B.A. Part–I
(SEMESTER-I)

MATHEMATICS : PAPER -II
DIFFERENTIAL EQUATIONS

SECTION : A & B

Department of Distance Education
Punjabi University, Patiala

Lesson No. :

Note : Students can download the syllabus from department's website www.pbidde.org

B.A. PART - I **MATHEMATICS : PAPER–II**
(SEMESTER-I) **DIFFERENTIAL EQUATIONS**

LESSON NO. 1.1 **Author : Dr. Chanchal**

LINEAR DIFFERENTIAL EQUATIONS OF FIRST ORDER-I

Structure :

Objectives

The prime objective of this lesson is to study the basic features of an ordinary differential equation such as order and degree, types of solution, how to form a differential equation etc. Further, this lesson also deals with the solutions of differential equation with first order and first degree.

I. Introduction

Firstly, we introduce the concept of differential equation as :A differential equation is an equation which involves differential coefficients or differentials. For example :

(i) $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2=0$ (ii) $\boxed{\frac{dy}{dx}}=\sin x$

These differential equations are of two types :

The one in which differential coefficients,called derivatives are w.r.t. a single independent invariable, called the ordinary differential equation and the other in which differential coefficients are w.r.t. more than one independent variables, called the partial differential equations. But, in this unit, we confine ourselves to the study of ordinary differential equations only.

II. Order and Degree of a Differential Equation

Order of a Differential Equation : The order of a differential equation is the order of the highest differential coefficient occuring in it. For example :

Order of $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 2 = 0$ is 2.

In simple words, Order of a differential equation is defined as the order of the highest order derivative of the dependent vairable with respect to the independent variable involved in the given differential equation.

On the basis of degree, the differential equation can be classified as linear and non-linear as :

A differential equation is said to be linear if the unkown function and all of its derivatives occuring in the equation occur only in the first degree and are not multiplied together.

The differential equations $\frac{dy}{dx} = \sin x, \frac{d^2y}{dx^2} + y = 0$ are linear whereas

$\left(\frac{d^2y}{dx^2}\right)^2 + x^2\left(\frac{dy}{dx}\right)^3 = 0$ is non-linear.

It should be noted that a linear differential equation is always of the first degree but every differential equation of the first degree need not be linear. For example, the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y^2 = 0$ is not linear, though its degree is 1.

Example 1 : Write the order and degree of the differential equation $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{5}{2}}}{\frac{d^2y}{dx^2}} = 2$

Sol. The given differential equation is $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{5}{2}}}{\frac{d^2y}{dx^2}} = 2$

This can be written as

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{5}{2}} = 2\,\frac{d^2y}{dx^2} \quad \text{or} \quad \left[1+\left(\frac{dy}{dx}\right)^2\right]^5 = 4\left(\frac{d^2y}{dx^2}\right)^2$$

$\therefore$ degree of differential equation is 2 and order is also 2.

III. Solution of a Differential Equation

A solution of a differential equation is a relation between the variables such that this relation and the derivatives obtained from this relation satisfy the given differential equation.

Solution of a differential equation is also called integral of the differential equation.

The solution of a differential equation is further classified into following three types:

Classification :

General (or Primitive) Solution : The solution of a differential equation which involves as many arbitrary constants as the order of the differential equation, is called the general solution. It is also called complete solution.

Particular Solution :

A particular solution of a differential equation is that which contains no arbitrary constant and is obtained from the general solution by giving particular values to the arbitrary constants.

Singular Solution :

A singular solution of the differential equation is that which contains no arbitrary constant and cannot be obtained from the general solution by giving particular values to the arbitrary constants.

IV. Formation of a Differential Equation

We follow the method given below to form the differential equation of an equation in x and y.

Step I : Write down the given equation.

Step II : Differentiate it w.r.t. x, as many times as the number of arbitrary constants.

Step III : Eliminate the arbitrary constants from the given equation and equations obtained in Step II.

The resulting equation is the required differential equation.

Example 2 : Form the differential equation of the family of curves

$$y = Ax + \frac{B}{x}$$

Sol. The given equation is $y = Ax + \frac{B}{x}$ ... (1)

$$\therefore \quad \frac{dy}{dx} = A - \frac{B}{x^2} \qquad \text{... (2)}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{2B}{x^3} \qquad \text{... (3)}$$

From (3), $B = \frac{x^3}{2}\frac{d^2y}{dx^2}$

From (2), $\frac{dy}{dx} = A - \frac{x}{2}\frac{d^2y}{dx^2} \Rightarrow A = \frac{x}{2}\frac{d^2y}{dx^2} + \frac{dy}{dx}$

Putting value of A and B in (1), we get,

$$y = \frac{x^2}{2}\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \frac{x^2}{2}\frac{d^2y}{dx^2} \text{ or } x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

which is required differential equation.

V. Differential Equations of First Order and First Degree

A general differential equation of first order and first degree is an equation of the form $\frac{dy}{dx} = f(x, y)$ or M dx + N dy = 0 where M and N may be both functions of x and y.

Existance and Uniqueness Theorem

If f(x, y) and $\frac{dy}{dx}$ and are continuous functions of x and y in a region D of the xy-plane and if P $(x_0, y_0) \in D$, then there exists one and only one function ϕ (x), say, which is in some neighbourhood of P (contained in D) is a solution of the differential equation $\frac{dy}{dx} = f(x, y)$ and is such that $\phi(x_0) = y_0$.

Now, differential equations with first order and first degree are of several types followed by a special rule or method for solving them. In the coming subsections, we study that types alongwith their methods.

V.(a) Variables Separable

If in an equation it is possible to get all the functions of x and dx to one side, and all the functions of y and dy to the other, the variables are said to be separable.

Thus, in the equation $\frac{dy}{dx} = XY$ where X is a function of x only and Y is a function of y only, the variables are separable as this equation can be written as $\frac{dy}{Y} = X\ dx$.

Example 3 : Solve the differential equation $\tan y \cdot \frac{dy}{dx} = \sin(x + y) + \sin(x - y)$.

Sol. The given differential equation is

$$\tan y . \frac{dy}{dx} = \sin(x + y) + \sin(x - y) \text{ or } \tan y . \frac{dy}{dx} = 2 \sin x \cos y$$

Separating the variables, we get,

$$\frac{\tan y}{\cos y} dy = 2 \sin x\, dx \text{ or } \sec y \tan y\, dy = 2 \sin x\, dx$$

Integrating, $\int \sec y \tan y\, dy = 2 \int \sin x\, dx$

$\therefore$ $\sec y = -2 \cos x + c$, which is the required solution.

Note : We may be given some equations of the form $\frac{dy}{dx} = f(ax + by + c)$, which are originally, not in variables separable form, but can be reduced to that form and solved under the following rule :

Rule : Equations Reducible to Variable Separable

To solve $\frac{dy}{dx} = f(ax + by + c)$

(i) Put $ax + by + c = t$.

(ii) Separate the variables and integrate.

(iii) In the solution put $t = ax + by + c$.

V.(b) Homogeneous Equations

Firstly, we define a homogeneous function of n^{th} degree in x and y as :-

Def. : A homogeneous function of the nth degree in x and y is that which can be put in the form $x^n f\left(\frac{y}{x}\right)$.

Consider $f(x,y)=\dfrac{x^3+y^3}{x^2+y^2}=\dfrac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x^2\left[1+\left(\frac{y}{x}\right)^2\right]}=x\,\phi\left(\dfrac{y}{x}\right)$

$\therefore$ f (x, y) is a homogeneous function of degree 1.

Now, A homogeneous differential equation of the first degree is an equation of the form $\dfrac{dy}{dx}=\dfrac{f_1(x,y)}{f_2(x,y)}$ where f_1 (x, y) and f_2 (x, y) are homogeneous functions of the same degree in x and y.

Rule : In order to solve such an equation, we follow the rule :

(i) Put y = vx.

(ii) Separate the variables and integrate

(iii) In the solution, put $v=\dfrac{y}{x}$.

Note : Method to solve $\dfrac{dy}{dx}=f\left(\dfrac{y}{x}\right)$ is the same.

As we have discussed in case of variables separable, we may need to solve some equations which are not homogeneous but can be made homogeneous. Such equations, which are reducible to homogenous are further of two types as :-

Type I : $\dfrac{dy}{dx}=\dfrac{ax+by+c}{a'x+b'y+c'}$ when $\dfrac{a}{a'}\neq\dfrac{b}{b'}$

Rule : (i) Put x = x' + h, y = y' + k, where h, k are constants.

(ii) Put the constant terms in the numerator and denominator of R.H.S. each equal to zero and determine h and k.

(iii) Solve the resulting homogeneous equation in x' and y'.

(iv) In the solution, put x' = x – h, y' = y – k and substitute the values of h and k determined above.

Type II : $\dfrac{dy}{dx}=\dfrac{ax+by+c}{a'x+b'y+c'}$ when $\dfrac{a}{a'}=\dfrac{b}{b'}$

Rule : (i) Put ax + by = t

(ii) Separate the variables t and x

(iii) Integrate and put t = ax + by.

Example 4 : Solve $\frac{dy}{dx}=\frac{x+y+4}{x+y-6}$

Sol. The given differential equation is $\frac{dy}{dx}=\frac{x+y+4}{x+y-6}$ ... (1)

Comparing (1) with $\frac{dy}{dx}=\frac{ax+by+c}{a'x+b'y+c'}$, we get,

$$\frac{a}{a'}=\frac{b}{b'}=1$$

Put x + y = t or $1+\frac{dy}{dx}=\frac{dt}{dx}\Rightarrow\frac{dy}{dx}=\frac{dt}{dx}-1$ in (1), we get

$$\frac{dt}{dx}-1=\frac{t+4}{t-6} \quad \text{or} \quad \frac{dt}{dx}=\frac{t+4}{t-6}+1$$

$$\therefore \quad \frac{dt}{dx}=\frac{2t-2}{t-6}$$

Separating the variables, we get,

$$\frac{t-6}{2t-2}dt=dx \quad \text{or} \quad \frac{1}{2}\int\frac{t-6}{t-1}dt=\int 1\,dx$$

$$\therefore \quad \frac{1}{2}\int\left(1-\frac{5}{t-1}\right)dt=\int 1\,dx$$

$$\therefore \quad \frac{1}{2}[t-5\log|t-1|]=x+c \quad \text{or} \quad \frac{1}{2}[x+y-5\log|x+y-1|]=x+c.$$

VI. Self Check Exercise

1. Solve the following differential equations :

(i) $\sec^2x \tan y\, dx + \sec^2y \tan x\, dy = 0$

(ii) $\frac{dy}{dx}=\frac{x(2\log x+1)}{\sin y+y\cos y}$

(iii) $x \frac{dy}{dx} = y - x \cos^2 \frac{y}{x}$

(iv) $(2x + y + 1)\, dx + (4x + 2y - 1)\, dy = 0$

VII. Suggested Readings :

1. R.K. Jain, S.R.K. Lyengar, Advanced Engineering Mathematics, Narosa Publishing House.
2. Rai Singhania : Ordinary and Partial Differential Equations, S. Chand & Company, New Delhi.
3. Zafar Ahsan, Differential Equations and their Applications, Prentice-Hall of India Pvt. Ltd., New Delhi - 2nd Ed.

B.A. PART - I **MATHEMATICS : PAPER–II**
(SEMESTER-I) **DIFFERENTIAL EQUATIONS**

LESSON NO. 1.2 **Author : Dr. Chanchal**

LINEAR DIFFERENTIAL EQUATIONS OF FIRST ORDER-II

Structure :

Objectives

I. Linear Equation

II. Exact Differential Equation

III. Self Check Exercise

IV. Suggested Readings

Objectives :

We have already discussed about the two types of differential equations of first order and first degree in lesson 1.1. Here, in this lesson, we discuss about the other remaining tyes viz. linear equation and exact differential equation and the equations that can be reduced to the above types.

I. Linear Equation

The standard form of a linear equation of the first order is $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x. This equation is also known as Leibnitz's equation. The solution of such a linear equation is given by :

$$y \,.\, e^{\int P dx} = \int \theta e^{\int P dx}\, dx + c$$

Here, the term $e^{\int P dx}$ is called integrating factor and is denoted by I.F.

Example 1 : Solve $(1 + y^2)\, dx = (\tan^{-1}y - x)\, dy$.

OR

Solve $\left(1+y^2\right)+\left(x-e^{\tan^{-1} y}\right)\frac{dy}{dx}=0$.

Sol. The given equation is $(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy$

or $\left(1+y^2\right)\frac{dx}{dy} = \tan^{-1} y - x$ or $\left(1+y^2\right)\frac{dx}{dy} + x = \tan^{-1} y$

or $\frac{dx}{dy} + \frac{1}{1+y^2} x = \frac{\tan^{-1} y}{1+y^2}$

Comparing with $\frac{dx}{dy} + Px = Q$, we get, $P = \frac{1}{1+y^2}$, $Q = \frac{\tan^{-1} y}{1+y^2}$

$\therefore$ I.F. = $e^{\int P\,dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$

$\therefore$ solution of given equation is

$$x.e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} e^{\tan^{-1} y}\, dy + c \qquad \ldots (1)$$

$$\left[\because x\,.\,e^{\int P\,dy} = \int Q\,.\,e^{\int P\,dy} dy + c\right]$$

Put $\tan^{-1} y = t$, $\therefore$ $\frac{1}{1+y^2} dy = dt$

$\therefore$ $I = \int te^t dt = te^t - \int 1.e^t\, dt = te^t - e^t = (t-1)\,e^t = (\tan^{-1} y - 1)\,e^{\tan^{-1} y}$

$\therefore$ from (1), $xe^{\tan^{-1} y} = e^{\tan^{-1} y}(\tan^{-1} y - 1) + c$.

Like the previous cases, there may be equations which may not be linear but can be reduced to linear form and solved accordingly. Such type of equations are discussed below :

1. Bernoulli's Equation : An equation of the form $\frac{dy}{dx} + Py = Qy^n$ where P, Q are functions of x is not linear, but it can be reduced to linear and solved accordingly under the following rule :

Rule : (i) Divide throughout by y^n.

(ii) Put $y^{1-n} = t$.

(iii) Solve the linear equation in t and then put $t = y^{1-n}$.

