Centre for Distance and Online Education

Punjabi University, Patiala

Class : B.A.III (Mathematics) **Semester : V**
Paper : II (Mathematical Methods-I) **Unit-I**
Medium : English

Lesson No.

Department website : www.pbidde.org

B.A. / B.Sc. (MATHEMATICS) Semester - V

PAPER-II: MATHEMATICAL METHODS-I

Maximum Marks: 50 **Pass Percentage: 35%**
Maximum Time: 3 Hrs

INSTRUCTIONS FOR THE PAPER SETTER

The question paper will consist of three sections A, B and C. Sections A and 8 will have four questions each from the respective sections of the syllabus and Section C will consist of one compulsory question having eight short answer type questions covering the entire syllabus uniformly. Each question in sections A and B will be of 7.5 marks and Section C will be of 20 marks.

INSTRUCTIONS FOR THE CANDIDATES

Candidates are required to attempt five questions in all selecting two questions from each of the Section A and B and compulsory question of Section C.

Section-A

Laplace Transforms: Definition of Laplace transforms, Linearity property, Piecewise continuous function, Existence of Laplace transform, Functions of exponential order and of class A, First and second shifting theorems of Laplace transform, Change of scale property, Laplace transform of derivatives, Initial value problem, Laplace transform of integrals, Multiplication by t, Division by t, Laplace transform of periodic functions and error function, Beta functions and Gamma functions, Definition of inverse Laplace transforms, Linearity property, First and second shifting theorems of inverse Laplace transforms, Change of scale property, Division by p, Convolution theorem, Heaviside’s expansion formula (with proofs and applications).

Section-B

Applications of Laplace Transforms: Applications of Laplace transforms to the solution of ordinary differential equations with constant coefficients and variable coefficients, Simultaneous ordinary differential equations, Second order partial differential equations (Heat equation, Wave equations and Laplace equation).

RECOMMENDED BOOKS:

1. Shanthi Narayan and P. K. Mittal : Scope as in A Course of Mathematical Analysis published by S. Chand and Company.
2. A. R. Vasishtha : Scope as in Integral Transforms published by Krishna Prakashan Media Pvt. Ltd. Meerut.

B.A. PART - III **PAPER-II**

MATHEMATICAL METHODS-I

LESSON NO. 1.1 **Author : Dr. Chanchal**

LAST UPDATED ON JUNE, 2023

LAPLACE TRANSFORMS–I

1.1.1 Objectives

In this lesson, our prime objectives are :

- to study Laplace Transforms as a special case of an integral transform which is an improper integral.
- to study an importat theorem based upon the existence of Laplace Transform.
- to derive some useful Laplace Transform.

1.1.2 Introduction

From our earlier studies, we are already familiar with the concept of improper integrals. Now, to define the Laplace transform, we firstly define the concept of integral transform as An improper integral of type $\int_{-\infty}^{\infty} H(s.t)\, f(t)\, dt$ is called **integral transform** of f(t) if it is convergent and is denoted by F (s) or T (f(t)).

Note :

(i) Here the function H (s, t) is known as **Kernel** of the transform.

(ii) Here x is parameter (real of complex), which is independent of t.

In particular, if $H(s, t) = \begin{cases} e^{-st}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

Then $F(s) = \int_{-\infty}^{\infty} H(s,t)\, f(t)\, dt$

$$= \int_{-\infty}^{0} H(s,t)\, f(t)\, dt + \int_{0}^{\infty} H(s,t)\, f(t)\, dt$$

$$= \int_{-\infty}^{0} 0\, f(t)\, dt + \int_{0}^{\infty} e^{-st}\, f(t)\, dt$$

$$= 0 + \int_{0}^{\infty} e^{-st} f(t)\, dt$$

i.e. $F(s) = \int_{0}^{\infty} e^{-st}\, f(t)$ dt is transform, known as **LAPLACE TRANSFORM** and it may be defined as

Let f be a real valued function of the real vairable t, defined over $(-\infty, \infty)$ such that $f(t) = 0$ for $t < 0$

Then the function F of s, defined as

$F(s) = \int_{0}^{\infty} e^{-st}\, f(t)\, dt;$ is called the **Laplace Transform** of f and is denoted as L({f(t)).

1.1.3 Existence of Laplace Transform

Firstly, we define function of exponential order and pieceuise continuous function as :

Definition : Function of exponential order

A function f (x) is said to be of exponential order $\alpha > 0$ if $\lim_{x \to \infty} e^{-\alpha x} f(x)$ exists and is finite-number.

i.e. there exists a real number $K > 0$ such that

$$\left| e^{-\alpha x} f(x) \right| < K \ \forall\ x \geq M$$

or $|f(x)| < K e^{\alpha x} \ \forall \ x \geq M$.

For Example : $f(x) = x^n$ is of exponential order $\alpha > 0$ as $x \to \infty$, where $a \in N$.

A function f (x) is called **piecewise or sectionally continuous** on finite interval [a b] iff this interval can be divided into finite subintervals such that f(x) is continuous in each of the subinterval except at end points of these intervals.

Thus in piecewise continuity (i) L.H.L. and R.H.L. of function exists in each subinterval

(ii) Function has finite jumps at the end points of these subintervals.

Example Let $f(x) = \begin{cases} 1, 0 < x < 2 \\ 3, 2 < x < 3 \\ 5, x > 3 \end{cases}$.

Now, the existence of Laplace transform can be studied through the following theorem:

Theorem 1 : If f (t) is piecewise continuous on every finite interval in its domain $t \geq 0$ and is of exponential order α as $t \to \infty$, then prove that the Laplace transform of f(t) exists for all $s > \alpha$.

