}$ is L.I.

Again, for any $W + v \in \frac{V}{W}, v \in V$ means v is a linear combination of $w_1, ..., w_m, v_1, ..., v_n$.

i.e., $v = \alpha_1 w_1 + ... + \alpha_m w_m + \beta_1 v_1 + ... + \beta_n v_n \ \alpha_i, \beta_j \in F$

giving $W + v = W + (\alpha_1 w_1 + ... + \alpha_m w_m) + (\beta_1 v_1 + ... + \beta_n v_n)$

$= W + (\beta_1 v_1 + ... + \beta_n v_n)$

$= (W + \beta_1 v_1) + ... + (W + \beta_n v_n)$

$= \beta_1 (W + v_1) + \beta_2 (W + v_2) + ... + \beta_2 (W + v_n)$.

Hence S spans $\frac{V}{W}$ and is therefore a basis.

$\therefore \quad \dim \frac{V}{M} = n$

Thus $\dim \frac{V}{W} = \dim V - \dim W$.

Theorem 9 : If A and B are twosubspaces of a F.D.V.S. V then

$$\dim (A + B) = \dim A + \dim B - \dim (A \cap B).$$

Proof : We'ave already proved that

$$\frac{A + B}{A} \cong \frac{B}{A \cap B}$$

$$\therefore \quad \dim \frac{A + B}{A} = \dim \frac{B}{A \cap B}$$

$$\Rightarrow \quad \dim (A + B) - \dim A = \dim B - \dim (A \cap B)$$

or that $\dim(A+B) = \dim A + \dim B - \dim(A \cap B)$.

Remark : The reader should try to give an independent proof of the above theorem as an exercise.

Cor. : If $A \cap B = (0)$ then dim (A + B) = dim A + dim B

i.e., $\dim (A \oplus B) = \dim A + \dim B$.

2.2.4 Summary

In this lesson, we have gained knowledge about the linear dependence / independence of vectors of a vector space. In continuation to this, we have defined basis of a vector space and elaborated the conditions to be satisfied by the basis. Various important results and theorems alongwith their proofs have been discussed in this lesson.

2.2.5 Key Concepts

Linearly dependent vectors, Linearly independent vectors, Basis, Ordered basis, Standard basis, Finite dimensional vector space, Isomorphic vector spaces.

2.2.6 Long Questions

1. Extend the set S = {(1, 1, 0)} to form two different bases of $R^3(R)$.

2. Let S be a finite subset of a vector space V such that S is L.I. and every proper superset of S in V is L.D. Show that S is a basis of V.

3. If W_1 and W_2 are subspaces of R^4 and {(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0)}, {(0, 0, 0, 1), (0, 0, 1, 1), (0, 1, 1, 1)} are bases of W_1 and W_2 respectively, find a basis of $W_1 \cap W_2$.

4. Let V be a vector space over F. Assume that every linearly independent set in oV can be extended to a basis of V. Deduce that V has a basis.

5. Let F be a field. Let A = {(x, y, 0) | x, y, ∈ F}, B = {(0, y, z) | y, z ∈ F} be subspaces of $F^3(F)$. Find dimension of the subspace A + B.

2.2.7 Short Questions

1. Show that the following vectors are L.I.

(i) (1, 0, 0), (1, 1, 1), (1, 2, 3), in $R^3(R)$

(ii) (1, 2, –1), (2, 2, 1), (1, –2, 3) in $R^3(R)$

2. Show that the following vectors are L.D.

(i) (1, 1, 2), (–3, 1, 0), (1, –1, 1), (1, 2, –3) in $R^3(R)$

(ii) (1, 1, 2), (1, 2, 5), (5, 3, 4) in $R^3(R)$

3. Show that {1, i} forms a basis of C (R).

2.2.8 Suggested Readings

1. P. B. Bhattacharya, S. K. Jain & S. R. Nagpaul : A First Course in Linear Algebra, New Age International (P) Ltd.

2. Gilbert Strang : Linear Algebra and its Applications, Cengage Learning Publishers (Fourth Edition)

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B.A. PART– I **MATHEMATICS : PAPER III**
(SEMESTER - I) **LINEAR ALGEBRA**

LESSON NO. 2.3 **AUTHOR : DR. CHANCHAL**

LINEAR TRANSFORMATIONS-I

2.3.1 Objectives

With the help of this lesson, the students would be able to get knowledge about

- Linear transformation and its important results
- Kernel of a Linear transformation called its Null space
- Fundamental theorem of homomorphism for vector spaces
- Rank and Nullity of a Linear transformation
- Sylvester's law

2.3.2 Linear Transformations (An Introduction)

Definition : Let V and U be two vector spaces over the same field F, then a mapping $T : V \to U$ is called a homomorphism or a linear transformation if

$$T(x + y) = T(x) + T(y) \text{ for all } x, y \in V$$
$$T(\alpha x) = \alpha T(x) \quad \alpha \in F$$

One can combine the two conditions to get a single condition

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \quad x, y \in V; \alpha, \beta \in F$$

It is easy to see that both are equivalent. If a homomorphism happens to be one-one onto also we call it an isomorphism, and say the two spaces are isomorphic. (Notation $V \cong U$).

Example 1 : Identity map $I : V \to V$, s.t.,

$$I(v) = v$$

and the zero map $O : V \to V$, s.t.,

$$O(v) = 0$$

are clearly linear transformations.

Example 2 : For a field F, consider the vector spaces F^2 and F^3. Define a map $T : F^3 \to F^2$, by

$$T(\alpha, \beta, \gamma) = (\alpha, \beta)$$

then T is a linear transformation as

for any $x, y \in F^3$, if $x = (\alpha_1, \beta_1, \gamma_1)$

$$y = (\alpha_2, \beta_2, \gamma_2)$$

then $T(x + y) = T(\alpha_1 + \alpha_2, \beta_1 + \beta_2, \gamma_1 + \gamma_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2)$

$$= (\alpha_1, \beta_1) + (\alpha_2, \beta_2) = T(x) + T(y)$$

and $T(\alpha x) = T(\alpha(\alpha_1, \beta_1, \gamma_1)) = T(\alpha\alpha_1, \alpha\beta_1, \alpha\gamma_1)$

$$= (\alpha\alpha_1, \alpha\beta_1) = \alpha(\alpha_1, \beta_1) = \alpha T(x)$$

Example 3 : Let V be the vector space of all polynomials in x over a field F. Define $T : V \to V$, s.t.,

$$T(f(x)) = \frac{d}{dx} f(x)$$

then $T(f + g) = \frac{d}{dx}(f + g) = \frac{d}{dx} f + \frac{d}{dx} g = T(f) + T(g)$

$$T(\alpha f) = \frac{d}{dx}(\alpha f) = \alpha \frac{d}{dx} f = \alpha T(f)$$

shows that T is a linear transformation.