2. General Equation : An equation of the form $f'(y)\frac{dy}{dx} + Pf(y) = Q$, where P and Q are functions, can be reduced to linear by substituting f(y) = t so that

$f'(y)\frac{dy}{dx} = \frac{dt}{dx}$ and the original equation is reduced to the linear form in variable 't' as :-

$$\frac{dt}{dx} = Pt + Q$$

II. Exact Differential Equation

The equation M dx + N dy = 0 (where M and N are functions of x and y), is said to be exact if M dx + N dy is the exact differential of a function of x and y, i.e., if

M dx + N dy = du, where u is a function of x and y.

For example : The differential equation sin x cos y dy + cos x sin y dx = 0 is an exact differential equation as

sin x cos y dy + cos x sin y dx = d (sin x cos y).

Art 1 : Find the necessary and sufficient condition that the equation M dx + N dy=0 (where M and N are functions of x and y with the condition that M, N, $\frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ are continuous functions of x and y) may be exact.

Proof : (i) Necessary Condition

Assume M dx + N dy = 0 is exact.

$\therefore$ M dx + N dy = du, where u is function of x and y.

But $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

$$\therefore \quad M\,dx + N\,dy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Equating coeffs. of dx and dy on both sides, $M = \frac{\partial u}{\partial x}$ and $N = \frac{\partial u}{\partial y}$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}, \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}$$

But $\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}$ $\left[\because \frac{\partial^2 u}{\partial x\,\partial y} \text{ and } \frac{\partial^2 u}{\partial y\,\partial x} \text{ are given to be continuous}\right]$

$\therefore \quad \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, which is the required necessary condition.

(ii) Condition is sufficient

Assume that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

We have to prove that M dx + N dy = 0 is exact.

Let $\int M\,dx = u$ ... (1)

where integration is performed on the supposition that y is constant.

$$\frac{\partial}{\partial x}\left[\int M\,dx\right] = \frac{\partial u}{\partial x}, \quad \text{or} \quad M = \frac{\partial u}{\partial x} \qquad \text{... (2)}$$

Also $\dfrac{\partial M}{\partial y} = \dfrac{\partial^2 u}{\partial y\,\partial x}$ ... (3)

But $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ (given) and $\dfrac{\partial^2 u}{\partial y\,\partial x} = \dfrac{\partial^2 u}{\partial x\,\partial y}$ (Assumption)

$\therefore \quad$ from (3), $\dfrac{\partial N}{\partial x} = \dfrac{\partial^2 u}{\partial x\,\partial y}$ or $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right)$

Integrating both sides w.r.t. x, regarding y as constant,

$N = \dfrac{\partial u}{\partial y} + f(y)$, (say) ... (4)

From (2) and (4), we get,

$$M\,dx + N\,dy = \frac{\partial u}{\partial x}dx + \left[\frac{\partial u}{\partial y} + f(y)\right]dy = \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + f(y)\,dy$$

$\therefore \quad$ M dx + N dy = du + f (y) dy ... (5)
which is an exact differential

[$\because$ f(y) dy is an exact differential as f(y) dy = d $\{\int f(y)\,dy\}$]

$\therefore \quad$ M dx + N dy = 0 is exact.

$\therefore \quad$ condition is sufficient.

Cor. If the condition is satisfied, solve the equation M dx + N dy = 0

Proof : The given equation is Mdx + Ndy = 0

or du + f (y) dy = 0 [$\because$ of (5)]

Integrating both sides, we get,

$$u + \int f(y)\, dy = c \quad \dots (6)$$

But $u = \int_{y\ \text{constant}} M\, dx$ [$\because$ of (1)]

and f (y) = terms in N not containing x

$\therefore$ from (6), we get, [$\because$ of (4)]

$$\int_{y\ \text{constant}} M\, dx + \int (\text{terms in N not containing x})\, dy = c$$

which is the required solution.

Example 6 : Show that the differential equation $2x \sin 3y\, dx + 3x^2 \cos 3y\, dy = 0$ is exact and hence solve it.

Sol. The given differential equation is $2x \sin 3y\, dx + 3x^2 \cos 3y\, dy = 0$

Comparing it with M dx + N dy = 0, we get,

$M = 2x \sin 3y,\ N = 3x^2 \cos 3y$

Now $\frac{\partial M}{\partial y} = 6x \cos 3y$ and $\frac{\partial N}{\partial x} = 6x \cos 3y$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ given equation is exact and its solution is

$$\int_{y\ \text{constant}} M\, dx + \int (\text{terms in N not containing x})\, dy = c$$

or $2 (\sin 3y) \int x\, dx + 0 = c$

Integrating Factor :

In case of linear differential equation, we have noticed a term integrating factor. Now, we define that term as :

Def. : An integrating factor (abbreviated I.F.) of a differential equation is a factor such that if the equation is multiplied by it, the resulting equation is exact.

Note : 1. The number of integrating factors of the equation Mdx + Ndy = 0 is infinite.

2. The integrating factors can be judged sometimes by insepction otherwise by the specific rules, as discussed below :

Integrating Factors by Inspection

	Group of terms	I.F.	Exact differential
1.	x dy – y dx	$\frac{1}{x^2}$	$d\left(\frac{y}{x}\right)$
		$\frac{1}{y^2}$	$d\left(-\frac{x}{y}\right)$
		$\frac{1}{xy}$	$d\left[\log\left\lvert\frac{y}{x}\right\rvert\right]$
		$\frac{1}{x^2+y^2}$	$d\left[\tan^{-1}\frac{y}{x}\right]$
2.	x dy + y dx	$\frac{1}{(xy)^n}$	$\begin{cases} d\left[\frac{-1}{(n-1)(xy)^{n-1}}\right], n \neq 1 \\ d(\log\lvert xy\rvert), n = 1 \end{cases}$
3.	x dx + y dy	$\frac{1}{(x^2+y^2)^n}$	$\begin{cases} d\left[\frac{-1}{2(n-1)(x^2+y^2)^{n-1}}\right], n \neq 1 \\ d\left[\frac{1}{2}\log(x^2+y^2)\right], n = 1 \end{cases}$

Five Rules for Finding Integrating Factors :

Rule I : If the equation M dx + N dy = 0 is homogeneous in x and y i.e., if M and N are homogeneous functions of the same degree in x and y, then $\frac{1}{Mx+Ny}$ is an I.F. provided Mx + Ny ≠ 0.

Note : 1. This method is suitable when Mx + Ny consists of only one term. Otherwise it is better to put y = vx.

2. If Mx + Ny = 0, then this method fails and the solution is given by $\therefore \left\lvert\frac{y}{x}\right\rvert = c$ or $\lvert y\rvert = c\lvert x\rvert$.

Rule II : If an equation M dx + N dy = 0 is of the form

$f_1(x\,y)\,y\,dx + f_2(x\,y)\,x\,dy = 0$, then $\dfrac{1}{Mx - Ny}$ is an I.F. provided $Mx - Ny \neq 0$.

Note : This rule fails if M x – Ny = 0 and the solution is given by $|xy| = c$.

Rule III : If in an equation M dx + N dy = 0, $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$ is a function of x only say f(x),

then $e^{\int f(x)\,dx}$ is an I.F.

Rule IV : If in an equation M dx + N dy = 0, $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$ is a function of y only say f(y),

then $e^{\int f(y)\,dy}$ is an I.F.

Rule V : If an equation is

$x^a y^b (my\,dx + nx\,dy) + x^{a'} y^{b'} (m'\,y\,dx + n'x\,dy) = 0$, then $x^h y^k$ is an I.F. where

$$\frac{a+h+1}{m} = \frac{b+k+1}{n}, \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}.$$

Example 3 : Solve $y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0$.

Sol. The given differential equation is

$$y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0 \qquad \ldots (1)$$

which is of the form $f_1(x\,y)\,y\,dx + f_2(x\,y)\,x\,dy = 0$

$\therefore$ comparing with M dx + N dy = 0, we get,

$M = y(xy + 2x^2y^2)$, $N = x(xy - x^2y^2)$

$$\therefore \quad \text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{xy\,(xy + 2x^2y^2 - xy + x^2y^2)} = \frac{1}{3x^3y^3}$$

$\therefore$ multiplying both sides of (1) by $\dfrac{1}{3x^3y^3}$, we get,

$$\frac{1+2xy}{3x^2y}\,dx + \frac{1-xy}{3xy^2}\,dy = 0$$

which is exact and its solution is

$$\frac{1}{3y}\int \frac{1}{x^2}\,dx + \frac{2}{3}\int \frac{1}{x}\,dx + \frac{1}{3}\int -\frac{1}{y}\,dy = c$$

or $$\frac{-1}{3xy} + \frac{2}{3}\log|x| - \frac{1}{3}\log|y| = c$$

or $$\frac{-1}{xy} + \log\frac{x^2}{|y|} = 3c = C, \text{ say}.$$

Example 4 : Solve the differential equation $(x + 2y^3)\dfrac{dy}{dx} = y$.

Sol. Given differential equation is

$$(x+2y^3)\frac{dy}{dx} = y \quad \text{or } (x+2y^3)\,dy = y\,dx$$

or $y\,dx - (x + 2y^3)\,dy = 0$ ... (1)

Comparing (1) with $M\,dx + N\,dy = 0$, we get, $M = y$, $N = -(x + 2y^3)$

$$\frac{\partial M}{\partial y} = 1, \frac{\partial N}{\partial x} = -1$$

Now $$\frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = \frac{-1-1}{y} = \frac{-2}{y} = f(y)$$

$\therefore$ $$\text{I.F.} = e^{\int f(y)\,dy} = e^{\int \frac{-2}{y}dy} = e^{-2\log y} = e^{\log y^{-2}} = y^{-2} = \frac{1}{y^2}$$

Multiplying both sides of (1) by $\dfrac{1}{y^2}$, we get

$$\frac{1}{y}dx - \left(\frac{x}{y^2} + 2y\right)dy = 0$$ which is exact and its solution is

$$\int_{y\ \text{constant}} \frac{1}{y}\,dx - \int 2y\,dx = c$$

or $\frac{x}{y} - y^2 = c$ or $x - y^3 = cy$

which is the required solution.

III. Self Check Exercise

1. $(x + y^3)\frac{dy}{dx} = y$

2. $(x+1)\frac{dy}{dx} + 1 = e^{x-y}$

3. Solve $x\,dx + y\,dy \frac{a^2(xdy - ydx)}{x^2 + y^2}$

4. Solve the differential equation y (x + y + 1) dx + x (x + 3y + 2) dy = 0, y>0

5. Find the integrating factor and hence solve

$$\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right)dx + \frac{1}{4}(x + xy^2)\,dy = 0.$$

IV. Suggested Readings :

1. R.K. Jain, S.R.K. Lyengar, Advanced Engineering Mathematics, Narosa Publishing House.
2. Rai Singhania : Ordinary and Partial Differential Equations, S. Chand & Company, New Delhi.
3. Zafar Ahsan, Differential Equations and their Applications, Prentice-Hall of India Pvt. Ltd., New Delhi - 2nd Ed.

LESSON NO. 1.3 Author : Dr. Chanchal

LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH CONSTANT COEFFICIENTS

Structure :

Objectives

After studying the linear differential equations of first order, in this lesson, we will learn to find out the solutions of linear differential equations of order more than one. Such equations are of two types : the one with constant coefficients and the other with vairable coefficients. In this lesson, we focus on the methods for finding solutions of linear differential equations of higher order with constant coefficients.

I. Introduction

In this lesson, we will be studying in detail about the solutions of linear homogeneous and non-homogeneous equations of higher order with constant coefficients. Firstly, we introduce these equations as :-

The non-homogeneous linear differential equation of order n with constant coefficients is

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + P_n y = Q \quad \ldots (1)$$

where $P_0, P_1, P_2, \ldots, P_n$ are constants and Q is a function of x. Also $P_0 \neq 0$.

The corresponding homogeneous linear differential equation is

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + P_n y = 0 \cdot \text{ (Here, } = 0)$$

II. Solution of Homogeneous Linear Equation with Constant Coefficients

The given differential equation is

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + P_n y = 0,$$ where P_1, P_2, ..., P_n are real constants, $P_0 \neq 0$.

Step 1. Write the equation in the symbolic form (S.F.)

$$(P_0 D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots + P_n)\, y = 0$$

$$\left(\text{By putting } \frac{d}{dx} = D, \frac{d^2}{dx^2} = D^2 \ldots, \frac{d^n}{dx^n} = D^n\right)$$

Step 2. Write down the auxiliary equation (A. E.) as

$$P_0 D^n + P_1 D^{n-1} + \ldots + P_n = 0$$

[By equating to zero the symbolic coeff. of y]

and solve it for D as it is an ordinary algebraic quantity.

Step 3. From the roots of the A.E., write down the corresponding part of the complete solution (C.S.) as follows :

		Roots of A.E.	**Corresponding part of C.S.**
(a)	(i)	Two real and different roots m_1, m_2	$c_1 e^{m_1 x} + c_2 e^{m_2 x}$
	(ii)	Two real and equal roots m_1, m_1	$e^{m_1 x}(c_1 + c_2 x)$
	(iii)	Three real and equal roots m_1, m_1, m_1. α and so on.	$e^{\alpha x}(c_1 + c_2 x + c_3 x^2)$ αx
(b)	(i)	One pair of complex and different roots α ± iβ	$e^{ax}(c_1 \cos\beta x + c_2 \sin\beta x)$
	(ii)	Two pairs of complex and equal roots α ± iβ, α ± i β. and so on	$e^{ax}[(c_1+c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x]$

Example 1 : Solve $(D^2 + 1)^3 (D^2 + D + 1)^2 y = 0$.

Sol. The given differential equation is $(D^2 + 1)^3 (D^2 + D + 1)^2 y = 0$

The A.E. is $(D^2 + 1)^3 (D^2 + D + 1)^2 = 0$

Either $(D^2 + 1)^3 = 0$ or $(D^2 + D + 1)^2 = 0$

$\therefore D^2 = -1, -1, -1$ or $D = \frac{-1 \pm \sqrt{1-4}}{2}, \frac{-1 \pm \sqrt{1-4}}{2}$

$\therefore D = \pm i, \pm i, \pm i$ or $D = \frac{-1 \pm i\sqrt{3}}{2}, \frac{-1 \pm i\sqrt{3}}{2}$

$\therefore$ we have $D = 0 \pm i, 0 \pm i, 0 \pm i, -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i, -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$\therefore$ the C.S. is

$$y = e^{0x}[(c_1 + c_2 x + c_3 x^2) \cos x + (c_4 + c_5 x + c_6 x^2) \sin x]$$
$$+ e^{-\frac{x}{2}} \left[(c_7 + c_8 x) \cos \frac{\sqrt{3}}{2} x + (c_9 + c_{10} x) \sin \frac{\sqrt{3}}{2} x \right]$$

or

$$y = (c_1 + c_2 x + c_3 x^2) \cos x + (c_4 + c_5 x + c_6 x^2) \sin x$$
$$+ e^{-\frac{x}{2}} \left[(c_7 + c_8 x) \cos \frac{\sqrt{3}}{2} x + (c_9 + c_{10} x) \sin \frac{\sqrt{3}}{2} x \right].$$

III. Solution of Non-Homogeneous Linear Equation with Constant Coefficients

The given differential equation is

$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + P_n y = \theta,$ where $P_0, P_1, P_2, \ldots, P_n$ are constants, and Q is a function of x.