Proof : Given f (t) is piecewise continuous on every finite interval in its domain $t \geq 0$

$\Rightarrow$ f (t) is piecewise continuous on $[0\ t_0]$, for $t_0 > 0$

$\Rightarrow$ $e^{-st} f(t)$ is also piecewise continuous on $[0\ t_0]$

$\Rightarrow$ $e^{-st} f(t)$ is integrable on $[0\ t_0]$

(As e^{-st}, being exponented function is continuous)

$$\therefore \int_0^{t_0} e^{-st} f(t)\, dt \text{ exists } \forall\ t_0 > 0$$

By def. of improper integrals

$$\int_0^{\infty} e^{-st} f(t)\, dt = \lim_{t_0 \to \infty} \int_0^{t_0} e^{-st} f(t)\, dt$$

$$\text{Now } |F(s)| = \left| \int_0^{\infty} e^{-st}\ f(t)\, dt \right|$$

$$\leq \int_0^{\infty} \left| e^{-st} f(t) \right| dt \qquad \left(\because \left| \int_0^{\infty} G(t)\, dt \right| \leq \int_0^{\infty} |G(t)|\, dt \right)$$

$$= \int_0^\infty e^{-st}\,|f(t)|\,dt \qquad (\because e^{-st} > 0)$$

$$\le \int_0^\infty e^{-st}\,(K\,e^{\alpha t})\,dt \qquad (\because f(t) \text{ is of exponential order } \alpha)$$

$$= K\int_0^\infty e^{(\alpha-s)\,t}\,dt$$

$$= K\left(\lim_{t_0\to\infty}\int_0^{t_0} e^{(\alpha-s)\,t}dt\right) \qquad \text{(using (i))}$$

$$= K\lim_{t_0\to\infty}\left[\frac{e^{(\alpha-s)\,t}}{\alpha-s}\right]_0^{t_0}$$

$$= \frac{K}{\alpha-s}\,\underset{t_0\to\infty}{\mathrm{L\,im}}\left[e^{(\alpha-s)\,t_0} - e^{(\alpha-s)\,0}\right]$$

$$= \frac{K}{\alpha-s}\,\underset{t_0\to\infty}{\mathrm{L\,im}}\left[\frac{1}{e^{(s-\alpha)\,t_0}} - 1\right] \text{ for } s > \alpha\,\cdot$$

$$= \frac{K}{\alpha-s}\,\underset{t_0\to\infty}{\mathrm{L\,im}}\left[\frac{1}{e^{(s-\alpha)\,t_0}} - 1\right] \text{ for } s > \alpha\,\cdot$$

$$= \frac{K}{\alpha-s}\left(\frac{1}{\infty} - 1\right) = \frac{K}{\alpha-s}\,(0-1)$$

$$= \frac{K}{s-\alpha} \text{ for } s > \alpha$$

$\therefore$ $$|F(s)| \le \frac{K}{s-\alpha} \text{ for } s > \alpha$$

$\Rightarrow$ $F(s) = \int_0^\infty e^{-st}\,f(t)\,dt$ converges for $s > \alpha$ so that Laplace Transform of $f(t)$ exists.

1.1.4 Derivations of Some Useful Laplace Transforms

(i) $L(1) = \frac{1}{s}, s > 0$

Proof : By def. of Laplace Transform

$$L\,(1) = \int_0^\infty e^{-st}(1)\,dt$$

$$= \lim_{t\to\infty} \int_0^t e^{-st}\,dt = \lim_{t\to\infty} \left[\frac{e^{-st}}{-s}\right]_0^t$$

$$= -\frac{1}{s} \lim_{t\to\infty} (e^{-st} - e^{0})$$

$$= -\frac{1}{s} \lim_{t\to\infty} \left(\frac{1}{e^{st}} - 1\right)$$

$$= -\frac{1}{s} \left(\frac{1}{\infty} - 1\right) \text{ if } s > 0$$

$$= -\frac{1}{s} (0 - 1) = \frac{1}{s} \text{ if } s > 0$$

$\therefore$ $L(1) = \frac{1}{s}$ if $s > 0$.

(ii) $L(t^\alpha) = \frac{\lceil \alpha + 1}{s^{\alpha+1}}, s > 0$ and α is any real > -1. In particular, $L(t^n) = \frac{\lfloor n}{s^{n+1}}$.

Proof : By def. of Laplace Transform

$$L\,(t^\alpha) = \int_0^\infty e^{-st} t^\alpha dt$$

Put $s\,t = y \Rightarrow s\,dt = dy$

$$= \int_{y=0}^{y=\infty} e^{-y} \left(\frac{y}{s}\right)^\alpha \left(\frac{dy}{s}\right)$$

$$= \frac{1}{s^{\alpha+1}} \int_0^\infty e^{-y} y^\alpha dy$$

$$= \frac{\lceil(\alpha+1)}{x^{\alpha+1}} \text{ if } \alpha > -1$$

(By using def. of Gamma function)

$$\therefore \qquad L(t^{\alpha}) = \frac{\lceil(\alpha+1)}{x^{\alpha+1}} \text{ if } \alpha > -1 \text{ and } s > 0$$

In particular, when $\alpha = n = 0, 1, 2$.........

Then $L(t^{\alpha}) = \frac{\lceil n+1}{s^{n+1}} = \frac{\lfloor n}{s^{n+1}}$ if $n > -1, s > 0$.

(iii) $L(e^{\alpha t}) = \frac{1}{s-\alpha}$ if $s > \alpha$

Proof : By def. of Laplace Transform

$$L(e^{\alpha t}) = \int_0^{\infty} e^{-st} e^{\alpha t}\, dt = \int_0^{\infty} e^{(\alpha - s)t}\, dt$$

$$= \lim_{t\to\infty} \int_0^{t} e^{(\alpha-s)t} dt$$

$$= \lim_{t\to\infty} \left[\frac{e^{(\alpha-s)t}}{\alpha - s}\right]_0^t$$

$$= \frac{1}{\alpha - s} \lim_{t\to\infty} \left(e^{(\alpha-s)t} - e^{(\alpha-s)(0)}\right)$$

$$= \frac{1}{\alpha - s} \lim_{t\to\infty} \left(\frac{1}{e^{(s-\alpha)t}} - 1\right)$$

$$= \frac{1}{\alpha - s} \left(\frac{1}{\infty} - 1\right) \text{ if } s > \alpha$$

$$= \frac{1}{\alpha - s}(0-1) = \frac{1}{s-\alpha} \text{ if } s > \alpha$$

$$\therefore \qquad L(e^{\alpha t}) = \frac{1}{s-\alpha} \text{ if } s > \alpha.$$