In fact if $\theta : V \to V$ be defined such that

$$\theta(f) = \int_0^x f(t)\,dt$$

then θ will also be a linear transformation.

Example 4 : Consider the mapping

$$T : R^3 \to R, \text{ s.t.,}$$

$$T(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$$

then T is not a linear transformation.
Consider, for instance,

$$T((1,0,0)+(1,0,0)) = T(2,0,0) = 4$$

$$T(1,0,0)+T(1,0,0) = 1 \therefore 1 = 2.$$

In the following theorems, we take V and U to be vector spaces over the same field.

Theorem 1 : Under a homomorphism $T : V \to U$,

(i) T (0) = 0 (ii) T(–x) = – T(x).

Proof : T(0) = T(0 + 0) = T(0) + T(0)

$$\Rightarrow T(0) = 0$$

Again T(–x) + T(x) = T(–x + x) = T(0) = 0

$$\Rightarrow -T(x) = T(-x).$$

Definition : Let $T : V \to U$ be a homomorphism, then kernel of T is defined by

$$\text{Ker } T = \{x \in V \mid T(x) = 0\}$$

It is also called the null space of T.

Theorem 2 : Let $T : V \to U$ be a homomorphism, then Ker T is a subspace of V.

Proof : Ker $T \neq \varphi$, as $0 \in$ Ker T

Let $\alpha, \beta \in F$, $x, y \in$ Ker T be any elements

then $$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

$$= \alpha.0 + \beta.0 = 0 + 0 = 0$$

$$\Rightarrow \alpha x + \beta y \in \text{Ker } T.$$

Theorem 3 : Let $T : V \to U$ be a homomorphism, then

Ker T = {0} iff T is one-one.

Proof : Let Ker T = {0}. If T(x) = T(y)

then T(x) – T(y) = 0

$$\Rightarrow T(x - y) = 0$$

$\Rightarrow (x-y) \in \text{Ker } T = \{0\}$

$\Rightarrow x - y = 0$

$\Rightarrow x = y \Rightarrow T$ is $1-1$.

Conversely, let T be one-one
if $x \in$ Ker T be any element, then T(x) = 0

$\Rightarrow T(x) = T(0)$

$\Rightarrow x = 0$

$\Rightarrow \text{Ker } T = \{0\}$.

Definition : Let $T : V \rightarrow U$ be a linear transformation then range of T is defined to be

$$T(V) = \{T(x) \mid x \in V\} = \text{Range } T = R_T$$

$$= \{u \in U \mid u = T(v), v \in V\}$$

Theorem 4 : Let $T : V \rightarrow U$ be a L.T. (linear transformation) then range of T is subspace of U.

Proof : Since $T(0) = 0, 0 \in V$

$\therefore$ $T(0) \in \text{Range } T$

i.e., Range $T \neq \varphi$

Let $\alpha, \beta \in F, T(x), T(y) \in T(V)$ be any elements

then $x, y \in V$

Now $\alpha T(x) + \beta T(y) = T(\alpha x + \beta y) \in T(V)$

as $\alpha x + \beta y \in V$

Hence the result.

Note: T(V) = U iff T is onto.

Theorem 5 : Let $T : V \rightarrow U$ be a L.T. then

$$\frac{V}{\text{Ker } T} \cong \text{Range } T = T(V).$$

Proof : Let $T : V \rightarrow U$ and put Ker T = K, then K being a subspace of V, we can talk V/

K.

Define a mapping $\theta : V/K \to T(V)$, s.t.,

$$\theta(K + x) = T(x), x \in V$$

Then θ is well defined, one-one map as

$$K + x = K + y$$

$$\Leftrightarrow x - y \in K = \text{Ker } T$$

$$\Leftrightarrow T(x - y) = 0$$

$$\Leftrightarrow T(x) = T(y)$$

$$\Leftrightarrow \theta(K + x) = \theta(K + y)$$

If T (x) $\in$ T(V) be any element, then x $\in$ V and θ (K + x) = T(x), showing that θ is onto

Finally, θ ((K + x) + (K + y)) = θ (K + (x + y))

= T (x + y)

= T(x) + T(y)

$$= \theta(K + x) + \theta(K + y)$$

and $\theta(\alpha(K + x)) = \theta(K + \alpha x) = T(\alpha x) = \alpha T(x) = \alpha\theta(K + x)$

shows θ is a L.T. and hence an isomorphism.

Note : The above is called the Fundamental Theorem of homomorphism for vector space. If the map T is also onto, then we have proved $\frac{V}{\text{Ker } T} \cong U$.

Theorem 6 : If A and B be two subspaces of a vector space V(F), then

$$\frac{A + B}{A} \cong \frac{B}{A \cap B}.$$

Proof : A being a subspace of A + B and A $\cap$ B being a subsapce of B, we can talk of $\frac{A + B}{A}$ and $\frac{B}{A \cap B}$.

Define a map $\theta : B \to \frac{A + B}{A}$, s.t.,

$$\theta(b) = A + b, b \in B$$

Since $b_1 = b_2 \Rightarrow A + b_1 = A + b_2$, we find θ is well defined.

Again, as $\theta(\alpha b_1 + \beta b_2) = A + (\alpha b_1 + \beta b_2)$

$$= (A + \alpha b_1) + (A + \beta b_2)$$

$$= \alpha(A + b_1) + \beta(A + b_2)$$

$$= \alpha\theta(b_1) + \beta\theta(b_2)$$

θ is a L.T.