Step 1. Write the equation in the S.F.

$$(P_0 D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots + P_n) y = Q$$

Step 2. Write down the A.E.

$P_0 D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots + P_n = 0$, and solve it for D.

Step 3. From the roots of the A.E., write down the corresponding part of the C.F. by the same rule by which we write the C.S. if the R.H.S. of the given equation were zero, instead of Q.

Step 4. Find the particular integral given by, $P.I. = \frac{1}{f(D)} Q$

The methods to find P.I. are discussed in the succeding part.

Step 5. The C.S. is y = C.F. + P.I.

Note : 1. $\frac{1}{f(D)}$ is the inverse of the operater f(D)

2. $\frac{1}{f(D)}$ Q is the particular integral of the equation f(D) y = Q.

3. $\frac{1}{D}Q = \int Q\,dx,$ no arbitrary constants being added.

4. $\frac{1}{D-a}Q = e^{ax}\int Qe^{-ax}\,dx,$ no arbitrary constant being added.

also, $\frac{1}{(D-a)^2}e^{ax} = \frac{x^2}{2}e^{ax}$ and $\frac{1}{(D-a)^n}e^{ax} = \frac{x^n}{\underline{|n}}e^{ax}.$

III.(a) Five Rules for Finding Particular Integrals

Rule I : Rule to evaluate $\frac{1}{f(D)}e^{ax}, f(a) \neq 0$

1. Put D = a and we get $\frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)}$ provided f(a) ≠ 0

2. If f (a) = 0, then there is case of failure and in that case

$$\frac{1}{f(D)}e^{ax} = x\frac{1}{\frac{d}{dD}[f(D)]}e^{ax}.$$

Note : If by using the above rule, we again get zero in the denominator, we repeat the rule and so on.

Rule II : Rule to evaluate $\frac{1}{f(D^2)}\cos ax, \frac{1}{f(D^2)}\sin ax$

1. Put $D^2 = -a^2$ and we get,

$$\frac{1}{f(D^2)}\cos ax \text{ (or } \sin ax) = \frac{1}{f(-a^2)}\cos ax \text{ (or } \sin ax) \text{ provided } f(-a^2) \neq 0$$

2. If f $(-a^2) = 0$, then case of failure and we have,

$$\frac{1}{f(D^2)} \cos ax \text{ (or } \sin ax) = x \frac{1}{\frac{d}{dD}\left[f(D^2)\right]} \cos ax \text{ (or } \sin ax)$$

Rule III : Rule to evaluate $\frac{1}{f(D)} x^m$, where m is a positive integer.

1. To evaluate $\frac{1}{f(D)} x^m$, we resolve $\frac{1}{f(D)}$ into partial fractions and then expand each partial fraction in ascending powers of D.

For example : Consider the partial fraction $\frac{1}{D-a} x^m$ we can expand it as :-

$$\frac{1}{D-a} x^m = \frac{1}{-a+D} x^m = \frac{1}{-a\left(1-\frac{D}{a}\right)} x^m = -\frac{1}{a}\left(1-\frac{D}{a}\right)^{-1} x^m$$

$$= -\frac{1}{a}\left[1 + \frac{D}{a} + \frac{D^2}{a^2} + \ldots + \frac{D^m}{a^m} + \frac{D^{m+1}}{a^{m+1}} + \ldots\right] x^m$$

$$= -\frac{1}{a}\left[x^m + \frac{1}{a} m x^{m-1} + \frac{1}{a^2} m(m-1) x^{m-2} + \ldots + \frac{1}{a^m} \underline{|m} + 0\right]$$

$$= -\frac{1}{a}\left[x^m + \frac{m}{a} x^{m-1} + \frac{m(m-1)}{a^2} x^{m-2} + \ldots + \frac{\underline{|m}}{a^m}\right].$$

Example 2 : Solve the differential equation :

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y - 8(x^2 + e^{2x} + \sin 2x).$$

Sol. The given equation is $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y - 8(x^2 + e^{2x} + \sin 2x)$

or, in S.F., $(D^2 - 4D + 4)\, y = 8\,(x^2 + e^{2x} + \sin 2x)$

The A.E. is $D^2 - 4D + 4 = 0$, or $(D-2)^2 = 0$

$\therefore$ D = 2, 2

C.F. = $(c_1 + c_2x)\ e^{2x}$

$$\text{P.I.} = 8\left[\frac{1}{D^2 - 4D + 4}x^2 + \frac{1}{D^2 - 4D + 4}e^{2x} + \frac{1}{D^2 - 4D + 4}\sin 2x\right] \quad \dots (1)$$

Now $\dfrac{1}{D^2 - 4D + 4}x^2 = \dfrac{1}{4}\cdot\dfrac{1}{\left(1 - \dfrac{D}{2}\right)^2}x^2 = \dfrac{1}{4}\left(1 - \dfrac{D}{2}\right)^{-2}x^2$

$$= \frac{1}{4}\left[1 + 2\,.\,\frac{D}{2} + 3\,.\,\frac{D^2}{4} + \dots\right]x^2 = \frac{1}{4}\left[x^2 + Dx^2 + \frac{3}{4}D^2x^2 + \dots\right]$$

$$= \frac{1}{4}\left[x^2 + 2x + \frac{3}{2}\right] = \frac{1}{8}(2x^2 + 4x + 3)$$

and $\dfrac{1}{D^2 - 4D + 4}e^{2x} = \dfrac{1}{2^2 - 4(2) + 4}e^{2x} = \dfrac{1}{0}e^{2x}$

∴ the rule fails

∴ $\dfrac{1}{D^2 - 4D + 4}e^{2x} = x\,.\,\dfrac{1}{2D - 4}e^{2x} = x\,.\,\dfrac{1}{2(2) - 4}e^{2x} = x\,.\,\dfrac{1}{0}e^{2x}$

The rule fails again.

∴ $\dfrac{1}{D^2 - 4D + 4} = x^2\,.\,\dfrac{1}{2}e^{2x} = \dfrac{x^2e^{2x}}{2}$

and $\dfrac{1}{D^2 - 4D + 4}\sin 2x = \dfrac{1}{-4 - 4D + 4}\sin 2x = -\dfrac{1}{4}\,.\,\dfrac{1}{D}\sin 2x$

$$= \frac{1}{4}\,.\,\frac{\cos 2x}{2} = \frac{1}{8}\cos 2x$$

∴ from (1), P.I. = $8\left[\dfrac{1}{8}(2x^2 + 4x + 3) + \dfrac{x^2e^{2x}}{2} + \dfrac{1}{8}\cos 2x\right]$

$= 2x^2 + 4x + 3 + 4x^2e^{2x} + \cos 2x$

∴ C.S. = $(c_1 + c_2x)\,e^{2x} + 2x^2 + 4x + 3 + 4x^2e^{2x} + \cos 2x$.

Rule IV : Rule to evaluate $\dfrac{1}{f(D)}\left(e^{ax}\,V\right)$, where V is any function of x.

$$\frac{1}{f(D)}\left(e^{ax}\ V\right) = e^{ax}\ \frac{1}{f(D+a)}\ V$$

Example 3 : Solve the differential equation : $(D^2 - 4D + 4)\ y = e^{2x}\cos^2 x$.

Sol. The given equation in S.F. is $(D^2 - 4D + 4)\ y = e^{2x}\cos^2 x$

The A.E. is $D^2 - 4D + 4 = 0$, or $(D - 2)^2 = 0$

$\therefore$ $D = 2, 2$

C.F. $= (c_1 + c_2x)\ e^{2x}$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 4}\ e^{2x}\cos^2 x = e^{2x}\ .\ \frac{1}{(D+2)^2 - 4\,(D+2) + 4}(\cos^2 x)$$

$$= e^{2x}.\ \frac{1}{D^2}\left(\frac{1+\cos 2x}{2}\right) = \frac{1}{2}\ .\ e^{2x}\left[\frac{1}{D^2} + \frac{1}{D^2}\cos 2x\right]$$

$$= \frac{1}{2}\ e^{2x}\left[\frac{x^2}{2} + \frac{\cos 2x}{-4}\right] = \frac{1}{8}\ e^{2x}\ (2x^2 - \cos 2x)$$

$\therefore$ C.S. is, $y = (c_1 + c_2x)\ e^{2x} + \dfrac{1}{8}e^{2x}\ (2x^2 - \cos 2x)$.

Rule V : Rule to evaluate $\dfrac{1}{f(D)}(xV)$, where V is any function of x.

$$\frac{1}{f(D)}\ (xV) = x\ \frac{1}{f(D)}\ V + \frac{d}{dD}\left[\frac{1}{f(D)}\right] V$$

Example 4 : Solve the differential equation

$$\frac{d^2y}{dx^2} - 2\ \frac{dy}{dx} + y = xe^x \sin x$$

Sol. The given equation is $\dfrac{d^2y}{dx^2} - 2\ \dfrac{dy}{dx} + y = xe^x \sin x$

or, in S.F. $(D^2 - 2D + 1)\ y = x\ e^x \sin x$

$\therefore$ the A.E. is $D^2 - 2D + 1 = 0$, or $D = 1, 1$

$\therefore$ C.F. $= (c_1 + c_2x)\ e^x$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1} xe^x \sin x = \frac{1}{(D-1)^2} xe^x \sin x$$

$$= e^x \frac{1}{(D+1-1)^2} (x \sin x) = e^x \cdot \frac{1}{D^2} (x \sin x)$$

$$= e^x \left[x \cdot \frac{1}{D^2} \sin x + \frac{d}{dD}\left(\frac{1}{D^2}\right) \sin x \right]$$

$$= e^x \left[x \frac{\sin x}{-1} - \frac{2}{D^3} \sin x \right] = e^x \left[-x \sin x - \frac{2}{D(-1)} \sin x \right]$$

$= e^x [-x \sin x + 2 (-\cos x)] = e^x (-x \sin x - 2 \cos x)$

$= -e^x (x \sin x + 2 \cos x)$

$\therefore$ C.S. is, $y = (c_1 + c_2 x) e^x - e^x (x \sin x + 2 \cos x)$.

Example 5 : Solve the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \sin e^{-x}$.

Sol. The given equation is $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \sin e^{-x}$

or in S.F., $(D^2 - 3D + 2)\, y = \sin e^{-x}$

The A.E. is $D^2 - 3D + 2 = 0$, or $(D-1)(D-2) = 0$

$\therefore$ $D = 1, 2$

$\therefore$ C.F. $= c_1 e^x + c_2 e^{2x}$

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2} \sin e^{-x} = \frac{1}{(D-1)(D-2)} \sin e^{-x}$$

$$= \frac{1}{D-2} e^x \int (\sin e^{-x})\, e^{-x} dx \qquad \left[\frac{1}{D-\alpha} V = e^{\alpha x} \int V e^{-\alpha x} dx \right]$$

$$= \frac{1}{D-2} (e^x \cos e^{-x}) = e^{2x} \int e^x \cos e^{-x} . e^{-2x} dx$$

$$= e^{2x} \int e^{-x} \cos e^{-x} dx = -e^{2x} \sin e^{-x}$$

$\therefore$ C.S. is, $y = c_1 e^x + c_2 e^{2x} - e^{2x} \sin e^{-x}$.

IV. Method of Variation of Parameters

Let any given linear equation of 2nd order be

$$P_0y'' + P_1y' + P_2y = Q \qquad \text{... (1)}$$

where P_0 ($\neq 0$), P_1, P_2 are constants and Q is a function of x.

The corresponding homogeneous equation is

$$P_0y'' + P_1y' + P_2y = 0 \qquad \text{... (2)}$$

Let $y_c = c_1y_1 + c_2y_2$ be the general solution of (2) and therefore, the complementary solution of (1), where y_1, y_2 are L.I. functions of x over an open interval I.

Now we try to find a particular solution of (1) by considering

$$y = A(x)\, y_1 + B(x)\, y_2 \qquad \text{... (3)}$$

and determine the functions A and B so that (3) is a solution of (1).

Differentiating (3) w.r.t. x, we get,

$$y' = Ay_1' + By_2' + A'y_1 + B'y_2 \qquad \text{... (4)}$$

Choose $A'y_1 + B'y_2 = 0$ ... (5)

$\therefore$ (4) becomes, $y' = Ay_1' + By_2'$ ... (6)

Differentiating (6) w.r.t. x, we get,

$$y'' = Ay_1'' + By_2'' + A'\, y_1' + B'\, y_2' \qquad \text{... (7)}$$

Substituting the values of y,y', y" from (3), (6) and (7) in (1), we get,

$$P_0\,(A\, y_1'' + By_2'' + A'y_1' + B'y_2') + P_1\,(Ay_1' + By_2') + P_2\,(Ay_1 + By_2) = Q$$

or $$A\,(P_0y_1'' + P_1y_1' + P_2y_1) + B\,(P_0\, y_2'' + P_1y_2' + P_2y_2) + P_0\,(A'\, y_1' + B'\, y_2') = Q \qquad \text{... (8)}$$

$\because$ y_1, y_2 are solutions of (2)

$$\left.\begin{array}{l}\therefore P_0y_1'' + P_1y_1' + P_2y_1 = 0 \\ \text{and } P_0y_2'' + P_1y_2' + P_2y_2 = 0\end{array}\right] \qquad \text{... (9)}$$

Using (9), equation (8) becomes

$$P_0\,(A'\, y_1' + B'y_2') = Q$$

or $$A'y_1' + B'y_2' = \frac{1}{P_0}\,Q \qquad \text{... (10)}$$

The equation (5) and (10) will give us values of A' and B'

if $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$ and this is true as

$W(y_1, y_2) \neq 0$ due to the fact that y_1, y_2 are L.I. over I.

Now when the values of A', B' are known then with the use of integration, we can determine A and B.

Now with A and B determined, (4) gives us a particular solution of (1) and hence we can find the general solution of (1).