(iv) $L(\sinh \alpha t) = \dfrac{\alpha}{s^2 - \alpha^2}$ if $s > |\alpha|$

Proof : By def. $L(\sinh \alpha t) = L\left(\dfrac{e^{\alpha t} - e^{-\alpha t}}{2}\right)$

$$= \int_0^\infty e^{-st}\left(\frac{e^{\alpha t} - e^{-\alpha t}}{2}\right) dt$$

$$= \frac{1}{2}\int_0^\infty (e^{(\alpha - s)t} - e^{-(\alpha + s)t})\, dt$$

$$= \frac{1}{2} \operatorname*{Lim}_{t \to \infty} \int_0^t (e^{(\alpha - s)t} - e^{-(\alpha + s)})dt$$

$$= \frac{1}{2} \operatorname*{Lim}_{t \to \infty} \left(\frac{e^{(\alpha - s)t}}{\alpha - s} - \frac{e^{-(\alpha + s)t}}{-(\alpha + s)}\right)_0^t$$

$$= \frac{1}{2} \operatorname*{Lim}_{t \to \infty} \left(\frac{1}{\alpha - s}(e^{(\alpha - s)t} - 1) + \frac{1}{\alpha + s}(e^{-(\alpha + s)t} - 1)\right)$$

$$= \frac{1}{2} \operatorname*{Lim}_{t \to \infty} \left[\frac{1}{\alpha - s}\left(\frac{1}{e^{(s-\alpha)^t}} - 1\right) + \frac{1}{\alpha + s}\left(\frac{1}{e^{(s+\alpha)\,t}} - 1\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{\alpha - s}\left(\frac{1}{\infty} - 1\right) + \frac{1}{\alpha + s}\left(\frac{1}{\infty} - 1\right)\right] \text{ if } s > \alpha.\ s > -\alpha$$

$$= \frac{1}{2}\left[\frac{1}{\alpha - s}(0 - 1) + \frac{1}{\alpha + s}(0 - 1)\right] \text{ if } s > |\alpha|$$

$$= \frac{1}{2}\left[\frac{1}{s - \alpha} - \frac{1}{s + \alpha}\right] = \frac{\alpha}{s^2 - \alpha^2} \text{ if } s > |\alpha|$$

$\therefore$ $L(\sinh \alpha t) = \dfrac{\alpha}{s^2 - \alpha^2}$ if $s > |\alpha|$.

Now, the reader can easily prove trhat

(v) $L(\cos \alpha t) = \dfrac{s}{s^2 - \alpha^2}$ if $s > |\alpha|$

(vi) $L(\sin \alpha t) = \dfrac{\alpha}{s^2 + \alpha^2}$ if $s > 0$

Proof : $L(\sin \alpha\, t) = \int_0^\infty e^{-st} \sin \alpha\, t\, dt$

$$= \lim_{t\to\infty} \left(\int_0^t e^{-st} \sin \alpha t\, dt \right)$$

$$= \lim_{t\to\infty} \left[\frac{e^{-st}}{s^2 + \alpha^2} (-s \sin \alpha t - \alpha \cos \alpha t) \right]_0^t$$

$$\left[\text{using } \int e^{ax} \sin(bx + c)\, dx = \frac{e^{ax}}{a^2 + b^2} (d \sin(bx + c) - b \cos(bx + c)) \right]$$

$$= \lim_{t\to\infty} \frac{-1}{s^2 + \alpha^2} \left(e^{-st}(s \sin \alpha t + \alpha \cos \alpha t) - e^0 (s \sin 0 + \alpha \cos 0) \right)$$

$$= \lim_{t\to\infty} \frac{-1}{s^2 + \alpha^2} \left(\frac{s \sin \alpha t + \alpha \cos \alpha t}{e^{st}} - \alpha \right)$$

$$= -\frac{1}{s^2 + \alpha^2} (0 - \alpha) \text{ if } s > 0 \qquad \left[\because \lim_{t\to\infty} \left| \frac{s \sin \alpha t + \alpha \cos \alpha t}{e^{st}} \right| \le \lim_{t\to\infty} \frac{|s + \alpha|}{e^{st}} = 0 \right]$$

$$= \frac{\alpha}{x^2 + \alpha^2} \text{ if } x > 0$$

$\therefore \quad L(\sin \alpha t) = \dfrac{\alpha}{s^2 + \alpha^2}$ if $s > 0$.

(vii) $L(\cos \alpha t) = \dfrac{\alpha}{s^2 + \alpha^2}$ if $s > 0$

Write : Do yourself

1.1.5 Some Important Examples :

Example 1 : Find the Laplace Transform of f (t) = |t–2| + |t+2|, t ≥ 0 by first principles.

Sol. Here f (t) = | t – 2 | + | t + 2 |, t ≥ 0

$$= \begin{cases} -(t-2) + t + 2, & 0 \le t \le 2 \\ t - 2 + t + 2, & t > 2 \end{cases}$$

$$= \begin{cases} 4, & 0 \le t \le 2 \\ 2t, & t > 2 \end{cases}$$

$$\therefore \quad L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^2 e^{-st} f(t)\, dt + \int_2^\infty e^{-st} f(t)\, dt$$

$$= \int_0^2 e^{-st}(4)\, dt + \int_2^\infty e^{-st}(2t)\, dt$$

$$= 4\int_0^2 e^{-st} dt + 2 \underset{t\to\infty}{\mathrm{Lim}} \int_2^t te^{-st} dt$$

$$= 4\left(\frac{e^{-st}}{-s}\right)_0^2 + 2 \underset{t\to\infty}{\mathrm{Lim}} \left[\left(\frac{te^{-st}}{-s}\right)_2^t - \int_2^t \frac{e^{-st}}{-s} dt\right]$$

$$= -\frac{4}{s}(e^{-2s} - e^0) + 2 \underset{t\to\infty}{\mathrm{Lim}} \left[-\frac{1}{s}(te^{-st} - e2e^{-2s}) + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)_2^t\right]$$

$$= -\frac{4}{s}(e^{-2s} - 1) + 2 \underset{t\to\infty}{\mathrm{Lim}} \left[-\frac{1}{s}\left(\frac{t}{e^{st}} - 2e^{-2s}\right) - \frac{1}{s^2}\left(e^{-st} - e^{-2s}\right)\right]$$

$$= \frac{4\left(1-e^{-2s}\right)}{s} + 2\left(-\frac{1}{s}(0-2e^{-2s}) - \frac{1}{s^2}(0-e^{-2s})\right) \text{ if } s > 0$$

$$= \frac{4}{s} - \frac{4e^{-2s}}{s} + \frac{4e^{-2s}}{s} + \frac{2e^{-2s}}{s^2} \text{ if } s > 0$$

$$\left(\begin{array}{l} \because \underset{t\to\infty}{\mathrm{L\,im}} \frac{t}{e^{st}} = \underset{t\to\infty}{\mathrm{L\,im}} \frac{1}{se^{st}} = \frac{1}{\infty} = 0 \\ \text{and } \underset{t\to\infty}{\mathrm{L\,im}}\, e^{-st} = 0 \text{ if } s > 0 \end{array}\right)$$

$$= \frac{4}{s} + \frac{2e^{-2s}}{s^2} \text{ if } s > 0$$

$$= \frac{2}{s}\left(2 + \frac{e^{-2s}}{s}\right) \text{ if } s > 0 \cdot$$

1.1.6 Summary :

In this lesson, we have defined Laplace Transforms and proved its existence. Further, Laplace Transforms of some special functions have been derived with the help of first principles.