For any $A + x \in \frac{A+B}{A}$, we find $x \in A + B$

$$\Rightarrow x = a + b, a \in A, b \in B$$

$$A + x = A + (a + b)$$

$$= (A + a) + (A + b) = A + (A + b)$$

$$= A + b = \theta(b).$$

Showing that b is the required pre image of A + x under θ and thus θ is onto. Hence by Fundamental theorem

$$\frac{A+B}{A} \cong \frac{B}{\text{Ker } \theta}.$$

We claim $\text{Ker } \theta = A \cap B$

Indeed $x \in \text{Ker } \theta \Leftrightarrow \theta(x) = A$

$$\Leftrightarrow A + x = A$$

$$\Leftrightarrow x \in A, \text{ also } x \in \text{Ker } \theta \subseteq B$$

$$\Leftrightarrow x \in A \cap B$$

Hence $\frac{A+B}{A} \cong \frac{B}{A \cap B}$

Note : By interchanging A and B, we get $\frac{B+A}{B} \cong \frac{A}{B \cap A}$

i.e., $$\frac{A+B}{A} \cong \frac{B}{A \cap B}.$$

Cor.: If A + B is the direct sum then as $A \cap B = \{0\}$

we get $$\frac{A}{(0)} \cong \frac{A \oplus B}{B}$$

But $\frac{A}{(0)} \cong A$ gives us $A \cong \frac{A \oplus B}{B}$.

Theorem 7 : Let W be a subspace of V, then $\exists$ an onto L.T. $\theta : V \to \frac{V}{W}$ such that Ker $\theta = W$.

Proof : The proof is left as an exercise for the reader.

Theorem 8 : A L.T. $T : V \to V$ is one-one iff T is onto.

Proof : Let $T : V \to V$ be one-one. Let dim V = n.

Let $\{v_1, v_2, ..., v_n\}$ be a basis of V, then $\{T(v_1), ..., T(v_n)\}$ will also be a basis of V as

$$\alpha_1 T(v_1) + \alpha_2 T(v_2) + ... + \alpha_n T(v_n) = 0$$

$$\Rightarrow T(\alpha_1 v_1 + ... + \alpha_n v_n) = T(0) \; (T \text{ a L.T.})$$

$$\Rightarrow \alpha_1 v_1 + ... + \alpha_n v_n = 0 \; (T \text{ is } 1-1)$$

$$\Rightarrow \alpha_i = 0 \text{ for all i}$$

thus $T(v_1)$, ... $T(v_n)$ are L.I. and as dim V = n the result follows (Theorem done earlier)

Let now $v \in V$ be any element

then $$v = a_1 T(v_1) + a_2 T(v_2) + ... + a_n T(v_n) \quad a_i \in F$$

$$= T(a_1 v_1 + ... + a_n v_n)$$

$$= T(v') \text{ for some } v'$$

Hence T is onto.

Conversely, let T be onto.

Here again we show that if $\{v_1, v_2, ... v_n\}$ is a basis of V then so also i

$\{T(v_1), T(v_2),..., T(v_n)\}$

For any $v \in V$, since T is onto, $\exists$ some $v' \in V$ s.t.,

$T(v') = v$

2.3.3 Rank and Nullity of a Linear Transformation

Definition : Let $T: V \to W$ be a L.T.

then we define Rank of T = dim Range T = r(T)

Nullity of T = dim Ker T = v(T).

Theorem 9 : (Sylvester's Law) : Let $T: V \to W$ be a L.T., then

Rank T + Nullity T = dim V.

Proof : Let $\{x_1, x_2,..., x_m\}$ be a basis of Ker T then $\{x_1, x_2,..., x_m\}$ being L.I. in Ker T will be L.I: in V. Thus it can be extended to form a basis of V.

Let $\{x_1, x_2,..., x_m, v_1, v_2,...., v_n\}$ be the extended basis of V.

Then dim Ker T = nullity of T = m

dim V = m + n

we show $\{T(v_1), T(v_2),..., T(v_n)\}$ is a basis of Range T

Now $\alpha_1 T(v_1) + \alpha_2 T(v_2) + ... + \alpha_n T(v_n) = 0$

$\Rightarrow T(\alpha_1 v_1 + ... + \alpha_n v_n) = 0$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 .... + \alpha_n v_n \in \text{Ker } T$

$\Rightarrow \alpha_1 v_1 + ... + \alpha_n v_n = \beta_1 x_1 + ... + \beta_m x_m$

or $\alpha_1 v_1 + ... + \alpha_n v_n + (-\beta_1) x_1 + ... + (-\beta_m) x_m = 0$

$\Rightarrow \alpha_1 = \alpha_2 = ... = \beta_1 = ... = \beta_m = 0$

i.e., $\{T(v_1), T(v_2),..., T(v_n)\}$ is L.I.

Now if T(V) $\in$ Range T be any element then as v $\in$ V

$v = a_1 x_1 + ... + a_m x_m + b_1 v_1 + ... + b_n v_n \quad a_i, b_j \in F$

$T(v) = a_1 T(x_1) + ... + a_m T(x_m) + b_1 T(v_1) + ... + b_n T(v_n)$

$= 0 + ... + 0 + b_1 T(v_1) + ... + b_n T(v_n) \quad [\text{as } x_i \in \text{Ker } T]$

or that T(v) is a linear combination of $T(v_1), \ldots, T(v_n)$
which, therefore, form a basis of Range T.
$\therefore$ dim Range T = n = rank T
which proves the theorem.

Theorem 10 : If $T : V \to V$ be a L.T. Show that the following statements are equivalent.

(i) Range T $\cap$ Ker T = {0}

(ii) If T (T(v)) = 0 then T(v) = 0, v $\in$ V

Proof : (i) $\Rightarrow$ (ii)

$$T(T(v)) = 0 \Rightarrow T(v) \in \text{Ker } T$$

Also $T(v) \in$ Range T (by definition)

$\Rightarrow$ T(v) = 0

(ii) $\Rightarrow$ (i)

Let x $\in$ Range T $\cap$ Ker T

$\Rightarrow$ $x \in \text{Range } T \text{ and } x \in \text{Ker } T$

$\Rightarrow$ $x = T(v) \text{ for some } v \in V$

and T(x) = 0

$$x = T(v) \Rightarrow T(x) = T(T(v))$$

$\Rightarrow$ $0 = T(T(v))$

$\Rightarrow$ T(v) = 0 (given condition)

$\Rightarrow$ v = 0.

2.3.4 Summary

In this lesson, we have gained knowledge about the linear transformations and many important results alongwith proofs. To define nullity , we have defined the kernel or null space of a linear transformation. To define rank, we have introduced range space of a linear transformation. An important law concerning rank, nullity and dimension of a vector space called Sylvester's law is elaborated with its statement and proof. Many simple examples are given in the lesson for a better understanding of concepts.