Example 6 : Solve $\frac{d^2y}{dx^2} + a^2y = \tan ax,$ using method of variation of parameters.

Sol. The given equation is $\frac{d^2y}{dx^2} + a^2y = \tan ax$

or $(D^2 + a^2)\, y = \tan ax$ … (1)

The corresponding homogeneous equation is $(D^2 + a^2)\, y = 0$ … (2)

Its A.E. is $D^2 + a^2 = 0$ or $D^2 = -a^2$

$\therefore$ $D = \pm ai = 0 \pm ai$

$\therefore$ the complementary solution of (1) is

$y_c = c_1 \cos ax + c_2 \sin ax$

Now we seek a particular solution of (1) by variation of parameters.

Let $y = A \cos ax + B \sin ax$ … (3)

Differentiating (3) w.r.t. x, we get,

$y' = A' \cos ax + B' \sin ax - A\, a \sin ax + B\, a \cos ax$ … (4)

Choose $A' \cos ax + B' \sin ax = 0$ … (5)

$\therefore$ (4) becomes, $y' = -A\, a \sin ax + B\, a \cos ax$ … (6)

Differentiating w.r.t. x, we get

$y'' = -A'\, a \sin ax + B'\, a \cos ax - Aa^2 \cos ax - Ba^2 \sin ax$ … (7)

Substituting the values of y, y'' from (3) and (7) in (1), we get

$-A'\, a \sin ax + B'\, a \cos ax - Aa^2 \cos ax - B\, a^2 \sin ax$

$+Aa^2 \cos ax + Ba^2 \sin ax = \tan ax$

… (8)

or $-A'\, a \sin ax + B'\, a \cos ax = \tan ax$

Now we try to find values of A' and B' from (5) and (8).

Multiplying (5) by asin ax and (8) by cos ax and adding, we get

$B'\, a\, (\sin^2 ax + \cos^2 ax) = \tan ax \cos ax$

or $B'\, a = \sin ax$ or $B' = \frac{1}{a} \sin ax$

Also from (5), $A' = -\frac{B' \sin ax}{\cos ax} = -\frac{1}{a} \frac{\sin^2 ax}{\cos ax} = -\frac{1}{a} \left(\frac{1 - \cos^2 ax}{\cos ax} \right)$

$= -\frac{1}{a} (\sec ax - \cos ax) = \frac{1}{a} \cos ax - \frac{1}{a} \sec ax$

$\therefore$ Integrating w.r.t. x, we get,

$$A = \frac{1}{a}\frac{\sin ax}{a} - \frac{1}{a^2}\log\left|\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right| = \frac{1}{a^2}\sin ax - \frac{1}{a^2}\log\left|\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right|$$

Also $B' = \frac{1}{a}\sin ax \Rightarrow B = -\frac{1}{a^2}\cos ax$

Putting values of A and B in (3), we get,

$$y = \left[\frac{1}{a^2}\sin ax - \frac{1}{a^2}\log\left|\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right|\right]\cos ax - \frac{1}{a^2}\cos ax \sin ax$$

which is particular solution of (1)

∴ General solution of (1) is

$y = c_1 \cos ax + c_2 \sin ax$

$$+\left[\frac{1}{a^2}\sin ax - \frac{1}{a^2}\log\left|\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right|\right]\cos ax - \frac{1}{a^2}\cos ax \sin ax \cdot$$

V. Method of Undetermined Coefficients

This method is used for finding the P.I. of a linear differential equation F(D) y = X, where X contains terms in some special forms, as tabulated below

S.No.		Special Form of X	Trial Solution y for P.I.
1.		x^n or $a_n X^n$	$A_0 + A_1x + \ldots\ldots\ldots + A_n x^n$
	or	$a_0+a_1x + a_2x^2 + \ldots. + a_nx^n$	
2.		e^{ax} or pe^{ax} ....Ae^{ax}	
3.		$a_n\, x^n\, e^{ax}$	$e^{ax}(A_0A_1x + A_2x^2 + \ldots..+ A_nx^n)$
	or	$e^{ax}(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n)$	
4.		p sin ax or q cos ax	A sin ax + B cos ax
	or	p sin ax + q cos ax	
5.		pe^{bx} sin ax or qe^{bx} cos ax	e^{bx} (A sin ax + B cos ax)
	or	e^{bx} (p sin ax + q cos ax)	
6.		x^n sin ax or a_nx^n sin ax	$(A_0 + A_1x + \ldots..+ A_nx^n)$ sin ax +
	or	x^n cos ax	$(A'_0 + A'_1 X + \ldots. + A'_nx^n)$ cos ax
	or	a_nx^n cos ax	
	or	$(a_0 + a_1x + \ldots.. + a_nx^n)$cos ax	
	or	$(a_0 + a_1x + \ldots.. + a_nx^n)$ sin ax	

Note. In the above table, n is a positive integer and a_0, a_1 .... a_n, p, q, a, b, A_0, A_1, ... A_n, A'_0, A'_1, ... A'_n are constants. The constants occuring in second column are known and the constants occurding in third column are determined by substituting the trial solution in the given equation i.e., from the identity F(D) y = X.

Example 7 : Solve $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+4y=2x^2+3e^{-x}$ by method of undetermined coefficients.

Sol. The given differential equation is $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+4y=2x^2+3e^{-x}$ ... (1)

In S.F., $(D^2 + 2D + 4)\, y = 2x^2 + 3e^{-x}$

The A.E. is $D^2 + 2D + 4 = 0$

or $D=\frac{-2\pm\sqrt{4-16}}{2}=\frac{-2\pm i2\sqrt{3}}{2}=-1\pm i\sqrt{3}$

$\therefore$ C.F. $= e^{-x}\,(c_1\cos\sqrt{3x}+c_2\sin\sqrt{3x})$

None of the terms in $X = 2x^2 + 3e^{-x}$ is present in C.F.

$\therefore$ the trial solutions corresponding to x^2 is $A_0 + A_1x + A_2x^2$ and to e^{-x} is $A_3\, e^{-x}$ respectively.

$\therefore$ trial solution for P.I. is

$y = (A_0 + A_1x + A_2x^2) + A_3\, e^{-x}$

$\frac{dy}{dx}=A_1+2A_2x-A_3e^{-x},\ \frac{d^2y}{dx^2}=2A_2+A_3e^{-x}$

Putting the value of $\frac{d^2y}{dx^2},\frac{dy}{dx}, y$ in (1), we get

$2A_2 + A_3e^{-x} + 2A_1 + 4A_2x - 2A_3e^{-x} + 4A_0 + 4A_1x + 4A_2x^2 + 4A_3e^{-x} = 2x^2 + 3e^{-x}$

Equating coefficients of like terms on both sides, we have

Coefficient of e^{-x} ; $A_3 - 2A_3 + 4A_3 = 3 \Rightarrow A_3 = 1$

Coefficient of x^2 ; $4A_2 = 2 \Rightarrow A_2 = \frac{1}{2}$

Coefficient of x ; $4A_2 + 4A_1 = 0 \Rightarrow A_1 = -\frac{1}{2}$

Constant terms ; $2A_2 + 2A_1 + 4A_0 = 0 \Rightarrow A_0 = 0$

$\therefore$ P.I. $=0-\frac{1}{2}x+\frac{1}{2}x^2+e^{-x}=-\frac{1}{2}x+\frac{1}{2}x^2+e^{-x}$

$\therefore$ the C.S is given by

$$y = \text{C.F.} + \text{P.I.} = e^{-x}(c_1\cos\sqrt{3x}+c_2\sin\sqrt{3x})-\frac{1}{2}x+\frac{1}{2}x^2+e^{-x}.$$

VI. Self Check Exercise

1. Solve $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = 0$
2. Solve $(D^3 + 1)\, y = 3 + e^{-x}$.
3. Solve $(D^2 + D + 1)\, y = (1 + \sin x)^2$
4. Solve $(D^2 - 4D + 4)\, y = x^2 + e^x + \cos 2x$
5. Solve $(D^4 - 1)\, y = x \sin x$
6. Solve $(D^2 + 3D + 2)\, y = \sin e^x$ by the method of variation of parameters.

VII. Suggested Readings :

1. R.K. Jain, S.R.K. Lyengar, Advanced Engineering Mathematics, Narosa Publishing House.
2. Rai Singhania : Ordinary and Partial Differential Equations, S. Chand & Company, New Delhi.
3. Zafar Ahsan, Differential Equations and their Applications, Prentice-Hall of India Pvt. Ltd., New Delhi - 2nd Ed.

LESSON NO. 1.4 **Author : Dr. Chanchal**

LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH VARIABLE COEFFICIENTS

Structure :

Objectives

In continuation with the previous lesson no. 2, in this lesson, we will study the methods for finding the solutions of linear differential equations of higher order with variable coefficients.

I. Introduction

An equation of the form

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots P_n y = Q$$

where $P_0, P_1, P_2, \dots, P_n$ and Q are functions of x, is called a linear differential equation with variable coefficients.

In this lesson, we will discuss various methods of solving some well known linear differential equations with variable coefficients such as Cauchy's linear equation, Legendre's linear equation and some other types of equations.

II. Cauchy's Linear Equation

A linear equation of the form

$$P_0 x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + P_n y = Q(x)$$

where P_0, P_1 ,.... , P_n are real constants and Q(x) is a function of x, is called Cauchy's linear equation.

Such an equation can be solved under the following rule :

Rule : Working rule to solve Cauchy's linear equation

Step 1. Put $x = e^z$ i.e., $z = \log x, x > 0$

Step 2. Put $\frac{d}{dz} = \theta$ so that

$$x D = \theta, x^2 D^2 = \theta(\theta - 1), \ldots, x^n D^n = \theta(\theta - 1)(\theta - 2)\ldots\left(\theta - \overline{n-1}\right)$$

Step 3. Putting these in the given equation, we get,

$\left[P_0 \theta(\theta - 1)\ldots\left(\theta - \overline{n-1}\right) + P_1 \theta(\theta - 1)\ldots\left(\theta - \overline{n-2}\right) + \ldots + P_n\right] y = Q(e^z)$ which is linear equation with constant coeffs and solve for y in terms of z.

Step 4. Put z = log x to get the required solution.

Example 1 : Solve : $x^3 \frac{d^3 y}{dx^3} + 3x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + 8y = 65 \sin(\log x)$

Sol. The given differential equation is

$$x^3 \frac{d^3 y}{dx^3} + 3x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + 8y = 65 \sin(\log x)$$

or, in S.F.

$$(x^3 D^3 + 3x^2 D^2 + xD + 8)\, y = 65 \sin(\log x) \qquad \ldots (1)$$

Putting $x = e^z$, or $z = \log x$, $x > 0$

and $x D = \theta$, $x^2 D^2 = \theta(\theta - 1)$ and $x^3 D^3 = \theta(\theta - 1)(\theta - 2)$

where $\frac{d}{dz} = \theta$, in (1), we get

$$[\theta(\theta - 1)(\theta - 2) + 3\theta(\theta - 1) + \theta + 8]\, y = 65 \sin z$$

or $\quad \theta^3 - 3\theta^2 + 2\theta + 3\theta^2 - 3\theta + \theta + 8)\, y = 65 \sin z$

or $\quad (\theta^3 + 8)\, y = 65 \sin z \qquad \ldots (2)$

A.E. is $\theta^3 + 8 = 0$

or $\quad (\theta + 2)(\theta^2 - 2\theta + 4) = 0$

$\therefore \quad \theta = -2, \dfrac{2 \pm \sqrt{4-16}}{2} = -2, \dfrac{2 \pm 2\,i\sqrt{3}}{2} = -2, 1 \pm \sqrt{3}\, i$

$\therefore \quad \text{C.F.} = c_1 e^{-2z} + e^z\,(c_2 \cos\sqrt{3}z + c_3 \sin\sqrt{3}z)$

and $\quad \text{P.I.} = \dfrac{1}{\theta^3 + 8}(65 \sin z) = 65 \,.\, \dfrac{1}{\theta(-1) + 8} \sin z \qquad \left[\because \theta^2 = -1^2\right]$

$= 65 \,.\, \dfrac{-\theta - 8}{(-\theta - 8)(-\theta + 8)} \sin z = 65 \dfrac{-\theta - 8}{\theta^2 - 65} \sin z$

$= 65 \,.\, \dfrac{1}{-1-64}(-\theta \sin z - 8 \sin z) = -(-\cos z - 8 \sin z) = 8 \sin z + \cos z$

$\therefore$ the general solution is

y = C.F. + P.I.

or $\quad y = c_1 e^{-2z} + e^z\,(c_2 \cos\sqrt{3}z + c_3 \sin\sqrt{3}z) + 8 \sin z + \cos z$

or $\quad y = c_1 x^{-2} + x\,[c_2 \cos(\sqrt{3}\log x) + c_2 \sin(\sqrt{3}\log x)]$

$+8 \sin(\log x) + \cos(\log x)$

$\left[\because e^z = x\right]$

III. Legendre's Linear Equation

A linear equation of the form

$$P_0 (a+bx)^n \frac{d^n y}{dx^n} + P_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + .. + P_n y = Q(x) \qquad \dots (1)$$

where P_0, P_1, P_2 ,...., P_n are real constants and Q(x) is a function of x, is called Legendre's linear equation.

Working method to solve Legendre's linear equation is given below :

Rule : Working rule to solve Legendre's linear equation

Step 1. Put $a + bx = e^z$, i.e., $z = \log(a + bx)$, $a + bx > 0$

Step 2. Put $\dfrac{d}{dx} = \theta$, so that

$$(a + bx) D = b\theta,\ (a + bx)^2 D^2 = b^2\theta(\theta - 1),\ \dots,\ (a + bx)^n D^n = b^n\theta(\theta - 1)\dots\left(\theta - \overline{n-1}\right)$$

Step 3. Putting in the given equation, we get,

$$\left[P_0 b^n \theta(\theta-1)\ldots\left(\theta-\overline{n-1}\right)+P_1 b^{n-1}\theta\,(\theta-1)\ldots\left(\theta-\overline{n-2}\right)+\ldots+P_n\right]y=Q\left(\frac{e^z-a}{b}\right)$$

which is a linear equation with constant coefficients and solve for y in terms of z.

Step 4. Put $z = \log(a + bx)$ to get the required solution.

Example 2 : Solve $(1 + x)^2 y_2 + (1 + x)\, y_1 = 2\cos[\log(1 + x)]$.