1.1.7 Key Concepts :

Integral Trasfoms, Kernel, Laplace Transform, Function of Exponential order, Piecewise continuous function.

1.1.8 Long Questions :

1. Describe the technique of finding Laplace Transform of a function using first principles.

1.1.9 Short Questions :

1. Define Integral Transform
2. Define function of Exponential order.
3. Define Piecewise continuous function.
4. Discuss the existence of Laplace Transform.

1.1.10 Suggested Readings :

1. A.R. Vasushtha & Dr. R.K. Gupta, Integral Transforms by Krishna Prakashan Media Pvt. Ltd. Meerut.

LESSON NO. 1.2 **Author : Dr. Chanchal**

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LAPLACE TRANSFORMS–II

1.2.1 Objectives

The prime objective of this lesson is to study some important properties of Laplace Transforms.

1.2.2 Some Important Properties of Laplace Transforms

1.2.2.1 Theorem 1 (Linearity Property of Laplace Transforms)

Let $f_1(t)$ and $f_2(t)$ be any functions of t, where $t \geq 0$ and their Laplace transforms exist. Show $L(a_1 f_1(t) + a_2 f_2(t)) = a_1 L(f_1(t)) + a_2 L(f_2(t))$ for any constants a_1, a_2.

Proof : The proof is left as an exercise for the reader.

1.2.2.2 Theorem 2 (First Shifting Theorem) :

If F (s) is the Laplace transform of f (t) for $t \geq 0$ and α is any number (real or complex) Prove $F(s - \alpha)$ is Laplace transform of $e^{\alpha t} f(t)$

OR

If $L(f(t)) = F(s)$ for $t \geq 0$

Prove $L(e^{\alpha t} f(t)) = F(s - \alpha)$, where α is any real or complex

Proof : We know $F(s) = L\,(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$

$$\therefore \quad L\,(e^{\alpha t} f(t)) = \int_0^\infty e^{-st} e^{\alpha t} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-\alpha)t}\, f(t)\,dt$$

$$= \int_0^\infty e^{-ut} f(t)\,dt \text{ where } s - \alpha = u$$

$= F\,(u) = F\,(s - \alpha)$,

$\Rightarrow$ $F\,(s - \alpha)$ is the Laplace transform of $e^{\alpha t}\, f(t)$.

1.2.2.3 Theorem 3 (Second Shifting Theorem) :

If F (s) is Laplace transform of f(t) form $t \geq 0$ and α is any number (real or complex)

Prove that, the function $g(t) = \begin{cases} f(t-\alpha), & t > \alpha \\ 0\ , & t < \alpha \end{cases}$ has Laplace transform $e^{\alpha t}$ F(s)

Prove if $g(t) = \begin{cases} f(t-\alpha), & t > \alpha \\ 0\ , & t < \alpha \end{cases}$ then $L\,(g(t)) = e^{-\alpha t} F(s)$.

Proof : We know F (s) = L (f (t)) $= \int_0^\infty e^{-st} f(t)\,dt$

and given function is $g(t) = \begin{cases} f(t-\alpha), & t > \alpha \\ 0\ , & t < \alpha \end{cases}$

$$\therefore \quad L(g(t) = \int_0^\infty e^{-st} g(t)\,dt$$

$$= \int_0^\alpha e^{-st} g(t)\,dt + \int_\alpha^\infty e^{-st} g(t)\,dt$$

$= \int_0^{\alpha} e^{-st} 0 \, dt + \int_{\alpha}^{\infty} e^{-st} \, f(t-\alpha)\, dt$ (By def. of g (t))

$= 0 + \int_{\alpha}^{\infty} e^{-st} f(t-\alpha)\, dt$ Put $t - \alpha = V \Rightarrow dt = dV$

$= \int_{V=0}^{V=\infty} e^{-s\,(V+\alpha)} f\,(V)\, dV$

$= e^{-\alpha s} \int_0^{\infty} e^{-sV} \, f\,(V)\, dV$ (By changing variable V by t)

$= e^{-\alpha s} \int_0^{\infty} e^{-st} f(t)\, dt$

$= e^{-\alpha s} F(s)$

$\Rightarrow$ Laplace transform of g(t) is $e^{-\alpha s}$ F(s)

Another Form of above Theorem

If L (f(t) = F (s) for $t \geq 0$

and $\alpha \geq 0$, real, Prove L (f (t – α) h (t – α)) = $e^{\alpha s}$ F(s)

where $h\,(t-\alpha) = \begin{cases} 1, & t > \alpha \\ 0, & t < \alpha \end{cases}$ or $h(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$.

1.2.2.4 Theorem 4 (Change of Scale Property) :

If L (f (t)) = F(s) for $t \geq 0$

Prove for any positive constant α,

(i) $L\,(f(\alpha t) = \frac{1}{\alpha} F\left(\frac{s}{\alpha}\right)$ (ii) $L\left(f\left(\frac{1}{\alpha}\right)\right) = \alpha \, F\,(\alpha s)$

Proof : We know $F(s) = L\,(f(t)) = \int_0^{\infty} e^{-st} f(t)\, dt$

(i) $\therefore L(f(\alpha\, t)) = \int_0^{\infty} e^{-st} (\alpha t)\, dt$ Put $\alpha t = V \Rightarrow \alpha dt = dV$

$$= \int_{V=0}^{V=\infty} e^{-s\frac{V}{\alpha}} f(V) \frac{dV}{\alpha}$$

$$= \frac{1}{\alpha} \int_0^{\infty} e^{-\left(\frac{s}{\alpha}\right)V} f(V)\, dV = \frac{1}{\alpha} \int_0^{\infty} e^{-\frac{s}{\alpha}t} f(t)\, dt$$

(Changing variable V by t)

$$= \frac{1}{\alpha} F\left(\frac{s}{\alpha}\right)$$

Now, the reader can easily prove the (ii) part.

1.2.3 Some Important Examples

Example 1 : Evaluate L {$\sinh^3 2r$}.