2.3.5 Key Concepts

Homomorphism, Isomorphism, Linear transformation, Kernel, Null space, Fundamental theorem of homomorphism, Range, Rank, Nullity, Dimension, Sylvester's law

2.3.6 Long Questions

1. Let dim V = n, T : V $\to$ V be a L.T. such that Range T = Ker T. show that n is even. Prove that T : $R^2 \to R^2$, s.t., $T(x_1, x_2) = (x_2, 0)$ is such a L.T.

2. Find the L.T. From $R^3 \to R^3$ which has its range the subspace spanned by (1, 0, –1),(1, 2, 2).

3. Show that the linear transformation $T : R^3 \to R^3$ defined by $T(x_1, x_2, x_3) = (2x_1, x - x_2, 5x_1 + 4x_2 + x_3)$ is invertible.

4. Let T be a L.T. from R^7 onto a 3-dimensional subspace of R^5. Show that dim Ker T = 4.

2.3.7 Short Questions

1. Find range, rank, Ker and nullity of the L.T. defined by

(i) $T : R^2 \to R^2$ s.t., $T(x_1, x_2) = (x_1 + x_2, x_1)\left[R^2, 2(0), 0\right]$

(ii) $T : R^2 \to R^2$ s.t., $T(x_1, x_2) = (x_1 + x_2, x_1 - x_2, x_2)$

(iii) $T : R^3 \to R^2$ s.t., $T(x_1, x_2, x_3) = (x_1 + x_2, x_1 - x_3)$

$$\left[(1,1,0)(1,-1,1), 2(0), 0\right]$$

(iv) The zero and the identify linear transformations

(v) $T : R^3 \to R^2$ s.t., $T(x_1, x_2, x_3) = (x_1 - x_2, 2x_3 - x_1)$.

2.3.8 Suggested Readings

1. P. B. Bhattacharya, S. K. Jain & S. R. Nagpaul : A First Course in Linear Algebra, New Age International (P) Ltd.
2. Gilbert Strang : Linear Algebra and its Applications, Cengage Learning Publishers (Fourth Edition)

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LINEAR TRANSFORMATIONS-II

2.4.1 Objectives

With the help of this lesson, the students would be able to get knowledge about

- Algebra of linear transformations to discuss binary operations on it
- Linear operator and linear functional
- Inverse of a linear transformation
- Matrix related to a linear transformation

2.4.2 Algebra of Linear Transformations

2.4.2.1 is Sum of Linear Transformations

Let V and W be two vector spaces over the same field F. Let T : V $\to$ W and S : V $\to$ W be two linear transformations. We define T + S, the sum of T and S by

$$T + S : V \to W, \text{s.t.}$$

$$(T + S)v = T(v) + S(v), v \in V$$

Then T + S is also a L.T. from V $\to$ W as

$$(T + S)(\alpha x + \beta y) = T(\alpha x + \beta y) + S(\alpha x + \beta y)$$

$$= \alpha T(x) + \beta T(y) + \alpha S(x) + \beta S(y)$$

$$= \alpha(T+S)x + \beta(T+S)y$$

Again for $\alpha \in F$, we define the product of a L.T. $T : V \to W$ with α, by $(\alpha T) : V \to W$ s.t.,

$(\alpha T)v = \alpha(T(v))$.

It is easy to see that αT is a also a L.T. from $V \to W$. Let Hom (V, W) be the set of all linear transformations from $V \to W$. Then we show Hom (V, W) forms a vector space over F under the addition and scalar multiplication as defined above.

We have already seen that when $T, S \in$ Hom (V, W), $\alpha \in F$ then $T + S$, $\alpha T \in$ Hom (V, W), thus closure holds for these operations. We verify some of the other conditions in the definition.

$$(T+S)v = T(v) + S(v) = S(v) + T(v) = (S+T)w \text{ for all } v \in V$$

$$\Rightarrow \quad T + S = S + T \text{ for all } S, T \in \text{Hom}(V, W)$$

The map $O : V \to W$, s.t., $O(v) = 0$ is a L.T. and

$$(T+O)v = T(v) + O(v) = T(v) = (O+T)v \text{ for all } v$$

This O is zero of Hom (V, W)

For any $T \in$ Hom (V, W), the map $(-T) : V \to W$, s.t.,

$$(-T)\ v = -T\ (v)$$

will be additive inverse of T.

Again, $$[\alpha(T+S)]v = \alpha[(T+S)v] = \alpha[T(v) + S(v)] = \alpha T(v) + \alpha S(v)$$

$$= (\alpha T)v + (\alpha S)v = (\alpha T + \alpha S)v \text{ for all } v \in V$$

$$\Rightarrow \quad \alpha(T+S) = \alpha T + \alpha S$$

$$[(\alpha\beta)T]v = (\alpha\beta)T(v) = \alpha[\beta T(v)] = [\alpha(\beta T)]v \text{ for all } v$$

$$\Rightarrow \quad (\alpha\beta)\ T = \alpha(\beta T)$$

$$(1T)v = 1.T(v) = T(v) \text{ for all } v$$

$$\Rightarrow \quad 1\ .\ T = T$$

Hence one notices that Hom (V, W) forms a vector space over F.

Note : The notation L(V, W) is also used for denoting Hom (V, W).

2.4.2.2 is Product of Linear's Transformations

Definition : Product (composition) of two linear transformations

Let V, W, Z be three vector spaces over a field F

Let $T : V \to W$, $S : W \to Z$ be L.T.

We define $ST : V \to Z$, s.t.,

$$(ST)v = S(T(v))$$

then ST is a linear transformation (verify!), called product of S and T.

Note : TS may not be defined and even if it is defined it may not equal ST.

2.4.2.3 Linear operator and Linear Functional

Definition : A L.T. $T : V \to V$ is called a linear operator on V, whereas a L.T. $T : V \to F$ is called a linear functional. We use notation T^2 for T.T. and $T^n = T^{n-1}\,T$ etc.

Theorem 11 : Let T, T_1, T_2 be linear operators on V, and let $I : V \to V$ be the identity map $l(v) = v$ for all v (which is clearly a L.T.) then

(i) $IT = TI = T$

(ii) $T(T_1 + T_2) = TT_1 + TT_2$

$(T_1 + T_2)T = T_1T + T_2T$

(iii) $\alpha(T_1T_2) = (\alpha T_1)T_2 = T_1(\alpha T_2)\ \alpha \in F$

(iv) $T_1(T_2T_3) = (T_1T_2)\,T_3$.