Sol. The given differential equation is

$$(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}=2\cos[\log(1+x)]$$

or, in S.F., $[(1 + x)^2 D^2 + (1 + x) D] = 2\cos[\log(1 + x)]$ ... (1)

Put $1 + x = e^z$, or $z = \log(1 + x)$, $x > -1$

and $(1 + x) D = \theta$, $(1 + x)^2 D^2 = \theta(\theta - 1)$ where $\frac{d}{dz}=\theta$

From (1), we get,

$[\theta(\theta - 1) + \theta]\, y = 2\cos z$ or $\theta^2 y = 2\cos z$

The A.E. is $\theta^2 = 0 \Rightarrow \theta = 0, 0$

$\therefore$ C.F. $= (c_1 + c_z z)\, e^{0z} = c_1 + c_2 z$

$$\text{P.I.}=\frac{1}{\theta^2}(2\cos z)=2\frac{1}{\theta^2}\cos z=2\frac{1}{-1}\cos z=-2\cos z$$

$\therefore$ C.S. is $y = c_1 + c_2 z - 2\cos z = c_1 + c_2\log(1 + x) - 2\cos[\log(1 + x)]$.

IV. Exact Equation

A differential equation $f\left(\frac{d^ny}{dx^n},\frac{d^{n-1}y}{dx^{n-1}},\ldots,\frac{dy}{dx},y\right)=Q(x)$ is said to be exact if it can be obtained, simply by differentiation, from an equation of the next lower order $F\left(\frac{d^{n-1}y}{dx^{n-1}},\frac{d^{n-2}y}{dx^{n-2}},\ldots,\frac{dy}{dx},y\right)=\int Q(x)\,dx+c$, where c is an arbitrary constant.

Further, the NASC that the differential equation

$$P_0\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots\ldots+P_ny=Q$$

where $P_0 (\neq 0)$, P_1, P_2, ... P_n and Q are functions of x, may be exact, is given by

$$P_n-P'_{n-1}+P''_{n-2}-P'''_{n-3}+\ldots\ldots+(-1)^{n-1}P_0^{(n)}=0$$

Working method to test exactness. Write the coeffs. P_0, P_1, P_2, ..., P_n, representing the missing coeffs (if any) by zero and operate upon them by – D as shown under.

$$-D\left|\begin{array}{llll} P_0 & P_1 & P_2 & P_n \\ & -P_0' & -P_1' + P_0'' & -P_{n-1}'\ P_{n-2}'' - \ldots + (-1)^n\ P_0^{(n)} \\ \hline P_0 & P_1 - P_0' & P_2 - P_1' + P_0'' & P_n - P_{n-1}'' + P_{n-2}'' - \ldots + (-1)^n\ P_0^{(n)} \end{array}\right.$$

The equation is exact iff remainder

$$P_n - P'_{n-1} + P''_{n-2} - \ldots + (-1)^n P_0^{(n)} = 0$$

And the first integral is

$$P_0 \frac{d^{n-1}y}{dx^{n-1}} + (P_1 - P_0') \frac{d^{n-2}y}{dx^{n-2}} + (P_2 - P_1' + P_0'') \frac{d^{n-3}y}{dx^{n-3}} + \ldots$$

$$+ \left\{P_{n-1} - P_{n-2}' + \ldots + (-1)^{n-1} P_0^{(n-1)}\right\} y = \int Q\ dx + c\ .$$

Example 3 : Solve $[(x^3 - 4x)\ D^3 + (9x^2 - 12)\ D^2 + 18x\ D + 6]\ y = 0$.

Sol. The given differential equation is

$$[(x^3 - 4x)\ D^3 + (9x^2 - 12)\ D^2 + 18x\ D + 6]\ y = 0$$

or $$(x^3 - 4x)\frac{d^3y}{dx^3} + (9x^2 - 12)\frac{d^2y}{dx^2} + 18x\frac{dy}{dx} + 6y = 0 \qquad \ldots (1)$$

$$-D\left|\begin{array}{llll} x^3 - 4x & 9x^2 - 12 & 18x & 6 \\ & -3x^2 + 4 & -12x & -6 \\ \hline x^3 - 4x & 6x^2 - 8 & 6x & \underline{|0} \end{array}\right.$$

Here remainder is zero. Therefore (1) is exact and its first solution is

$$(x^3 - 4x)\frac{d^2y}{dx^2} + (6x^2 - 8)\frac{dy}{dx} + 6xy = c_1 \qquad \ldots (2)$$

$$-D\left|\begin{array}{lll} x^3 - 4x & 6x^2 - 8 & 6x \\ & -3x^2 + 4 & -6x \\ \hline x^3 - 4x & 3x^2 - 4 & \underline{|0} \end{array}\right.$$

Again the remainder is zero. Therefore (2) is exact and its first integral or a second integral of (1) is

$$(x^3-4x)\frac{dy}{dx}+(3x^2-4)\,y=c_1x+c_2 \qquad \dots (3)$$

$-D$	x^3-4x	$3x^2-4$
		$-3x^2+4$
	x^3-4x	$\lfloor 0$

Again the remainder is zero. Therefore (3) is exact and its first integral or a third integral of (1) is

$$(x^3-4x)\,y=c_1\frac{x^2}{2}+c_2x+c_3.$$

V.(a) Integrating Factor

Many times, the given equation may not be exact but it can be made exact by multiplying with the I.F., which can be determined under the following rules :

Rule 1 : If the coefficients P_0, P_1, P_2,...., P_n of non-exact linear equation are of the form kx^p or sum or difference of the terms of the above type, then we shall suppose x^m is an I.F. We shall multiply the given differential equation by x^m and apply the condition of exactness. This will give us the value of m and the I.F.

Rule 2 : If the coeffs. P_0, P_1, P_2, ..., P_n are trigonometrical functions of the form sin x, cos x, tan x etc., then by trial we will find some suitable trigonometrical function as I.F.

Example 4 : Solve the following equation :

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}(\tan x)+3y=3\tan^2 x\sec x.$$

Sol. The given equation is $\frac{d^2y}{dx^2}+2\frac{dy}{dx}(\tan x)+3y=3\tan^2 x\sec x$ ... (1)

Multiplying both sides of (1) by cos x, we get,

$$\cos x\frac{d^2y}{dx^2}+2\sin x\frac{dy}{dx}+3\cos x\,.\,y=3\tan^2 x \qquad \dots (2)$$

$-D$	cos x	2 sin x	3 cos x
		sin x	−3 cos x
	cos x	3 sin x	$\lfloor 0$

$\because$ remainder = 0

$\therefore$ (2) is exact and its first solution is

$$\cos x \frac{dy}{dx} + 3 \sin x . y = 3 (\tan x - x) + c_1$$

or $$\frac{dy}{dx} + 3 \tan x . y = 3 (\sec x \tan x - x \sec x) + c_1 \sec x \qquad \ldots (3)$$

$\because$ (3) is linear equation of first order in y

$\therefore$ comparing it with $\frac{dy}{dx} + Py = Q,$ we get,

$P = 3 \tan x,\ Q = 3 (\sec x \tan x - x \sec x) + c_1 \sec x$

$\therefore$ $\int P\, dx = 3 \int \tan x\, dx = -3 \log \cos x = \log (\cos x)^{-3}$

$\therefore$ $\text{I.F.} = e^{\int P\, dx} = (\cos x)^{-3} = \frac{1}{\cos^3 x}$

$\therefore$ solution of (3) i.e., that of (1) is

$$y . \frac{1}{\cos^3 x} = \int 3 (\sec x \tan x - x \sec x) \sec^3 x dx + c_1 \int \sec x . \sec^3 x dx + c_2$$

or $$y \sec^3 x = 3 \int \sec^4 x \tan x\, dx - 3 \int x \sec^4 x\, dx + c_1 \int \sec^4 x\, dx + c_2$$

$$= 3 . \frac{\sec^4 x}{4} - 3 \left[x \left(\tan x + \frac{\tan^3 x}{3} \right) \right]$$

$$- \int \left(\tan x + \frac{\tan^3 x}{3} \right) dx \Bigg] + c_1 \left(\tan x + \frac{\tan^3 x}{3} \right) + c_2$$

or $$y \sec^3 x = \frac{3}{4} \sec^4 x - 3 \left[x \left(\tan x + \frac{\tan^3 x}{3} \right) \right.$$

$$\left. + \frac{2}{3} \log | \cos x | - \frac{1}{6} \sec^2 x \right] + c_2 + c_1 \left(\tan x + \frac{\tan^3 x}{3} \right)$$

which is the required solution.

V. Differential Equation of the 2^{nd} Order

The equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ where P, Q and R are functions of x, is called the standard linear differential equation of the second order. If P = 0, the equation $\frac{d^2y}{dx^2} + Qy = R$ is called the normal form.

We discuss some methods below to solve such type of differential equations :

VI.(a) Method of Variation of Parameters

Example 5 : Solve the following equation :

$$(x + 2)\, y'' - (2x + 5)\, y' + 2y = (x + 1)\, e^x$$

given that its complementary solution is $c_1 (2x + 5) + c_2 e^{2x}$.

Sol. The given equation is $(x + 2)\, y'' - (2x + 5)\, y' + 2y = (x + 1)\, e^x$ ... (1)

Its complementary solution is $y_c = c_1 (2x + 5) + c_2 e^{2x}$

Now we seek a particular solution of (1) by variation of parameters.

Let $y = A (2x + 5) + Be^{2x}$ ... (2)

where A and B are functions of x.

Diff. (2) w.r.t. x, we get $y' = 2A + 2Be^{2x} + A' (2x + 5) + B' e^{2x}$ ... (3)

Choose $A' (2x + 5) + B' e^{2x} = 0$ ... (4)

$\therefore$ (3) becomes, $y' = 2A + 2Be^{2x}$ ... (5)

and $y'' = 2A' + 2B' e^{2x} + 4 Be^{2x}$ ... (6)

Substituting the values of y, y', y" form (2), (5), and (6) in (1), we get,

$$(x + 2) (2A' + 2B' e^{2x} + 4Be^{2x}) - (2x + 5) (2A + 2Be^{2x}) +2 [A (2x + 5) + Be^{2x}] = (x + 1)\, e^x \quad ... (7)$$

or $2 (x + 2) A' + 2 (x + 2) e^{2x} B' = (x + 1)\, e^x$

Multiplying (4) by 2 (x + 2) and (7) by 1 and subtracting, we get,

$A' [(2x + 5) . 2 (x + 2) - 2 (x + 2)] = - (x + 1)\, e^x$

or $2 (x + 2) (2x + 4) A' = - (x + 1)\, e^x$

or $4 (x + 2)^2 A' = - (x + 1)\, e^x$

$$\therefore \quad A' = -\frac{(x+1)\, e^x}{4\, (x+2)^2}$$

Again multiplying (4) by 2 (x + 2), (7) by (2x + 5) and subtracting we get,

$B' e^{2x} [2 (x + 2) - 2 (x + 2) (2x + 5)] = - (x + 1) (2x + 5)\, e^x$

or $-2 B' e^x (x + 2) (2x + 4) = - (x + 1) (2x + 5)$

$\therefore \qquad B' = \dfrac{(2x+5)(x+1)}{4(x+2)^2} e^{-x}$

$\therefore \qquad A = -\int \dfrac{(x+1)e^x}{4(x+2)^2} dx = -\dfrac{1}{4}\int e^x \left[\dfrac{x+1}{(x+2)^2}\right] dx$

$= -\dfrac{1}{4}\int e^x \left[\dfrac{(x+2)-1}{(x+2)^2}\right] dx = -\dfrac{1}{4}\int e^x \left[\dfrac{1}{x+2} - \dfrac{1}{(x+2)^2}\right] dx$

$= -\dfrac{1}{4} e^x \cdot \dfrac{1}{x+2} \qquad \left[\because \int e^x \{(f(x) + f'(x)\} dx = e^x f(x)\right]$

and $\quad B = \dfrac{1}{4}\int \dfrac{(2x+5)(x+1)}{(x+2)^2} e^{-x} = \dfrac{1}{4}\int \dfrac{2(x+2)^2 - (x+2) - 1}{(x+2)^2} e^{-x} dx$

$= \dfrac{1}{4} .2 \int e^{-x} dx - \dfrac{1}{4}\int \dfrac{1}{x+2} e^{-x} dx - \dfrac{1}{4}\int \dfrac{1}{(x+2)^2} e^{-x}$

$= \dfrac{1}{2}\dfrac{e^{-x}}{-1} - \dfrac{1}{4}\left[\dfrac{1}{x+2}\dfrac{e^{-x}}{-1} - \int \dfrac{-1}{(x+2)^2}\dfrac{e^{-x}}{-1} dx\right] - \dfrac{1}{4}\int \dfrac{1}{(x+2)^2} e^{-x} dx$

$= -\dfrac{1}{2} e^{-x} + \dfrac{1}{4(x-2)} e^{-x} + \dfrac{1}{4}\int \dfrac{1}{(x+2)^2} e^{-x} dx - \dfrac{1}{4}\int \dfrac{1}{(x+2)^2} e^{-x} dx$

$= -\dfrac{1}{2} e^{-x} + \dfrac{1}{4(x+2)} e^{-x}$

$\therefore$ particular solution of (1) is

$$y_p = \frac{-(2x+5)}{4(x+2)} e^x + \left[-\frac{1}{2} e^{-x} + \frac{e^{-x}}{4(x+2)}\right] e^{2x} = -e^x$$

$\therefore$ the general solution of (1) is $y = c_1 (2x + 5) + c_2 e^{2x} - e^x$.

VI.(b) Method of Changing the Independent Variable

Example 6 : Solve the following equation : $x^6 \frac{d^2y}{dx^2} + 3x^5 \frac{dy}{dx} + a^2y = \frac{1}{x^2}$.