Sol. Here $\sinh^3 2t = \left(\frac{e^{2t} - e^{-2t}}{2}\right)^3$

$$= \frac{1}{8}\left(\left(e^{2t}\right)^3 - \left(e^{-2t}\right)^3 - 3\, e^{2t} e^{-2t} \left(e^{2t} - e^{2t}\right)\right)$$

$$= \frac{1}{8}\left(e^{6t} - e^{-6t} - 3e^{2t} + 3e^{-2t}\right)$$

$$\therefore \quad L\,(\sinh^3 2t) = L\left(\frac{1}{8}\left(e^{6t} - e^{-6t} - 3e^{2t} + 3e^{-2t}\right)\right)$$

$$= \frac{1}{8} L\left(e^{6t}\right) - \frac{1}{8} L\left(e^{-6t}\right) - \frac{3}{8} L\left(e^{2t}\right) + \frac{3}{8} L\left(e^{-2t}\right)$$

$$= \frac{1}{8}\frac{1}{s-6} - \frac{1}{8}\frac{1}{s+6} - \frac{3}{8}\frac{1}{s-2} + \frac{3}{8}\frac{1}{s+2}$$

if $s > 6$, $s > -6$, $s > 2$, $s > -2$

$$= \frac{1}{8}\left(\frac{(s+6)-(s-6)}{(s-6)\,(s+6)}\right) - \frac{3}{8}\left(\frac{(s+2)-(s-2)}{(s-2)\,(s+2)}\right)$$

$$= \frac{12}{8(s^2-36)} - \frac{3}{8}\left(\frac{4}{s^2-4}\right) \text{ if } s > 6$$

$$= \frac{3}{2}\left(\frac{(s^2-4)-(s^2-36)}{(s^2-36)(s^2-4)}\right) \text{ if } s > 6$$

$$= \frac{3}{2}\left(\frac{32}{(s^2-36)(s^2-4)}\right) \text{ if } s > 6$$

$$= \frac{48}{(s^2-36)(s^2-4)} \text{ if } s > 6 \cdot$$

Example 2 : Find the Laplace transform of following functions of t, $t \geq 0$

$e^{-3/2t} \sin 6t \sin 2t$

Sol. We know L (sin 6t sin 2t) = L $\left(\frac{1}{2}(\cos 4t - \cos 8t)\right)$

$$= \frac{1}{2}(L(\cos 4t) - L(\cos 8t))$$

$$= \frac{1}{2}\left(\frac{s}{s^2+4^2} - \frac{s}{s^2+8^2}\right); s > 0$$

$$= \frac{s}{2}\left(\frac{s^2+64-s^2-16}{(s^2+16)(s^2+64)}\right); s > 0$$

$$= \frac{s}{2}\left(\frac{48}{(s^2+16)(s^2+64)}\right)$$

$$= \frac{24s}{(s^2+16)(s^2+64)}; s > 0$$

$\therefore$ Using First Shifting Theorem

$$L\,(e^{-3/2\,t}\sin 6t \sin 2t) = \frac{24\left(s-\left(-\frac{3}{2}\right)\right)}{\left\{\left(s-\left(-\frac{3}{2}\right)\right)^2+16\right\}\left\{\left(s-\left(-\frac{3}{2}\right)\right)^2+64\right\}}$$

$$= \frac{24\left(s+\frac{3}{2}\right)}{\left\{\left(s+\frac{3}{2}\right)^2+16\right\}\left\{\left(s+\frac{3}{2}\right)^2+64\right\}},\ s > -\frac{3}{2}.$$

Example 3 : Find the Laplace transform of $g(t) = \begin{cases} 0, & 0 < t < \frac{1}{2} \\ t+\frac{3}{2}, & t > \frac{1}{2} \end{cases}$ by Second Shifting Theorem.

Sol. Given Function is

$$g(t) = \begin{cases} 0, & 0 < t < \frac{1}{2} \\ (t+2)-\frac{1}{2}, & t > \frac{1}{2} \end{cases} \qquad \left[\begin{array}{l} \because \lim\limits_{t\to\infty} \dfrac{t+\frac{3}{2}}{e^{st}} = 0 \\ \text{By L}'\text{ Hospital Rule} \end{array}\right]$$

Here $f(t) = t + 2, \alpha = \frac{1}{2}$

We know L (f(t)) = L (t + 2) = L (t) + 2 L (1)

$$= \frac{1}{s^2} + \frac{2}{s} = \frac{1+2s}{s^2}$$

$\therefore$ By Second Shifting Theorem

$$L\,(g(t)) = e^{-\frac{1}{2}s}\left(\frac{1+2s}{s^2}\right) = \frac{2s+1}{s^2\, e^{\frac{1}{2}s}} \text{ if } s > 0.$$

Example 4 : Evaluate L (e^{10} sin 10 t cos 10 t), $t \geq 0$ by change of scale property.

Sol. Firstly, we shall evaluate

$L(e^t \sin t \cos t),\ t \geq 0$

We have $L(\sin t \cos t) = L\left(\frac{\sin 2t}{2}\right)$

$$= \frac{1}{2} L(\sin 2t) = \frac{1}{2}\left(\frac{2}{s^2 + 2^2}\right)$$

$$= \frac{1}{s^2 + 4},\ s > 0$$

∴ By First Shifting Theorem, we have

$$L(e' \sin i \cos t) = \frac{1}{(s-1)^2 + 4};\ s - 1 > 0$$

Further, Using Change of Scale Property

$$L(e^{10t} \sin 10t \cos 10t) = \frac{1}{10} \frac{1}{\left(\frac{s}{10} - 1\right)^2 + 4},\ \frac{s}{10} > 1$$

$$= \frac{10}{(s-10)^2 + 400},\ s > 10$$

$$= \frac{10}{s^2 - 20s + 500},\ s > 10.$$

1.2.4 Summary

In this lesson, we have studied various properties related to Laplace Transforms. The concepts are elaborated well with the help of some solved examples,

1.2.5 Key Concepts :

Linearity Property, first shifting theorem, second shifting theorem, change of scale property.