Proof : (i) Obvious.

(ii) $\left[T(T_1 + T_2)\right]x = T\left[(T_1 + T_2)x\right] = T\left[T_1(x) + T_2(x)\right]$

$= T(T_1(x)) + T(T_2(x)) = TT_1(x) + TT_2(x)$

$= (TT_1 + TT_2)x$

$\Rightarrow T(T_1 + T_2) = TT_1 + TT_2$

Other result follows similarly.

(iii) $\left[\alpha(T_1T_2)\right]x = \alpha\left[(T_1T_2)x\right] = \alpha\left[T_1(T_2(x))\right]$

$\left[(\alpha T_1)T_2\right]x = (\alpha T_1)\left[T_2(x)\right] = \alpha\left[T_1(T_2(x))\right]$

$\left[T_1(\alpha T_2)\right]x = T_1(\alpha T_2)x = T_1(\alpha T_2(x)) = \alpha T_i(T_2(x))\big]$

Hence the result follows.

(iv) Follows easily by definition.

See exercises for the generalised version of above theorem.

Problem 1 : Find the range, Rank, Ker and nullity of the linear transformation

$$T : R^3 \to R^3, \text{s.t.},$$

$$T(x, y, z) = (x + z, x + y + 2z, 2x + y + 3z)$$

Solution : Let (x, y, z) $\in$ Ker T be any element, then

$$T(x, y, z) = (0, 0, 0)$$

$$\Rightarrow (x + z, x + y + 2z, 2x + y + 3z) = (0, 0, 0)$$

$$\Rightarrow x + 0 + z = 0$$

$$x + y + 2z = 0$$

$$2x + y + 3z = 0$$

Giving $x = -z, -z + y + 2z = 0$ i.e., $y = -z$

Thus Ker T consists of all elements of the type (x, x, –x) = x (1, 1, –1) where x is any real no. i.e., Ker T is spanned by (1, 1, –1) which is L.I. Note (1, 1, –1) $\in$ Ker T

Hence dim (Ker T) = 1 = nullity of T

Again, from def. of T, we notice elements of the types (x + z, x + y + 2z, 2x + y + 3z) are in Range T.

Now $(x + z, x + y + 2z, 2x + y + 3z) = (x + 0 + z, x + y + 2z, 2x + y + 3z)$

$$= (x, x, 2x) + (0, y, y) + (z, 2z, 3z)$$

$$= x(1, 1, 2) + y(0, 1, 1) + z(1, 2, 3)$$

Thus Range T is spanned by {(1, 1, 2), (0, 1, 1), (1, 2, 3)}

Since (1, 1, 2) + (0, 1, 1) = (1, 2, 3) we find these vectors are L.D. So dim Range $T \le 2$

Again as (1, 1, 2) and (0, 1, 1) are L.I. we find

dim Range T = 2 = Rank T.

Problem 2 : Find the range, rank, Ker and nullity of the following linear transformations

(a) $T : R^2 \to R^3$ s.t., $T(x_1, x_2) = (x_1, x_1 + x_2, x_2)$

(b) $T : R^4 \to R^3$ s.t., $T(x_1, x_2, x_3, x_4) = (x_1 - x_4, x_2 + x_3, x_3 - x_4)$

Solution : (a) From definition of T, we notice elements of the type $(x_1, x_1 + x_2, x_2)$ will have pre images in R^2 i.e., elements of this type are in Range T.

Now $(x_1, x_1 + x_2, x_2) = (x_1 + 0, x_1 + x_2, 0 + x_2)$

$= (x_1, x_1, 0) + (0, x_2, x_2)$

$= x_1 (1,1,0) + x_2 (0,1,1)$

or that Range T is spanned by ((1,1,0), (0, 1, 1)} and since

$\alpha_1 (1,1,0) + \alpha_2 (0,1,1) = (0,0,0)$

$\Rightarrow \alpha_1 = \alpha_2 = 0$

these are L.I. and thus form a basis of Range T

$\Rightarrow$ Rank T = dim Range T = 2.

Again $(x_1, x_2) \in \text{Ker } T \Rightarrow T(x_1, x_2) = (0,0,0)$

$\Rightarrow (x_1, x_1 + x_2, x_2) = (0,0,0)$

$\Rightarrow x_1 = 0, x_1 + x_2 = 0, x_2 = 0$

$\Rightarrow x_1 = x_2 = 0$

$\Rightarrow \text{Ker } T = \{(0, 0)\}$

Also then nullity T = dim Ker T = 0.

(b) From definition of T, we find elements of the type $(x_1 - x_4, x_2 + x_3, x_3 - x_4)$ have pre image in R^4.

Now

$(x_1 - x_4, x_2 + x_3, x_3 - x_4) = (x_1 + 0 + 0 - x_4, 0 + x_2 + x_3 + 0, 0 + 0 + x_3 - x_4)$

$= x_1 (1,0,0) + x_2 (0,1,0) + x_3 (0,1,1) + x_4 (-1,0,-1)$

or that Range T is spanned by

$\{(1,0,0), (0,1,0), (0,1,1), (-1,0,-1)\}$

Since Range T is a subspace of R^3 which has dim 3 these four elements cannot form basis of Range T.

In fact these re L.D., elements as

$$(-1,0,-1)+(1,0,0)+(0,1,0)+(0,1,1)=(0,0,0)$$

If we consider three members

$$(1,0,0),(0,1,0),(0,1,1)$$

we notice $\alpha_1(i,0,0)+\alpha_2(0,1,0)+\alpha_3(0,1,1)=(0,0,0)$

$\Rightarrow$ $\alpha_i = 0$ for all i

or that (1, 0, 0), (0, 1, 0) (0, 1, 1) are L.I., and hence form basis of Range T

$\Rightarrow$ dim Range T = 3 = rank of T

one might notice here that as

$$(-1,0,-1)=-1(1,0,0)-1(0,1,0)-1(0,1,1)$$

the elements $(1,0,0),(0,1,0),(0,1,1)$ span Range T

Also then Range T = R^3

Again $(x_1,x_2,x_3,x_4)\in \text{Ker } T \Rightarrow T(x_1,x_2,x_3,x_4)=(0,0,0)$

$\Rightarrow$ $x_1 - x_4 = 0$

$x_2 + x_3 = 0$

$x_3 - x_4 = 0$

if we fix x_4, we get $x_1 = x_4, x_2 = -x_3 = -x_4, x_3 = x_4$

or that elements of the type $(x_4,-x_4,x_4,x_4)$ are in the Ker T

Problem 3 : Let T be a linear operator on V. If $T^2 = 0$, what can you say about the relation of the range of T to the null space of T? Give an example of linear operator T of R^2 such that $T^2 = 0$, but $T \neq 0$.