Sol. The given equation is $x^6 \frac{d^2y}{dx^2} + 3x^5 \frac{dy}{dx} + a^2y = \frac{1}{x^2}$

or $$\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} + \frac{a^2}{x^6}y = \frac{1}{x^8} \quad \ldots (1)$$

Comparing it with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$, we get, $P = \frac{3}{x}, Q = \frac{a^2}{x^6}, R = \frac{1}{x^8}$

Changing the independent variable from x to z, the equation (1) reduces to

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1 \quad \ldots (2)$$

where $P_1 = \frac{P\frac{dz}{dx} + \frac{d^2z}{dx^2}}{\left(\frac{dz}{dx}\right)^2}, Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}, R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$

We choose z in such a way that $Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{\frac{a^2}{x^6}}{\left(\frac{dz}{dx}\right)^2} = \text{constant}=a^2\text{(say)}$

$$\therefore \quad x^3\frac{dz}{dx} = 1, \text{ or } \frac{dz}{dx} = \frac{1}{x^3}$$

$$\therefore \quad z = -\frac{1}{2x^2} \quad \ldots (3)$$

Now $P_1 = \dfrac{\dfrac{3}{x}\cdot\dfrac{1}{x^3} + \left(-\dfrac{3}{x^4}\right)}{\left(\dfrac{1}{x^3}\right)^2} = 0$ and $R_1 = \dfrac{\dfrac{1}{x^8}}{\dfrac{1}{x^6}} = \dfrac{1}{x^2} = -2z$

$\therefore$ equation (2) reduces to $\dfrac{d^2y}{dz^2} + a^2y = -2z$

or $(D^2 + a^2)\, y = -2z$

A.E. is $D^2 + a^2 = 0$, or $D^2 = -a^2$ or $D = \pm\, ia$

$\therefore$ C.F. $= c_1 \cos az + c_2 \sin az = c_1 \cos\left(-\dfrac{a}{2x^2}\right) + c_2 \sin\left(-\dfrac{a}{2x^2}\right)$

$$\text{P.I.} = -\frac{1}{D^2 + a^2}\, 2z = -2\,\frac{1}{a^2\left(1 + \dfrac{D^2}{a^2}\right)}\, z$$

$$= -\frac{2}{a^2}\left(1 + \frac{D^2}{a^2}\right)^{-1} z = -\frac{2}{a^2}(1)\, z = -\frac{2z}{a^2} = \frac{1}{a^2x^2}$$

$\therefore$ complete solution is $y = c_1 \cos\dfrac{a}{2x^2} - c_2 \sin\dfrac{a}{2x^2} + \dfrac{1}{a^2x^2}$.

VI. Self Check Exercise

1. Solve the following differential equations :

   (i) $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = x$

   (ii) $(x^2D^2 - 3xD + 5)\, y = \sin(\log x)$

2. Solve $((1 + 2x)^2D^2 - 6\,(1 + 2x)\, D + 16)\, y = 8\,(1 + 2x)^2$

3. Solve $(2x^2 + 3x)\dfrac{d^2y}{dx^2} + (6x + 3)\dfrac{dy}{dx} + 2y = (x + 1)e^x$

4. Solve $x\dfrac{d^2y}{dx^2} - (2x - 1)\dfrac{dy}{dx} + (x - 1)\, y = 0$ by the method of variation of parameters.

5. Solve $\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - y \sin^2 x = 0$ by the change of independent variables.

VII. Suggested Readings :

1. R.K. Jain, S.R.K. Lyengar, Advanced Engineering Mathematics, Narosa Publishing House.
2. Rai Singhania : Ordinary and Partial Differential Equations, S. Chand & Company, New Delhi.
3. Zafar Ahsan, Differential Equations and their Applications, Prentice-Hall of India Pvt. Ltd., New Delhi - 2nd Ed.

LESSON NO. 2.1

Series Solutions of Differential Equations

Objectives

I. **Introduction**
II. **Power Series Method**
III. **Frobenius Method**
IV. **Solving Bessel's Differential Equation of Order n**
V. **Solving Legendre's Equation of Order n**
VI. **Solving Hermite's Differential Equation**
VII. **Self Check Exercise**

I. Introduction :

Firstly, we introduce some basic terms :

I Power Series : An infinite series of the form

$$\sum_{n=0}^{\infty} A_n(x-\alpha)^n = A_0 + A_1(x-\alpha) + A_2(x-\alpha)^2 + \ldots\ldots\ldots\ldots \text{ in } (x-\alpha),$$

is called power series about 'α'.

Where A_0, A_1, A_2 .... are real constants called coefficients of the power series and 'α' is known as centre of power series.

Moreover, the power series , $\sum_{n=0}^{\infty} A_n(x-\alpha)^n$, where A_i's are real is said to be convergent at $x = x_0$ iff

$\lim_{K\to\infty} \sum_{n=0}^{K} A_n(x-\alpha)^n$ exists finitely otherwise the series is called divergent.

$$P_0(x)\frac{d^2y}{dx^2}+P_1(x)\frac{dy}{dx}+P_2(x)y=0;\ P_0(x)\neq 0 \qquad \text{..........(1)}$$

Or $$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0 \qquad \text{............(2)}$$

in series, we have two methods :

1. Power Series Method
2. Robenius Method.

Now, we discuss these methods one by one.

2.1.2 Power Series Method

Existence Theorem :

If α is an ordinary point of equation (1) or (2), then each solution of (1) or (2) is analytic at $x=\alpha$ and may be expressed as a power series of $x-\alpha$, with radius of convergence $R>0$. [This method is used to solve equation (1) or (2) about an ordinary point α]

Steps Involved in Power Series Method :

Step I: Let $y = A_0 + A_1 (x-\alpha) + A_2 (x-\alpha)^2 \ldots\ldots = \sum_{n=0}^{\infty} A_n(x-\alpha)^n$ be a solution of (1), where A_0, A_1,. A_2, ... A_n are to be determined.

Step II: Calculate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ as

$$\frac{dy}{dx} = A_1 + 2A_2(x-\alpha) + 3A_3(x-\alpha)^2 + \ldots\ldots\ldots = \sum_{n=1}^{\infty} nA_n(x-\alpha)^{n-1}$$

and $$\frac{d^2y}{dx^2} = 2A_2 + 6A_3(x-\alpha) + 12A_4(x-\alpha)^2 + \ldots\ldots\ldots = \sum_{n=2}^{\infty} n(n-1)A_n(x-\alpha)^{n-2}$$

Step III: Substituting these values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

Case 4. When α-β is an integer and one value of k makes a coefficient of y indeterminate.

Now, we proceed to solve some important differential equations.

2.1.4 Bessel's Differential Equation of Order n (Case-I : when roots of indicial equation differ by a non-interger)

Solve $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\ y = 0$ where 2n is not on integer.

Solution : The given equation can be written as $\rightarrow \frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \frac{x^2 - n^2}{x^2} y = 0,$

Here $P(x) = \frac{1}{x}, Q(x) = \frac{x^2 - n^2}{x^2}$ are not analytic at x = 0 but xP(x) = 1, $x^2Q(x) = x^2-n^2$ are analytic at x=0. $\therefore$ x=0 is a regular singular point, so solution in series exist.

Step-I Let $y = x^k \sum_{i=0}^{\infty} a_i\ x^i$, where $a_0 \neq 0$ be a solution of given differential equation.

So, $$y = \sum_{i=0}^{\infty} a_i x^{K+i}$$

$$\Rightarrow \quad \frac{dy}{dx} = \sum a_i\ (k+i)\ x^{k+i-1} \text{ and}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \sum a_i\ (k+i)\ (k+i-1)\ x^{k+i-2}$$

Now using values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given differential equation, we have

$$x^2 \sum_{i=0}^{\infty} a_i\ (k+i)\ (k+i-1)\ x^{k+i-2} + x\sum_{i=0}^{\infty} a_i\ (k+i)\ x^{k+i-1} + (x^2 - n^2) \sum_{i=0}^{\infty} a_i\ x^{k+i} = 0$$

or $$\sum a_i\ (k+i)\ (k+i-1)\ x^{k+i} + \sum a_i\ (k+i)\ x^{k+i} + \sum a_i\ x^{k+2-i} - n^2 \sum a_i\ x^{k+i} = 0$$

$$y_1 = \left[\sum_{k=0}^{\infty} a_i\, x^{k+i}\right] k = a\, \left[a_0\, x^k + a_2\, x^{k+2} + a_4\, x^{k+4} + \dots\dots\dots\right] \begin{pmatrix} \because a_1 = a_3 = a_5 \\ = \dots\dots 0 \end{pmatrix}$$

Put k = n,

$$\therefore y_1 = x^n \left[a_0 - \frac{a_0}{(2n+2)(2)} x^2 + \frac{a_0}{(2n+2)(2n+4)(2)(4)} x^4 + \frac{-a_0}{(2n+2)(2n+4)(2n+6)(2)(4)(6)} x^6 + \dots\dots\right]$$

$$\Rightarrow \quad y_1 = x^n \left[a_0 - \frac{a_0}{2(n+1).2} x^2 + \frac{a_0}{2.2(n+1)(n+2)} \frac{x^4}{(1.2).2^2} + \frac{a_0}{2.2.2(n+1)(n+2)(n+3)} \frac{x^6}{(1.2.3)2^3}\right.$$

$$\Rightarrow \qquad y_1 = x^n a_0 \left[1 - \frac{\left(x^2/4\right)}{(n+1)} + \frac{\left(x^2/4\right)^2}{(n+1)(n+2)\,\underline{|2}} - \frac{\left(x^2/4\right)^3}{\underline{|3}\,(n+1)(n+2)(n+3)} + \dots\dots\dots\dots\right]$$

and

$$y_2 = x^{-n} a_0 \left[1 - \frac{\left(x^2/4\right)}{(-n+1)} + \frac{\left(x^2/4\right)^2}{(-n+1)(-n+2)} \frac{1}{\underline{|2}} - \frac{\left(x^2/4\right)}{(-n+1)(-n+2)(-n+3)} \frac{1}{\underline{|3}} + \dots\dots\right]$$

Moreover, the general solution is $y = Ay_1 + By_2$

Similarly, the students can easily find out the solutions of Bessil's Squations of order 0 (Case-II) and order 1 (Case-III) for n=0 and n=1 respectively.

2.1.5 Legendre's Equation of Order n (Case IV : Roots of indicial equation differ by an integer and make a coefficient indeterminate)

Solve $(1 - x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + n\,(n+1)\,y = 0$ ...... (1)

Assuming as in Bessel's equation

Step-I: Let $y = \sum a_i x^{k+i}$, $a_0 \neq 0$ be a trial solution of (1)

Similarly, $a_5 = \frac{a_3}{4.5}[3.4 - n(n+1)] = -\frac{a_3}{20}(n^2 + n - 12) = -\frac{a_3}{20}(n+4)(n-3)$

$= \frac{a_1}{6 \times 20}(n+4)(n+2)(n-1)(n-3) = \frac{a_1}{\angle 5}(n+4)(n+2)(n-1)(n-3)$ etc.

Now, on using these values in $y = x^k [a_0 + a_1 x + a_2 x^2 + ...]$, we get

$$y = x^0 \left[a_0 + a_1 x + \left(-\frac{a_0}{2}\right) n)(n+1)x^2 + \frac{a_1}{6}(-)x^3(n+1)(n-1)\right.$$

$$\left. + \frac{(n+3)(n+1)n(n-2)}{4!} a_0 x^4 + \frac{(n+4)(n+2)(n-1)(n-3)}{5!} a_1 x^5 + ...\right]$$

$$\Rightarrow \quad y = a_0\left[1 - \frac{1}{2}(n+1)nx^2 + \frac{(n+3)(n-1)(n)(n-2)}{4!}x^4 + ...\right] +$$

$$a_1 x\left[1 - \frac{x^2(n+2)(n-1)}{3!} + \frac{x^4}{\angle 5}(n+4)(n+2)(n-1)(n-3) + ...\right]$$

So, $$y = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n+3)(n-1)n(n-2)}{4!}x^4 + ...\right] + a_1\left[x - \frac{(n+2)(n-1)}{3!}x^3\right.$$

$$\left. + \frac{(x+4)(n+2)(n-1)(n-3)}{5!}x^5.......\right]$$

is the general solution of Legendre's Differential Equation of order ‘n’.

2.1.6 Hermite's Differential Equation (Case IV)

Solve $\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2p\,y = 0$ ..... (1)

where p is any constant.

Sol.: Step-I : Let the trial solution of (1) be $y = \left[a_0 x^k + a_1 x^{k+1} - a_2 x^{k+2} + ...\right]$, $a_0 \neq 0$

Now, $\frac{dy}{dx} = \left[a_0 k (x)^{k-1} + a_1 (k+1) x^k + a_2 (k+2) x^{k+1} + ...\right]$

$$y = a_0\left[1 - \frac{2p}{\angle 2}x^2 + 2^2\frac{p(p-2)}{\angle 4}x^4 + \ldots\ldots\ldots\ldots\right] + a_1\left[x - \frac{2(p-1)}{\angle 3}x^3 + 2^2\frac{(p-1)(p-3)}{\angle 5}x^5 + \ldots\ldots\right]$$

as the general solution of Hermite's differential equation.

2.1.7 Self Check Exercise :

1. Solve Bessel's equation of order 0.
2. Solve Bessel's equation of order 1.
3. Solve in series $2x\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} - 2y = 0$.
4. Solve in series $(1-x^2)\frac{d^2y}{dx^2} + 2y = 0$.

Case-III : If n = O, then we obtain Bessel's equation of order 'O' as :

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy=O$$

Whose general solution can be easily obtained by Frobenius method, given by

$$y = Au(x) + Bv(x)$$

Where

$$Au(x) = a_0\left(1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+.....\right)$$

$$Bv(x) = a_0\, u(x)\ \text{legx} + a_0\left(\frac{x^2}{2^2}-\left(1+\frac{1}{2}\right)\frac{x^4}{2^2.4^2}+.....\right)$$

Bessel's Functions

Let in equation (1) $a_0 = \dfrac{1}{2^n\sqrt{n+1}}$

where $\sqrt{n}=\int_0^\infty e^{-t}t^{n-1}dt$

Known as gamma function.

$$\therefore\ u(x) = \frac{x^n}{2^n\sqrt{n+1}}\left(1-\frac{x^2}{2^2(n+1)}+\frac{x^4}{2^4\angle 2(n+1)(n+2)}......\right)$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r x^{n+2r}}{2^{n+2r}\angle r\sqrt{(n+1)}(n+1)(n+2)...(n+k)}$$

$$\Rightarrow u(x) = =\sum_{r=0}^{\infty}\frac{(-1)^r x^{n+2r}}{2^{n+2r}\angle r\sqrt{(n+r+1)}}\ ....(3)$$

(using Property of gamma function $n\sqrt{n}=\sqrt{n+1}$ for any non-negative real n.)

The above u(x) given by equation (3) is a particular solution of Bessel's equation and is known as Bessel's function of first kind of order 'n' denoted by $J_n(x)$.