1.2.6 Long Questions :

1. Find Laplace transform of $\sin \sqrt{t},\ t \geq 0$.
2. Find Laplace Transform of $t^2 e^t \sin 4t$
3. Find the Laplace transform of following using second shifting Theorem

$$g(t) = \begin{cases} 0, & 0 < t < \frac{\pi}{2} \\ \sin t, & t > \frac{\pi}{2} \end{cases}$$

4. Evaluatge L (e' cosh t), $t \geq 0$. Hence evaluate L (e^{5t} cosh 5t).

1.2.7 Short Questions :

1. Using $L(e^{\alpha t}) = \frac{1}{s-\alpha}, s > \alpha$

    Evaluate (i) L (sin h α t) (ii) L (cosh α t)
    (iii) L (sin α t) (iv) L (cos α t)

2. State and prove Linearity property of Laplace Transforms.

1.2.8 Suggested Readings :

1. A.R. Vasushtha & Dr. R.K. Gupta, Integral Transforms by Krishna Prakashan Media Pvt. Ltd. Meerut.

B.A. PART - III **PAPER-II**

MATHEMATICAL METHODS-I

LESSON NO. 1.3 **Author : Dr. Chanchal**

LAST UPDATED ON JUNE, 2023

LAPLACE TRANSFORMS–III

1.3.1 Objectives

For this lesson, our main goal is to study Laplace Transforms of derivatives and integrals. In continuation to that, we can easily understand about the Laplace Transforms of functions multiplied by t and divided by t.

1.3.2 Introduction

From our previous lesson, we are already familiar with the basic concept of Laplace transforms and its properties. So, now we have the enough knowledge to understand the Laplace transforms of derivatives and integrals of functions, as discussed below.

1.3.3 Laplace Transforms of Derivatives and Integrals

For finding the Laplace transforms of derivatives, we introduce the following result:

Result 1 : Let f (t) be real function defined for $t \geq 0$

If f (t), f (t), f"(t), ... f^{n-1} (t) are continuous on $[0, \infty)$ and are of exponential order α and f"(t) is cont. or piecewise continuous on $[0, \infty)$. Then, Laplace transform of f"(t) exists and $L(f''(t)) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \ldots -sf^{n-2}(0) - f^{n-1}(0)$; $s > \alpha$ where $f''(t) = \frac{d^n f(t)}{dt^n}$.

In particular, (i.e., nth order derivative of f (t))

(i) $L(f'(t)) = s F(s) - f(0), s > \alpha$

(ii) $L(f''(t)) = s^2 F(s) - sf(0) - f'(0)$ for $s > \alpha$

(iii) $L(f'''(t) = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$

For providing this result, we will prove it for the first derivative, as given below:

Theorem 1 : Let f(t) be real and continuous function of exponential order α on $[0, \infty)$. Also f'(t) be continuous or piecewise continuous function on $[0, \infty)$. Then prove that Laplace transform of f'(t) exists and $L(f'(t)) = s F(s) - f(0)$ for $s > \alpha$ if $F(s) = L(f(t))$.

Proof : Case I. When f' is continuous on $[0, \infty)$

$$\text{Then } L(f'(t)) = \int_0^\infty e^{-st} f'(t)\,dt \qquad \text{(by def. of Laplace transform)}$$

$$= \lim_{t\to\infty} \int_0^t e^{-st} f'(t)\,dt$$

$$= \lim_{t\to\infty} \left(e^{-st} f(t)\right)_0^t - \int_0^t -se^{-st} f(t)\,dt$$

$$= \lim_{t\to\infty} \left(\left\{e^{-st} f(t) - f(0)\right\} + s\int_0^t e^{-st} f(t)\,dt\right)$$

$$= \lim_{t\to\infty} e^{-st} f(t) - f(0) + s \lim_{t\to\infty} \int_0^t e^{-st} f(t)\,dt$$

$= 0 - f(0) + s\,L(f(t))$

$= s F(s) - f(0)$, if $s > \alpha$

$$\left[\begin{array}{l} \because \text{Given } f(t) \text{ is cont. of exponential order } \alpha \\ \therefore |f(t)| < ke^{\alpha t} \text{ for some scalar } k \\ \Rightarrow |e^{-st} f(t)| = e^{-st}\,|f(t)| < e^{-st}(ke^{\alpha t}) \\ \qquad\qquad\qquad \le \dfrac{K}{e^{(s-\alpha)t}} \to 0 \\ \qquad\qquad\qquad \text{as } t \to \infty \\ \Rightarrow \lim_{t\to\infty} e^{-st} f(t) = 0 \end{array}\right.$$

Case II. When f' is piecewise cont. on $[0, \infty)$

$\Rightarrow$ f' is piecewise cont. on [0 t]
where t is any positive number

$\Rightarrow$ f'(t) is discontinuous at t_1, t_2...........t_n
where $0 < t_1 < t_2 .... < t_n \le t$

$$\Rightarrow \quad L\,(f\,'(t)) = \int_0^{\infty} e^{-st} f\,'(t)\,dt = \underset{t\to\infty}{L\,im} \int_0^{t} e^{-st} f\,'(t)\,dt$$

$$= \underset{t\to\infty}{L\,im} \left\{ \int_0^{t_1} e^{-st} f\,'(t)\,dt + \int_{t_1}^{t_2} e^{-st} f\,'(t)\,dt ... + \int_{t_n}^{t} e^{-st} f\,'(t)\,dt \right\}$$

$$= \underset{t\to\infty}{L\,im} \left\{ (e^{-st} f(t))_0^{t_1} - \int_0^{t_1} -se^{-st} f(t)\,dt + (e^{-st} f(t))_{t_1}^{t_2} \right.$$

$$\left. -\int_{t_1}^{t_2} -se^{-st} f(t)\,dt + ... + (e^{-st} f(t))_{t_n}^{t} - \int_{t_n}^{t} -se^{-st} f(t)\,dt \right\}$$

$$= \underset{t\to\infty}{L\,im} \left\{ (e^{-st_1} f(t_1) - f(0) + s\int_0^{t} \frac{1}{e^{st}} f(t)\,dt \right.$$

$$+ (e^{-st_2} f(t_2) - e^{-st_1} f(t_1)) + s\int_{t_1}^{t_2} e^{-st} f(t)\,dt + ...$$

$$\left. + (e^{-st} f(t) - e^{-st_n} f(t_n) + s\int_{t_n}^{t} e^{-st} f(t)\,dt \right\}$$

$$= \underset{t\to\infty}{L\,im} \left\{ e^{-st_1} f(t_1) - f(0) + e^{-st_2} f(t_2) - e^{-st_1} f(t_1) + ... + e^{-st} f(t) \right.$$

$$\left. -e^{-st} f(t_n) + s\int_0^{t_1} e^{-st} f(t)\,dt + \int_{t_1}^{t_2} e^{-st} f(t)\,dt + ... \right\}$$

$$= \underset{t\to\infty}{L\,im} \left(e^{-st} f(t) - f(0) + s\left(\int_0^{t} e^{-st} f(t)\,dt + ... \right) \right) + \int_{t_n}^{t} e^{-st} f(t)\,dt$$

[Using property of definite integrals]

= s F(s) – f(0) if s > α [As explained in case (i)]

$\therefore$ L (f'(t)) = s F (s) – f (0), s > α

Hence the result.