Solution : $T^2 = 0 \Rightarrow T^2(v) = 0$ for all $v \in V$

$\Rightarrow T(T(V)) = 0$

$\Rightarrow T(v) \in \text{Ker } T$ for all $v \in V$

$\Rightarrow$ range $T \subseteq$ Ker T.

Define $T : R^2 \to R^2$, such that

$$T(x_1,x_2)=(x_2,0)$$

then T is a linear operator (Verify!)

Since $T(2,2) = (2,0) \neq (0,0)$

$T \neq 0$

But $T^2(x_1, x_2) = T(T(x_1, x_2)) = T(x_2\ 0) = (0, 0)$

$\Rightarrow T^2 = 0.$

Problem 4 : Let T be a linear operator on V and let Rank T^2 = Rank T then show that Rank T $\cap$ Ker T = {0}.

Solution : $T : V \to V, T^2 : V \to V$

Rank T^2 = dim V – dim Ker T^2

$\Rightarrow \dim \text{Ker } T = \dim \text{Ker } T^2$

We claim Ker T = Ker T^2

$x \in \text{Ker } T \Rightarrow T(x) = 0 \Rightarrow T^2(x) = T(0) = 0$

$\Rightarrow x \in \text{Ker } T^2 \Rightarrow \text{Ker } T \subseteq \text{Ker } T^2$

$\Rightarrow \text{Ker } T = \text{Ker } T^2$ (as they have same dim)

Now $x \in \text{Range } T \cap \text{Ker } T \Rightarrow x \in \text{Range } T$ and $x \in \text{Ker } T$

$\Rightarrow T(x) = 0, x = T(y)$ for some $x \in V$

$\Rightarrow T(T(y)) = 0$

$\Rightarrow T^2(y) = 0$

$\Rightarrow y \in \text{Ker } T^2 = \text{Ker } T$

$\Rightarrow T(y) = 0 \Rightarrow x = 0$

$\Rightarrow \text{Ker } T \cap \text{Range } T = \{0\}.$

2.4.3 Invertible Linear Transformations

We recall that a map $T : V \to W$ is invertible iff it is 1-1 onto, and inverse of T is the map $T^{-1} : W \to V$ such that

$$T^{-1}(y) = x \Leftrightarrow T(x) = y$$

We show that inverse of a (1-1 onto) L.T. is also a L.T. Let $T : V \to W$ be 1-1 onto L.T. and $T^{-1} : W \to V$ be its inverse.

We have to prove

$$T^{-1}(\alpha w_1 + \beta w_2) = \alpha T^{-1}(w_1) + \beta T^{-1}(w_2) \ \alpha, \beta \in F, w_1, w_2 \in W$$

Since T is onto, for $w_1, w_2 \in W, \exists\, v_1, v_2 \in V$ such that $T(v_1) = w_1,\ T(v_2) = w_2$

$$\Leftrightarrow v_1 = T^{-1}(w_1),\ v_2 = T^{-1}(w_2)$$

Now $$T^{-1}(\alpha w_1 + \beta w_2) = T^{-1}(\alpha T(v_1) + \beta T(v_2))$$

$$= T^{-1}(T(\alpha v_1) + T(\beta v_2))$$

$$= T^{-1}(T(\alpha v_1 + \beta v_2))$$

$$= \alpha v_1 + \beta v_2$$

$$= \alpha T^{-1}(w_1) + \beta T^{-1}(w_2).$$

Definition : A L. T. $T : V \to W$ is called non-singular if Ker T = {0} i.e. if T is 1.1.

Theorem 12 : A linear transformation $T : V \to W$ is non singular iff T carries each L.I. subset of V onto a L.I. subset of W.

Proof : Let T be non-singular and $\{v_1, v_2 ..., v_n\}$ be a L.I. subset of V. we show $\{T(v_1), T(v_2) ..., T(v_n)\}$ is L.I. subset of W.

Now $$\alpha_1 T(v_1) + \alpha_2 T(v_2) + ... + \alpha_n T(v_n) = 0 \ \alpha_i \in F$$

$$\Rightarrow T(\alpha_1 v_1 + ... + \alpha_n v_n) = 0$$

$$\Rightarrow \alpha_1 v_1 + ... \alpha_n v_n \in \text{Ker } T = \{0\}$$

$$\Rightarrow \alpha_1 v_1 + ... \alpha_n v_n = 0$$

$\Rightarrow \alpha_i = 0$ for all i as $v_1, v_2 ..., v_n$ are L.I.

Conversely, let $v \in \text{Ker } T$ be any element

Then T(v) = 0

$\Rightarrow \{T(v)\}$ is not L.I. in W

$\Rightarrow$ v is not L.I. in V. (by hypothesis)

$\Rightarrow v = 0 \Rightarrow \text{Ker } T = \{0\}$

$\Rightarrow$ T is non singular.

Theorem 13 : Let $T : V \to W$ be a L.T. where V and W are two F.D.V.S. with same dimension. Then the following are equivalent.

(i) T is invertible

(ii) T is non singular (i.e., T is 1-1)

(iii) T is onto (i.e. Range T = W)

(iv) If $\{v_1, v_2, ..., v_n\}$ is a basis of V then

$\{T(v_1), T(v_2), ..., T(v_n)\}$ is a basis of W.

Proof : $(i) \Rightarrow (ii)$ follows by definition.

$(ii) \Rightarrow (iii)$ T is non-singular

$\Rightarrow \text{Ker } T = \{0\}$

$\Rightarrow \dim \text{Ker } T = 0$

Since dim Range T + dim Ker T = dim V, we get

dim Range T = dim V

$\Rightarrow$ dim Range T = dim W (given condition)

But Range T being a subspace of W, we find

Range T = W

$(iii) \Rightarrow (i)$ T onto means Range T = W

$\Rightarrow$ dim Range T = dim W = dim V

and as dim Range T + dim Ker T = dim V, we get

dim Ker T = 0

$\Rightarrow$ Ker T = {0}

or that T is 1-1 and as it is onto T will be invertible.