If :- We have $J_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{\angle r}\frac{1}{\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$

$$x\frac{d}{dx}(J_n(x))=x\sum_{r=0}^{\infty}\frac{(-1)^r\left(\frac{n+2r}{2}\right)\left(\frac{x}{2}\right)^{n+2r-1}}{\angle r\,\Gamma(n+r+1)}$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r\ (n+2r)\left(x/2\right)^{n+2r}}{\angle r\ \Gamma\ (n+r+1)}$$

$$=\sum_{r=0}^{\infty}\ \frac{(-1)^r\ (2n+2r-n)\left(x/2\right)^{n+2r}}{\angle r\ \Gamma(n+r+1)}\quad\left[\begin{aligned}\because n+2r&=2n-n+2r\\&=2n+2r-n\end{aligned}\right]$$

$$=\sum_{r=0}^{\infty}\ \frac{(-1)^r 2\left(x/2\right)^{n+2r}}{\angle r\ \Gamma\ (n+r)}-n\sum_{r=0}^{\infty}\ \frac{(-1)^r\ \left(x/2\right)^{n+2r}}{\angle r\ \Gamma\ (n+r+1)}$$

$$=+x\sum_{r=0}^{\infty}\ \frac{(-1)^r\ \left(x/2\right)^{n-1+2r}}{\angle r\Gamma((n-1)+r+1)}\ -n\sum_{r=0}^{\infty}\ \frac{(-1)^r\ \left(x/2\right)^{n+2r}}{\angle r\ \Gamma\ (n+r+1)}$$

$$=x\ j_{n-1}(x)-nJ_n(x)$$

$$\therefore\qquad x\frac{d}{dx}(J_n(x))=xJ_{n-1}(x)-nJ_n(x).\qquad \text{........II}$$

3. Prove that $2\frac{d}{dx}(J_n(x))=J_{n-1}(x)-J_{n+1}(x).$

It can be proved very easily by adding recurrence formula 1 and 2,

However, the above recurrence formula can be proved independently also as

From the definition of $J_n(x)$, we have

$$= x\sum_{r=0}^{\infty}\frac{(-1)^r (x/2)^{n+2r-1}}{\angle r\ \Gamma((n+1)+r+1)} + \sum_{t=0}^{\infty}\frac{(-1)^t 2(x/2)^{n+2t+1}(x/2)}{\angle t\,\Gamma(n+1)+t}$$

$$= x\ J_{n-1}(x) + x\ J_{n+1}(x)$$

$$= x\left[J_{n-1}(x) + J_{n+1}(x)\right] \qquad \text{.......(IV)}$$

5. $\frac{d}{dx}\ x^{-n}J_n(x)\ = -x^{-n}J_{n+1}(x)$

Now

$$\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -n\ x^{-n-1}J_n(x) + x^{-n}J_n^{'}(x)$$

$$= -x^{-n}\left[\frac{n}{x}J_n(x) - J_n^{'}(x)\right]$$

From recurrence formula 1

$$= -x^{-n}J_{n+1}(x) \qquad \text{.........(V)}$$

6. $\frac{d}{dx}\ x^{n}J_n(x)\ = x^{n}J_{n-1}(x)$

Consider $\frac{d}{dx}\ x^{n}J_n(x)\ = x^{n}J_n^{'}(x) + nx^{n-1}.J_n(x)$

$$= x^{n}\left[J_n^{'}(x) + \frac{n}{x}J_n(x)\right]$$

On using recurrence formula 2, we get

$$= x^{n}J_{n-1}(x).$$

Theorem : Show that

(i) $J_{-n}(x) = (-1)^n J_n(x)$ $\left(\begin{array}{l}\text{where } J_{-n}(x) \text{ is called Bessel's} \\ \text{function of first kind of order '} -n\text{'.}\end{array}\right)$

(ii) $J_n(-x) = (-1)^n J_n(x)$.

Using $J_1(x) = -J_0'(x)$, $J_1'(x) = -J_0''(x)$

We have

$$x\,J_1'(x) = -J_0'(x) - x\,J_2(x)$$

$$\Rightarrow \quad -x\,J_0''(x) + J_0'(x) = -x\,J_2(x)$$

$$\Rightarrow \quad J_2 = J_0'' - \frac{1}{x}J_0'$$

The result no (iii) can also be proved very easily.

Example : Prove that

(i) $$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x,$$

(ii) $$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x,$$

Solution :

(i) Using the definition of $J_n(x)$

$$J_n(x) = \frac{x^n}{2^n\Gamma(n+1)}\left[1 - \frac{x^2}{2.(2n+2)} + \frac{x^4}{2.4.(2n+2)(2n+4)}\cdots\cdots\right]$$

On putting $n = -1/2$

$$J_{-1/2}(x) = \frac{x^{-1/2}}{2^{-1/2}\Gamma(-1/2)}\left[1 - \frac{x^2}{2.(1)} + \frac{x^4}{2.4(1.3)}\right] = \sqrt{\frac{2}{x\pi}}\left(1 - \frac{x^2}{\angle 2} + \frac{x^4}{\angle 4}\right)\Bigg]\left(\because \sqrt{1/2} = \sqrt{\Pi}\right)$$

$$= \sqrt{\frac{2}{x\pi}}\cos x$$

Similarly (ii) part, can also be proved.

2.2.3 Generating Function for $J_n(x)$

Which by def. is $J_n(x)$

Also when we compare the coefficient of Z^{-n} we get $J_{-n}(x)$, so we can write

$$\exp.\left(\frac{x}{2}\left(Z-\tfrac{1}{Z}\right)\right)=\sum_{n=-\infty}^{\infty} Z^n J_n(x) \qquad ....(6)$$

Example: Prove that

$$\frac{d}{dx}\ J_n^2+J_{n+1}^2\ =2\left[\frac{n}{x}J_n^2-\frac{n+1}{x}J_{n+1}^2\right]$$

Solution :

$$\frac{d}{dx}\ J_n^2+J_{n+1}^2\ =2\ J_n^{(x)}\,J_n'(x)+J_{n+1}^{(x)}\,J_{n+1}'(x)\ ....(i)$$

From recurrence formula I, we have

$$J_n'\ =\frac{n}{x}J_n(x)-J_{n+1}(x)$$

and from Recurrence relation II, we have

$$J_n'(x)=\frac{-n}{x}J_n^{(x)}+J_{n-1}(x)$$

$$\Rightarrow \qquad J_{n+1}'(x)=\frac{-(n+1)}{x}J_{n+1}^{(x)}+J_n(x)$$

Putting these values of J_n' and J_{n+1}' in (i),

We have

$$\frac{d}{dx}\ J_n^2+J_{n+1}^2\ =2\left(\left[J_n\left(+\frac{n}{x}J_n-J_{n+1}\right)\right]\right.$$

$$\left.+J_{n+1}\left[\frac{-(n+1)}{x}J_{n+1}+J_n\right]\right)$$

$$\left(\because (J_0)_{x=0} = 1, (J_1)_{x=0} = (J_2)_{x=0} = ...= 0\right)$$

Hence $J_0^2 + 2\left(J_1^2 + J_2^2 + J_3^2 + ...\right) = 1$

2.2.4 Orthogonality of Bessel's Functions :

Theorem : Prove that the Bessel functions $J_n(\lambda_1 x)$, $J_n(\lambda_2 x)$ ------- are orthogonal w.r.t. x on [0, R] where R is any fixed +ve real and λ_1, λ_2, ... are roots of $J_n(\lambda R) = 0$ $\left(n \varepsilon IN \cup \{0\}\right)$.

Proof :

Consider $J_n(\lambda_\ell x)$, $J_n(\lambda_m x)$, $\lambda_\ell \neq \lambda_m$ and *l*, m = 1, 2, -----.

Also differential equation satisfied by $J_n(u)$ is

$$u^2 \frac{d^2y}{du^2} + u\frac{dy}{du} + \left(u^2 - n^2\right) y = 0 \quad .... \text{(i)}$$

Put $u = \lambda_m x \Rightarrow x = \frac{1}{\lambda_m} u \Rightarrow \frac{dx}{du} = \frac{1}{\lambda_m}$

$$\therefore \quad \frac{dy}{du} = \frac{dy}{dx}.\frac{dx}{du} = \frac{1}{\lambda_m}\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{du^2} = \frac{d}{du}\left(\frac{dy}{du}\right) = \frac{d}{du}\left(\frac{1}{\lambda_m}\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{\lambda_m}\frac{dy}{dx}\right).\frac{dx}{du} = \frac{1}{\lambda_m^2}\frac{d^2y}{dx^2}$$

So the d.e. X becomes

$$\frac{1}{\lambda_m^2}\lambda_m^2 x^2 \frac{d^2y}{dx^2} + \frac{1}{\lambda_m}\lambda_m x \frac{dy}{dx} + \left(\lambda_m^2 x^2 - n^2\right) y = 0$$

$$\Rightarrow \quad x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(\lambda_m^2 x^2 - n^2\right) y = 0 \quad ....\text{(ii)}$$

$$\Rightarrow \quad R\ J_n(\lambda_\ell x)J_n'(\lambda_m x) - J_n(\lambda_m x)J_n^1(\lambda_\ell x)\Big|_{x=R}$$

$$= (\lambda_\ell^2 - \lambda_m^2)\int_0^R xJ_n(\lambda_\ell x)J_n(\lambda_m x)dx$$

Since $\lambda_\ell,\ \lambda_m$ are roots of $J_n(\lambda R) = 0$

$$\Rightarrow J_n(\lambda_m R) = J_n(\lambda_\ell R) = 0$$

$$\Rightarrow (\lambda_\ell^2 - \lambda_m^2)\int_0^R x\,J_n(\lambda_\ell x)J_n(\lambda_m x)dx = 0(\lambda_\ell \neq \lambda_m)$$

$$\Rightarrow \int_0^R x\,J_n(\lambda_\ell x)J_n(\lambda_m x)dx = 0$$

$\Rightarrow$ Bessel's functions $J_n(\lambda_\ell x)$ and $J_n(\lambda_m x)$ OR in general $J_n(\lambda_1 x), J_n(\lambda_2 x), \ldots\ldots\ldots\ldots$ etc. are orthogonal w.r.t x on [0, R].

2.2.5 Self-Check Exercise

1. $\dfrac{J_2}{J_1} = \dfrac{1}{x} - \dfrac{J_0''}{J_0'}$

2. $\dfrac{d}{dx}\left(x^n J_n(\alpha x)\right) = \alpha x^n J_{n-1}(\alpha x)$

3. $\lim\limits_{x\to 0}\dfrac{J_n(x)}{x^n} = \dfrac{1}{2^n\Gamma(n+1)}$

4. $\int J_0(x)J_1(x)dx = \dfrac{-J_0^2(x)}{z} + C$

$$\frac{d^2y}{dx^2} = \sum a_r (r+k)(r+k-1)x^{r+k-2}$$

Using these values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in (1), we have

$$(1-x^2)\Sigma a_r (k+r)(k+r-1)x^{k+r-2} \; -2x\,\Sigma a_r (k+r)x^{k+r-1} + n(n+1)\Sigma a_r x^{k+r} = 0$$

$\Rightarrow$ $$\Sigma a_r (k+r)(k+r-1)x^{k+r-2} - \Sigma a_r (k+r)(k+r-1)x^{k+2} \; -2\,\Sigma a_r (k+1)x^{k+r} + n(n+1)\,\Sigma a_r x^{k+r} = 0$$

$\Rightarrow$ $$\Sigma a_r (k+r)(k+r-1)x^{k+r-2} \; +\Sigma a_r \left[(n+1)^n - 2(k+r) - (k+r)(k+r-1)\right]x^{k+r} = 0$$

Equation to zero the lowest power coeff. i.e. x^{k-2}

$a_0 k(k-1) = 0 \Rightarrow a_0 \neq 0, k(k-1) = 0$ (indical eq.)

which gives the value k = 0, k = 1

Comparing Coeff. of x^{k-1}

$a_1 k(k + 1) = 0$

For k = 0 a_1 is arbitrary

for k = 1 $2a_1 = 0 \Rightarrow a_1 = 0$

on comparing coeff. of x^{k+r} :

$(k + r + 2)(k + r + 1)a_r + 2$

$= [(k + r)(k + r - 1) + 2(k + 2) - n(n + 1))]a_r$

$= [(k + r)^2 + (k + r) - n(n + 1)]a_r$

$= (k + r + n + 1)(k + r - n)a_r$

$$a_{r+2} = \frac{(k+r+n+1)(k+r-n)}{(k+r+2)(k+r+1)}a_r$$

Now $$a_2 = \frac{(k+n+1)(k+0-n)}{(k+2)(k+1)}a_0 = \frac{(k+n+1)(k-n)}{(k+1)(k+2)}a_0$$

$$a_3 = \frac{(k+n+2)(k+1-n)}{(k+3)(k+2)}a_1$$

$$a_4 = \frac{(k+3+n)(k+3-n)}{(k+4)(k+3)}a_2 \; = \frac{(k+3+n)(k+2+n)(k+0-n)(k+2-n)}{(k+1)(k+2)(k+3)(k+4)}a_0$$

$$y=\frac{1.3.5-(2n-1)}{\angle n}\left[x_n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4}+......\right] \quad \text{— (C)}$$

and is denoted by P_n (x), called Legendre's function of first kind.

To define Legendre's functions, we obtain the power series solution of Legendre's equation in descending powers of x.

We assume that :

$y = a_0x^K + a_1x^{K-1} + a_2x^{K+2} + ...+ a_nx^{K-n} +........$

is trial solution of the given Legendre's equation.

$$\frac{dy}{dx}=a_0kx^{K-1}+a_1(k-1)x^{K-2}+a_2(k-2)x^{K-3}+.....+(k-n)a_nx^{K-n-1}+.......$$

$$\frac{d^2y}{dx^2}=a_0k(k-1)x^{K-2}+a_1(k-1)(k-2)n^{K-3}+a_2(k-2)(k-3)x^{K-4}+.........$$

$$+(k-n)(k-n-1)a_nx^{K-n-2}+........$$

Using these values in Legendre's differential equation, we get

$$(1-x^2)\left[a_0k(k-1)x^{K-2}+a_1(k-1)(k-2)x^{K-3}+........\right]$$

$$-2x\left[a_0kx^{K-1}+a_1(k-1)x^{K-2}+a_2(k-2)x^{K-3}+.....\right]$$

$$+n(n+1)\left[a_0x^K+a_1x^{K-1}+a_2x^{K-2}+.......\right]=0$$

Now equate to zero, the coefficient of x^k, x^{k-1}, x^{k-2}, ... etc. we get

x^k : $-a_0k(k-1)-2ka_0+n(n+1)a_0=0 \Rightarrow [k(k-1)+2K-n(n+1)]a_0=0$

$\Rightarrow$ $[k^2+k-n(n+1)]a_0=0$

$\Rightarrow$ $(k+n+1)(k-n)a_0=0$

Since $a_0\neq 0$, $\boxed{k=-1-n, k=n}$

x^{k-1} : $-a_1(k-1)(k-2)-2(k-1)a_1+n(n+1)a_1=0$

$[(k-1)(k-2)+2(k-1)-n(n+1)]a_1=0$

$\Rightarrow$ $(k-1+n+1)(k-1-n)a_1=0$

required solution of Legendre's equation

It should be noted that as we have taken n to be a +ve integer, (C) becomes a terminating power series and this (C) is also known as Legendres polynomials of degree n.