Now, to understand the Laplace transform of integral, we study the following theorem:

Theorem 2 : Let f (t) (for t ≥ 0) be real and continuous function on [0, ∞).

If L (f (t)) = F(s), then prove Laplace transform of $\int_0^t f(z)\, dz$ exists

and $L\left[\int_0^t f(z)\, dz\right) = \frac{F(s)}{s}$ or $\frac{L\,(f(t))}{s}$

Sol. Given L (f(t) = F(s)

and let $H(t) = \int_0^t f(z)\, dz$

$$\therefore \quad H(0) = \int_0^0 f(z)\, dz = 0$$

Now $H'(t) = \frac{d}{dt}\left(\int_0^t f(z)\, dz\right)$

$$= \int_0^t \frac{\partial}{\partial t}(f(z))\, dz + \frac{dt}{dt} f(t) - 0$$

(Using Leibnitz Rule of differentiation under integral sign)

= 0 + f(t) – 0 = f (t)

$\Rightarrow$ H'(t) = f(t)

We know

L(H'(t)) = s L(H(t)) – H (0)

$\Rightarrow$ L (f(t) = s L (H (t) – 0 (By Theorem 1)

$$\Rightarrow \quad L(H(t)) = \frac{L\,(f(t))}{s}$$

$$\Rightarrow \quad L\left(\int_0^t f(z)\, dz\right) = \frac{L\,(f(t))}{s} \text{ or } \frac{F(s)}{s}$$

Now the reader can easily prove the result :

Cor : Prove $L\left(\int_0^t f(z)\,dz\right) = -\frac{F(s)}{s} + \frac{1}{s}\int_0^\infty f(z)\,dz$.

1.3.4 Multiplication and Division by 't'

On the basis of Laplace transforms of derivatives and integrals, we can easily understand the following results :

Result 2 : Let f (t) be real and piecewise continuous function of exponential order α on on $[0, \infty)$ and if L (f (t)) = F (s)

Then, $\frac{dF(s)}{ds} = -L(tf(t))$

and further $\frac{d^nF(s)}{ds^n} = (-1)^n\ L\,(t^n f(t))$ for $n = 1, 2$..........

Hint : The reader can easily prove this result with the help of Principle of mathematical induction.

Result 3 : Let f (t) be real and piecewise continuous function on each interval in $[0, \infty)$ and is of exponential order α.

If L (f (t)) = F(s). Then prove $L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s)$ ds if integral is convergent.

Proof : Given L (f (t) = F(s)

$$\Rightarrow \qquad F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Taking integrals on both sides w.r.t. s from s = s to s = ∞ we get

$$\int_s^\infty F(s)\,ds = \int_{s=s}^{s=\infty}\left(\int_{t=0}^{t=\infty} e^{-st}\ f(t)\,dt\right)ds$$

$$= \int_{t=0}^{t=\infty}\left(\int_{s=s}^{s=\infty} e^{-st} ds\right) f(t)\,dt$$

($\because$ s and t are independent so order of integration can be interchanged)

$$= \int_{t=0}^{t=\infty}\left(\frac{e^{-st}}{-t}\right)_s^\infty f(t)\,dt$$

$$= \int_{t=0}^{t=\infty} -\frac{1}{t}\left(\underset{s\to\infty}{L\,im}\, e^{-st} - e^{-st}\right) f(t)\, dt$$

$$= \int_{0}^{\infty} -\frac{1}{t}\,(0 - e^{-st})\, f(t)\, dt$$

$$\left(\because \underset{s\to\infty}{L\,im}\, e^{-st} = \underset{s\to\infty}{L\,im}\, \frac{1}{e^{st}} = 0\right)$$

$$= \int_{0}^{\infty} e^{-st}\, \frac{f(t)}{t}\, dt$$

$$= L\left(\frac{f(t)}{t}\right) \qquad \text{(Using def. of Laplace Transform)}$$

$$\Rightarrow \qquad \int_{s}^{\infty} F(s)\, ds = L\left(\frac{f(t)}{t}\right)$$

Hence the result.

1.3.5 Laplace Transform of Periodic Function

Theorem 3 : Let f(t) be periodic function with period T i.e. f (t + nT) = f(t) for $n \in N$.

then Prove that $L(f(t)) = \int_{0}^{T} \frac{e^{-st}}{1-e^{-sT}}\, f(t)\, dt$

Sol. We have $L(f(t)) = \int_{0}^{\infty} e^{-st} f(t)\, dt$

$$= \int_{0}^{T} e^{-st} f(t)\, dt + \int_{T}^{2T} e^{-st}\, f(t)\, dt + \int_{2T}^{3T} e^{-st} f(t)\, dt + \ldots\ldots..$$

$$= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)\,T} e^{-st} f(t)\, dt \qquad \text{Put } t = y + n\,T \Rightarrow dt = dy$$

$$= \sum_{n=0}^{\infty} \int_{y=0}^{y=T} e^{-s\,(y+nT)} f\,(y + nT)\, dy$$

($\because$ f is periodic with period T so f (y + nT) = f(y))

$$= \sum_{n=0}^{\infty} \int_0^T e^{-sy} e^{-snT} f(y)\, dy$$

$$= \sum_{n=0}^{\infty} e^{-snT} \int_0^T e^{-sy} f(y)\, dy$$

$$= (1 + e^{-sT} + e^{-2sT} + \ldots\ldots \text{ to } \infty)\left(\int_0^T e^{-sy} f(y)\, dy\right)$$

(Change variable y by t)

$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt = \int_0^T \frac{e^{-st}}{1-e^{-sT}} f(t)\, dt$$

Hence the result.