$(i) \Rightarrow (iv)$ T is invertible $\Rightarrow$ T is 1-1 onto

i.e., T is an isomorphism.

$(iv) \Rightarrow (i)$

Let $\{T(v_1), ..., T(v_n)\}$ be basis of W where $\{v_1, ... v_n\}$ is basis of V. Any $w \in W$

can be put as

$$w = \alpha_1 T(v_1) + \ldots + \alpha_n T(v_n)$$

$$= T(\alpha_1 v_1 + \ldots + \alpha_n v_n) = T(v) \text{ for some } v \in V$$

$\therefore$ T is onto. Thus (iii) holds.

Hence (i) holds.

Problem 5 : Let T be a linear operator on R^3, defined by

$$T(x_1, x_2 x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3)$$

show that T is invertible and find the rule by which T^{-1} is defined.

Solution: $T : R^3 \to R^3$

Let $T(x_1, x_2, x_3) \in \text{Ker}\, T$ be any element

Then $T(x_1, x_2, x_3) = (0, 0, 0)$

$$\Rightarrow (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3) = (0, 0, 0)$$

$$\Rightarrow 3x_1 = 0, x_1 - x_2 = 0, 2x_1 + x_2 + x_3 = 0$$

$$\Rightarrow x_1 = x_2 = x_3 = 0 \text{ or that Ker } T = \{(0, 0, 0)\}$$

$\Rightarrow$ T is non singular and thus invertible (See theorem 8)

Now If (z_1, z_2, z_3) be any element of R^3, then (x_1, x_2, x_3) will be its image under T if

$$T(x_1, x_2, x_3) = (z_1, z_2, z_3)$$

$$\Rightarrow 2x_1 = z_1$$

$$x_1 - x_2 = z_2$$

$$2x_1 + x_2 + x_3 = z_2$$

Which give $x_1 = \dfrac{z_1}{3}, x_2 = \dfrac{z_1}{3} - z_2, z_3 = z_3 - z_1 + z_2$

Hence $T^1 : R^3 \to R^3$ is defined by

$$T^1 : (z_1, z_2, z_3) = \left(\frac{z_1}{3}, \frac{z_1}{3} - z_2, z_3 - z_1 + z_2\right).$$

Problem 6 : If T : V → V is a L.T., such that T is not onto then show that there exists some $0 \neq v$ in V s.t., T(v) = 0.

Solution : Since T is not onto, it is not 1-1 (theorem done)

Suppose $\exists$ no $0 \neq v \in V$ V s.t. T (v) = 0

Then T(v) = 0 only when v = 0

$\Rightarrow$ Ker T = {0} $\Rightarrow$ T is 1-1, a contradietion.

2.4.4 Matrix of a Linear Transformations

Let . U(F), V(F) be vector spaces of dimension n and m respectively, Let $\beta = \{u_1, ...u_n\}, \beta' = \{v_1, ..., .v_m\}$ be their ordered basis respectively. Suppose $T : U \to V$ is a linear transformation. Since $T(u_1), ..., T(u_n) \in V$ and $\{v_1, ..., v_m\}$ spans V, each T(u_i) is a linear combination of vectors $v_1, ..., v_m$.

Let $T(u_1) = \alpha_{11}v_1 + ...\alpha_{m1}v_m$

$T(u_2) = \alpha_{12}v_1 + ...\alpha_{m2}v_m$

....

$T(u_n) = \alpha_{1n}v_1 + ...\alpha_{mn}v_m$

where each $a_{ij} \in F$. Then the m × n matrix

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \cdots & \alpha_{1n} \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \cdots & \alpha_{mn} \end{bmatrix}$$

is called matrix of T with repsect to ordered basis β, β' respectively. A is uniquely determined by T as each $\alpha_{ij} \in F$ is uniquely determined. We write

$$A = [T]_{\beta,\beta'}$$

Theorem 14 : Hom (U,V) = $M_{n\times N}$ (F)

Proof : Define θ $Hom(U,V) \to m_{m\times n}(F)$, s.t.,

$$\theta(T) = [T]_{\beta,\beta'}$$

Where $\beta=\{u_1,....u_n\},\beta'=\{v_1,.....v_m\}$ are ordered basis of U, V respectively. θ is well defined as $[T]_{\beta,\beta'}$. is uniquely determinec by T

It is not difficult to verify that θ is a linear transformation.

Let $\theta(S)=\theta(T), ST\in H(U,V)$

Then $[S]_{\beta,\beta'}=[T]_{\beta,\beta'}$

$\Rightarrow (a_{ij})=(b_{ij})$

$\Rightarrow a_{ij}=b_{ij}$ for all i, j

$\Rightarrow S(u_j)=\sum_{i=1}^{m}a_{ij}v_i=\sum_{i=1}^{m}b_{ij}v_i=T(u_j)$ for all j = 1, ...n

$\Rightarrow S=T\Rightarrow\theta$ is $1-1$

Let $A=(a_{ij})_{m\times n}\in M_{m\times n}(F)$. Then $\exists$ a linear transformation T $\in$ H (U, V) s.t.,

$$T(u_j)=\sum_{i=1}^{m}a_{ij}v_i= \text{ for } j=1,.....n$$

$A=[T]_{\beta,\beta'}=\theta(T)\Rightarrow\theta$ is onto.

Hence θ is an isomorphism and so Hom $(U,V)\cong M_{m\times n}(F)$

Cor .: dim Hom (U, V) = mn

Proof : S = set of al m ×n matrices with only one entry I and all other entries zero, is a basi of $M_{m\times n}(F)$.

Clealy, $o(S)=mn\Rightarrow \dim M_{m\times n}(F)=mn$

dim Hom (U, V) = mn .

Problem 7 : Let T be a linear operator on C^2 defined by $T(x_1,x_2)=(x_1,0)$ Let $\beta=\{\in_1=(1,0,)\in_2(0,1)\},\beta'=\{\alpha_1=(1,i),\alpha_2=(-i,2)\}$ be ordered basis for C^2. What is the matrix of T relative to the pair β,β'?