Case-II :- Also from (B), if we take $a_0 = \dfrac{\angle n}{1.3.5(2n+1)}$ and n is a +ve integer

then (B) takes the following form and is denoted by $Q_n(x)$, called Legendre's function of second kind.

$$\therefore \qquad Q_n(x) = \frac{\angle n}{1.3.5(2n+1)}\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2(2n+3)(2n+5)}x^{-n-5} + ...\right]$$

which is a non-termianting power series when n is taken as +ve integer.

($\therefore$ The general solution of Legendre's Equation is $y = AP_n(x) + BQ_n(x)$ where A and B are arbitrary constants.)

Note :- In $P_n(x)$....... we had taken a_0 as

$$\frac{1.3.5......(2n+1)}{\angle n} \text{ or } \frac{1.2.3.4.....(2n-1)(2n)}{(2.4.............2n)\angle n} = \frac{\angle 2n}{2^n(\angle n)^2} \Rightarrow a_0 = \frac{\angle 2n}{2^n(\angle n)^2}$$

$$\Rightarrow \qquad P_n(x) = \frac{\angle 2n}{2^n(\angle n)^2}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} + ...\right]$$

$$= \frac{\angle 2n}{2^n(\angle n)^2}x^n - \frac{\angle(2n-2)}{2^n\angle(n-1)\angle(n-2)}x^{n-2} + \frac{\angle(2n-4)}{2^n\angle(n-2)\angle(n-4)}x^{n-4} + ...]$$

$$\Rightarrow \qquad P_n(x) = \sum_{r=0}^{t}\frac{(-1)^r\angle(2n-2r)}{2^n\angle r\angle(n-r)\angle(n-2r)}x^{n-2r}$$

$$\text{where } t = \begin{cases} \dfrac{n-1}{2} & \text{if n is odd} \\ \dfrac{n}{2} & \text{if n is even or zero} \end{cases}$$

Therefore, the above $P_n(x)$ represents the particular solution of Legendre's equation.

We now find $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$ etc.

$\Rightarrow \qquad P_3(x) = \frac{1}{2}\left(5x^2 - 3\right)x$

$P_4(x)$

For $\quad n = 4,\ r = 0,\ 1,\ 2$

$$P_4(x) = \frac{(-1)^0(\angle 8-1)}{2^4 \angle 0(\angle 4-0)(\angle 4-0)}x^{4-0} + \frac{(-1)^1(\angle 8-2)}{2^4 \angle 1(\angle 4-1)(\angle 4-2)}x^{4-2} + \frac{(-1)^2(\angle 8-4)}{2^4 \angle 2(\angle 4-2)(\angle 4-4)}x^{4-4}$$

$$= \frac{\angle 8}{16\ \angle 4\ \angle 4}x^4 - \frac{\angle 6}{2^4\ \angle 3\ \angle 2}x^2 + \frac{\angle 4}{2^4\ \angle 2\ \angle 2\ \angle 0}.1$$

$$= \frac{8\times 7\times 6\times 120}{24\times 24\times 16}x^4 - \frac{720}{16\times 16\times 2} - x^2 + \frac{24}{4\times 16}$$

$$= \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$$

$\Rightarrow \qquad P_4(x) = \frac{1}{8}\left(35x^4 - 15x^2 + 3\right)$

Similarly we can find the values of $P_5(x)$ etc.

2.3.2 Rodrigue's Formula

Prove that : $P_n(x) = \frac{1}{2^n \angle n}.\frac{d^n\left(x^2-1\right)^n}{dx^n}$

Proof. Taking $y = (x^2 - 1)^n$

$$\frac{dy}{dx} = 2n\,x\left(x^2-1\right)^{n-1} = \frac{2nx\left(x^2-1\right)^n}{\left(x^2-1\right)}$$

$$\left(x^2-1\right)\frac{dy}{dx} - 2nx\left(x^2-1\right)^n = 0$$

$$\left(x^2-1\right)\frac{dy}{dx} - 2nxy = 0$$

Now differentiate the above result w.r.t. x, (n + 1) times by using Leibnitz theorem.

$$+{}^nc_2\left(\frac{d^{n-2}}{dx^{n-2}}(x-1)^n\right)\left(n(x-1)(x+1)^{n-2}\right)+\ldots\ldots\ldots$$

$$+{}^nc_{n-1}n(x-1).^{n-1}\angle n(x+1)+{}^nc_n\angle n.(x-1)^n$$

on taking x = 1

$$\left(\frac{d^ny}{dx^n}\right)_{x=1}=\angle n\,.\,2^n$$

So, $A=2^n\angle n$

Now, $\frac{d^ny}{dx^n}=A\,P_n(x)=2^n\angle n\,P_n(x)$

$$=2^n\angle n\,P_n(x)$$

$\Rightarrow$ $P_n(x)=\frac{1}{2^n\angle n}\frac{d^n}{dx^n}\left(x^2-1\right)^n$

which is the required Roderigue's formula for Legendre's functions $P_n(x)$.

We can also find Legendre's polynomials $P_0(x)$, $P_1(x)$, $P_2(x)$ etc. from Rodrigue's formula as :

Now $P_n(x)=\frac{1}{2^n\angle n}\frac{d^n}{dx^n}\left(x^2-1\right)^n$

$\therefore$ $P_0(x)=\frac{1}{2^0\angle 0}\frac{d^0}{dx^0}\left(x^2-1\right)^0=1\Rightarrow P_0(x)=1$

$$P_1(x)=\frac{1}{2^1\angle 1}\frac{d^1}{dx^1}\left(x^2-1\right)^1=\frac{1}{2}.2x=x\Rightarrow P_1(x)=x$$

$$P_2(x)=\frac{1}{2^2\angle 2}\frac{d^2}{dx^2}\left(x^2-1\right)^2=\frac{1}{8}.\frac{d}{dx}\left(4x\left(x^2-1\right)\right)=\frac{1}{8}\left(12x^2-4\right)=\frac{\left(3x^2-1\right)}{2}$$

$$P_3(x)=\frac{1}{2^3\angle 3}\frac{d^3}{dx^3}\left(x^2-1\right)^3=\frac{1}{48}.\frac{d^3}{dx^3}\left(x^6-3x^4+3x^2-1\right)=\frac{1}{48}\left(120x^3-72x\right)$$

Rodrigue's formula is given by -

Sol. $P_n(x) = \frac{1}{2^n \angle n} \frac{d^n}{dx^n}(x^2-1)^n$

$\Rightarrow$ $P_n'(x) = \frac{1}{2^n \angle n} \frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n = \frac{1}{2^n \angle n} \frac{d^{n+1}}{dx^{n+1}}(x-1)^n (x+1)^n$

Using Leibnitz's theorem :

$$= \frac{1}{2^n \angle n}\Bigg[(x-1)^n \frac{d^{n+1}}{dx^{n+1}}(x+1)^n + {}^{n+1}c_1{}^1 \frac{d^n}{dx^n}(x+1)^n .n(x-1)^{n-1}$$

$${}^{n+1}c_2 \frac{d^{n-1}}{dx^{n-1}}(x+1)^n .n(n-1)(x-1)^{n-2} + \ldots\ldots + {}^{n+1}c_n \underline{\frac{d}{dx}(x+1)^n \frac{d^n}{dx^n}(x-1)^n}$$

$$+(x+1)^n \frac{d^{n+1}}{dx^{n+1}}(x-1)^n\Bigg] \qquad \left[\begin{matrix}\text{Obtained from the 2nd}\\ \text{last term}\end{matrix}\right]$$

at x = 1,

$$P_n^1(1) = \frac{1}{2^n \angle n}.(n+1).n.(2)^{n-1} \angle n = \frac{n(n+1)}{2}$$

$\Rightarrow$ $P_n^1(1) = \frac{n(n+1)}{2}. = \frac{n}{2} \times (n+1)$

Similarly we can prove that $P_n'(-1) = -\frac{n(n+1)}{2}$ (this will be obtained from second term from the beginning.)

2.3.3 Generating Function for $P_n(x)$

Prove that $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$ where $|x| \leq 1$ and $|t| < 1$

Sol. Now, $(1-2xt+t^2)^{-1/2} = \left(1-(2xt-t^2)^{-1/2}\right)$

Solution :

Legendre's differential equation satisfied by $P_n(x)$ is

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n\left(n+1\right)y=0$$

or $$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dy}{dx}\right]+n\left(n+1\right)y=0.......(A)$$

Similarly, differential equation satisfied by $P_m(x)$ is :

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dy}{dx}\right]+m\left(m+1\right)y=0......(B)$$

Using y = $P_n(x)$, and y = $P_m(x)$
(A) and (B) become :

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP_n\left(x\right)}{dx}\right]+n\left(n+1\right)P_n\left(x\right)=0.....(i)$$

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP_m\left(x\right)}{dx}\right]+m\left(m+1\right)P_m\left(x\right)=0.....(ii)$$

Now, multiply (1) by $P_m(x)$ and (ii) by $P_n(x)$ and subtract

$$P_m\left(x\right)\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP_n\left(x\right)}{dx}\right]-P_n\left(x\right)\frac{d}{dx}\left[1-x^2\,\frac{d\,P_m\left(x\right)}{dx}\right]$$

$$=\left[m\left(m+1\right)-n\left(n+1\right)\right]P_m\left(x\right)P_n\left(x\right)$$

$\Rightarrow$ $$P_n\left(x\right)\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP_m}{dx}\right]-P_m\left(x\right)\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP_n}{dx}\right]-\left(m\left(m+1\right)-n\left(n+1\right)\right)P_m\left(x\right)P_n\left(x\right)$$

on integrating both sides w.r.t. x between –1 & 1, we have

$$\int_{-1}^{1}\left[P_n\left(x\right)\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP_m}{dx}\right]-P_m\left(x\right)\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP_n}{dx}\right]\right]dx$$

$$=-\left[m\left(m+1\right)-n\left(n+1\right)\right]\int_{-1}^{1}\left[P_m\left(x\right)P_n\left(x\right)dx\right]$$

$\Rightarrow$

$$\int_{-1}^{1}\left(1-2xt+t^2\right)^{-1}ax=\Sigma t^{2n}\int_{-1}^{1}\left(\left(P_n\left(x\right)\right)^2 dx+0\right)$$ (from the previous result)

Now we evaluate LHS

i.e. $\int_{-1}^{1}\left(1-2xt+t^2\right)^{-1}dx$

$$=\int_{-1}^{1}\frac{1}{\left(1-2xt+t^2\right)}dx$$

$$=\frac{-1}{2t}\int_{-1}^{1}\frac{-2t}{\left(1-2xt+t^2\right)}dx$$

$$=\frac{-1}{2t}\left[\log\left(1-2xt+t^2\right)\right]_{-1}^{1}$$

$$=\frac{-1}{2t}\left[\log\left(1-2xt+t^2\right)-\log\left(1+2t+t^2\right)\right]$$

$$=\frac{-1}{2t}\left[\log\left(1-t^2\right)-\log\left(1+t\right)^2\right]$$

$$=\frac{1}{t}\log\left(\frac{1+t}{1-t}\right)$$

$$=\frac{1}{t}\left[t-\frac{t^2}{2}+\frac{t^3}{3}.........\right)+\left[t-\frac{t^2}{2}+\frac{t^3}{3}.........\right)$$

$$=\frac{1}{t}\left[2t-\frac{2t^3}{3}+\frac{2t^5}{5}........\right]$$

$$=2+\frac{2}{3}t^2+\frac{2}{5}t^4+.......+\frac{2}{\left(2n+1\right)}t^{2n}+....$$

$$-\frac{-1}{2}\left(1-2xt+t^2\right)^{-1/2}(-2x+2t)=\left(1-2xt+t^2\right)\Sigma nP_n(x)t^{n-1}$$

$$(x-t)\Sigma P_n(x)t^n=\left(1-2xt+t^2\right)\Sigma P_n(n)t^{n-1}$$

$$=\Sigma nP_n(n)t^{n-1}-2x\Sigma nP_n(x)t^n+\Sigma nP^n(n)t^{n+1}$$

$$\left[xP_nt^n-\Sigma P_n(x)t^{n+1}=\Sigma nP_n(x)t^{n+1}-2n\Sigma n(x)t^n+\Sigma nP_n(n)t^{n+1}\right]$$

$$x\Sigma(2n+1)P_nt^n=\Sigma nP_n(x)t^n+\Sigma(n+1)P_n(x)t^{n+1} \quad \text{.... (ii)}$$

To get (A), we compare the coeff. of t^{n-1},

$$x(2n-2+1)P_{n-1}(x)=(n-1)P_n(x)+(n-1)P'_{n-2}(x)$$

$$x(2n-1)P_{n-1}(x)=nP_n(x)+(n-1)P_{n-2}(x) \quad \text{(A)}$$

Recurrence relation (A) is valid when $n \geq 2$

If we compare the coefficients of t^n in (ii) we arrive at B.

If we differentiate (i) w.r.t. 'x', to get

$$-\frac{1}{2}\left(1-2xt+t^2\right)^{-3/2}(-2t)=\Sigma P'_n(x)t^n \quad \text{...(iii)}$$

Also $$-\frac{1}{2}\left(1-2xt+t^2\right)^{-3/2}(-2x+2t)=\Sigma nP_n(x)t^{n-1} \quad \text{...(iv)}$$

multiply (iii) by (x – t) and (iv) by t and then equate to have :

$$(x-t)\Sigma P'_n(x)t^n=\Sigma nP_n(x)t^n$$

$$n\Sigma P'_n(x)-\Sigma P'_n(x)t^{n+1}=\Sigma nP_n(x)t^n$$

which when compared for coefficients of t^h gives relation D.

Now (iii) can be rewritten as :

$$\left(1-2xt+t^2\right)^{-3/2}=\sum_{n=0}^{\infty}P'_n(x)t^{n-1} \quad \text{...(v)}$$

Also (iv) can be rewritten as :

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