Solution : Now $T(\in_1) = T(0,0)$

$= (1, 0)$

$= a(1,i) + b(-i,2)$

$\Rightarrow a - bi = 1$ where $a, b \in C$

$ai + 2b = 0$

$\Rightarrow a = 2, b = -i$

$\Rightarrow T(\in_1) = 2\alpha_1 - i\alpha_2$

Also $T(\in_2) = T(0,1) = (0,0) = 0\alpha_1 + 0\alpha_2$

$$\therefore \quad [T]_{\beta\beta'} = \begin{bmatrix} 2 & 0 \\ -i & 0 \end{bmatrix}.$$

Proble 8 : Let T be linear operator on R^3, the matrix of which in the standard ordered basis is:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

Find a basis for the range of T and a basis for the null space of T.

Solution : Det $A = 1(4-3) - 2(1) + 1(1)$

$= 1 - 2 + 1 = 0$

$\therefore$ A is not invertible and so T is not invertible

Let $\{\in_1 = (1,0,0,), \in_2 = (0,1,0), \in_3 = (0,0,1)\}$

be standard ordered basis of R^3.

Let $(x_1, x_2, x_3) \in \text{Ker } T$

Then $T(x_1, x_2, x_3) = 0$

$\Rightarrow T(x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)) = 0$

$\Rightarrow T(x_1 \in_1 + x_2 \in_2 + x_3 \in_3) = 0$

$\Rightarrow x_1 T(\in_1) + x_2 T(\in_2) + x_3 T(\in_3) = 0$

$$\Rightarrow x_1(1,0,-1)+x_2(2,1,3)+x_3(1,1,4)=0$$

$$\Rightarrow (x_1+2x_2+x_3, x_2+x_3, -x_1+3x_2+4x_3)=0$$

$$\Rightarrow x_1+2x_2+x_3=0, x_2+x_3=0, -x_1+3x_2+4x_3=0$$

$$\Rightarrow x_1+x_2=0, x_2+x_3=0$$

$$\Rightarrow (x_1,x_2,x_3)=(-x_2,x_2,-x_3)$$

$$= x_2(-1,1,-1)$$

$\Rightarrow$ every element in Ker T is multiple of (–1, 1, –1)

$\Rightarrow$ Ker T is spanned by (–1, 1, –1)

Since (–1, 1, –1) $\neq$ 0, {(–1, 1, –1)} is basis of Ker T.

$\therefore$ dim Ket T = 1 $\Rightarrow$ dim Range T = 2

Since $T\in_1=(1,0,-1)$

$T\in_2=(2,1,3)$

belong to Range T and $aT\in_1+bT\in_2=0$

we find $a(1,0,-1)+b(2,1,3)=0$

$\Rightarrow b=0, a=0$

$\Rightarrow \{T\in_1, T\in_2\}$ is a linearly independent set in Range T. As dim Range T = 2, $\{(1,0-1),(2,1,3)\}$ is a basis of Range T.

Problem 9 : Let T be a linear operator on F^n and let A be the matrix of T in the standard ordered basis for F^n. Let W be the subspace of F^n spanned by the column vectors of A. Find a relation between W and T.

Solution : $T.F^n \rightarrow F^n$

Let $\beta=\{e_1=(1,0,0,.....0),......e_n=(0,0,......,1)\}$ be the standard ordered basis of F' and let

$$A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}$$

thus $T(e_1) = a_{11}e_1 + e_{21}e_2 + ... + a_{n1}e_n$

$T(e_2) = a_{12}e_1 + e_{22}e_2 + ... + a_{n2}e_2$

....

$T(e_n) = a_{1n}e_1 + a_{2n}e_2 + ... + a_{nn}e_2$

and also W is spanned by

$$\{(a_{11}, a_{21}, ..., a_{n1}), (a_{12}, a_{22}, ..., a_{n2}), ..., (a_{1n}, a_{2n}, ..., a_{nn})\}$$

We claim $T : F^n \to W$ is onto L.T.

For any $x \in F^n, x = \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n$

$\Rightarrow T(x) = \alpha_1 T(e_1) + \alpha_2 T(e_2) + ... + \alpha_n T(e_n)$

$\Rightarrow T(x) \in W \text{ as } T(e_1), T(e_2), ..., T(e_n) \in W$

Again, for any $w \in W, w = \beta_1 T(e_1) + \beta_2 T(e_2) + ... + \beta_n T(e_n)$

$= T(\beta_1 e_1 + \beta_2 e_2 + ... + \beta_n e_n)$

showing thet T is onto

$\Rightarrow$ Range T – W $\Rightarrow$ dim Range T = dim W

or that rank of T = dim W

which is the required relation between T and W.

2.4.5 Summary

In this lesson, we have discussed about the operations particularly sum and product of a linear transformation. For a linear transformation from V to V, we have defined linear operator. The concepts of invertible linear transformation concerning the inverse and matrix related to a linear transformation have been clearly elaborated. Many simple examples are given in the lesson for a better understanding of concepts.

2.4.6 Key Concepts

Sum of linear transformations, Product of linear transformations, Linear operator, Linear functional, Invertible linear transformation, Inverse, Matrix of linear transformation.

2.4.7 Long Questions

1. Let T be the linear transformation from R^3 into R^2 defined by

$T(x_1, x_2, x_3) = (x_1 + x_2, 2x_1 - x_3)$

(i) If β, β' are standard ordered basis for R^3 and R^2 respectively, find $[T]\beta, \beta$.

(ii) If $\beta = \{\alpha_1 = (1,0,-1), \alpha_2 = (1,1,1), \alpha_3 = (1,0,0)\}$

$\beta' = \{\beta_1 = (0,1), \beta_2 = (1,0)\}$.

2. Let T be the linear operator on R^3, the matrix of which in the standard ordered basis is $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Find a basis of Range T and Ker T.

3. Show that $T : R^3 \to R^2$ s.t, $T(x,t,z) = (y,x)$ is a L.T. Find the matrix representation of T for the standard ordered basis fo R^3 and $\{(0,1),(2,3)\}$ of R^2.

2.4.8 Short Questions

1. Define sum of linear transformations.

2. Define product of linear transformations.

3. Define linear operator and linear functional.

4. For what conditions, a linear transformation is said to be invertible?

2.4.9 Suggested Readings

1. P. B. Bhattacharya, S. K. Jain & S. R. Nagpaul : A First Course in Linear Algebra, New Age International (P) Ltd.

2. Gilbert Strang : Linear Algebra and its Applications, Cengage Learning Publishers (Fourth Edition)

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