1.3.6 Some Important Examples

Example 1 : Evaluate L (t sin α t)

Sol. L f(t) = t sin α t

$\Rightarrow$ f'(t) = sin αt + t (α) cos αt

and f'(t) = α cos α t + α (cos α t – (α sin α t) t)

= 2 α cos α t – α^2 t sin α t

Using L (f"(t)) = s^2 L (f(t)) – s f(0) – f' (0)

$\Rightarrow$ L (2 α cos α t – α^2(t sin α t) = s^2 L (t sin α t) – s (0) – 0

$\Rightarrow$ 2 α L (cos α t) – α^2 L (t sin α t) = s^2 L (t sin α t)

$$\Rightarrow \quad 2\alpha \frac{s}{s^2+\alpha^2} = (s^2+\alpha^2)\, L\,(t \sin \alpha t)$$

$$\Rightarrow \quad L\,(t \sin \alpha t) = \frac{2\alpha s}{(s^2+\alpha^2)^2}$$

By Second Method

We know $L\ (\sin \alpha\, t) = \dfrac{\alpha}{s^2+\alpha^2} = F(s)$

$$L\,(t \sin \alpha\, t) = (-1)\frac{dF(s)}{ds} = (-1)\frac{d}{ds}\left(\frac{\alpha}{s^2+\alpha^2}\right)$$

$$= (-1)\,\alpha\left(\frac{(-1)\,2s}{(s^2+\alpha^2)^2}\right) = \frac{2\alpha s}{(s^2+\alpha^2)^2}$$

$$\Rightarrow \qquad L\,(t \sin \alpha\, t) = \frac{2\alpha s}{(s^2+\alpha^2)^2}.$$

Example 2 : Show that $\int_0^\infty e^{-3t} t \cos t\, dt = \frac{2}{25}$

Sol. We know $L\,(\cos t) = \frac{s}{s^2+1} = F(s)$

$$\therefore \qquad L\,(t \cos t) = (-1)\frac{d}{ds}\left(\frac{s}{s^2+1}\right)$$

$$= (-1)\frac{1\,(s^2+1) - s\,(2s)}{(s^2+1)^2} = \frac{s^2-1}{(s^2+1)^2}$$

$$\Rightarrow \qquad \int_0^\infty e^{-st} t \cos t\, dt = \frac{s^2-1}{(s^2+1)^2}$$

Put s = 3

$$\Rightarrow \qquad \int_0^\infty e^{-3t} t \cos t\, dt = \frac{9-1}{(9+1)^2} = \frac{8}{100} = \frac{2}{25}.$$

Example 3 : Find Laplace transform for $t \geq 0$, of $(t^2 - 3t + 2) \sin 3t$

Sol. Here $f(t) = (t^2 - 3t + 2) \sin 3t$

We know $L\,(\sin 3t) = \frac{3}{s^2+9}$

Now by Result 4.2 $L\,(t \sin 3t) = (-1)\,\frac{d}{ds}\left(\frac{3}{s^2+9}\right)$

$$= (-3)\frac{d}{ds}(s^2+9)^{-1}$$

$$= (-3)(-1)(s^2+9)^{-2}\,(2\,s) = \frac{6s}{(s^2+9)^2}$$

and $L\,(t^2 \sin 3t) = (-1)^2 \dfrac{d^2}{ds^2}\left(\dfrac{3}{s^2+9}\right)$

$$= \frac{d}{ds}\left(\frac{-6s}{(s^2+9)^2}\right)$$

$$= -6\,\frac{1(s^2+9)^2 - 2\,(s^2+9)\,(2s)\,(s)}{(s^2+9)^4}$$

$$= -\frac{6(s^2+9)\,\{s^2+9-4s^2\}}{(s^2+9)\,(s^2+9)^3}$$

$$= \frac{-6\,(-3s^2+9)}{(s^2+9)^3} = \frac{18\,(s^2-3)}{(s^2+9)^3}$$

$\therefore$ $L\,((t^2 - 3t + 2)\sin 3t) = L\,(t^2 \sin 3t) - 3L\,(t \sin 3t) + 2L\,(\sin 3t)$

$$= \frac{18(s^2-3)}{(s^2+9)^3} - 3\,\frac{6s}{(s^2+9)^2} + 2\,\frac{3}{(s^2+9)}$$

$$= \frac{6\,\{3s^2 - 9 - 3s\,(s^2+9) + (s^2+9)^2\}}{(s^2+9)^3}$$

$$= \frac{6(s^4 - 3s^3 + 21s^2 - 27s + 72)}{(s^2+9)^3}.$$

Example 4 : Find Laplace Transform of

$$\frac{e^{-\alpha t}\sin\beta t}{t}$$

Sol. We know $L\,(\sin\beta t) = \dfrac{\beta}{s^2+\beta^2}$

By first shifting Theorem.

$$L\,(e^{-\alpha t}\sin\beta t) = \frac{\beta}{(s+\alpha)^2+\beta^2} = F(s)$$

$\therefore$ By Result 4.3

$$L\left(\frac{e^{-\alpha t}\sin\beta t}{t}\right) = \int_s^\infty \frac{\beta}{(s+\alpha)^2+\beta^2} = F(s)$$

$$= \lim_{u\to\infty}\int_s^u \frac{\beta}{(s+\alpha)^2+\beta^2}\,ds$$

$$= \lim_{u\to\infty}\beta\,.\,\frac{1}{\beta}\tan^{-1}\left(\frac{s+\alpha}{\beta}\right)\Bigg]_s^u$$

$$= \lim_{u\to\infty}\left(\tan^{-1}\frac{u+\alpha}{\beta} - \tan^{-1}\frac{s+\alpha}{\beta}\right)$$

$$= \lim_{u\to\infty}\tan^{-1}\frac{u+\alpha}{\beta} - \tan^{-1}\frac{s+\alpha}{\beta}$$

$$= \tan^{-1}\infty - \tan^{-1}\frac{s+\alpha}{\beta} = \frac{\pi}{2} - \tan^{-1}\frac{s+\alpha}{\beta}$$

$$= \cot^{-1}\frac{s+\alpha}{\beta}.$$

Example 5 : Evaluate $\int_0^\infty t^3 e^{-1}\sin t\,dt$.

Sol. $\int_0^\infty t^3 e^{-1}\sin t\,dt = \int_0^\infty e^{-t}(t^3\sin t)\,dt$, where $s = 1$

$= L\,(t^3 \sin\, t)$ (By def. of Laplace Transform)

$$= (-1)^3\frac{d^3}{ds^3}(L(\sin t))$$

$$= -\frac{d^3}{ds^3}\left(\frac{1}{s^2+1}\right) = -\frac{d^2}{ds^2}\left(\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right)$$

$$= -\frac{d^2}{ds^2}\left(-\frac{2s}{(s^2+1)^2}\right)$$

$$= 2\frac{d}{ds}\left(\frac{d}{ds}\left(\frac{s}{(s^2+1)^2}\right)\right)$$

$$= 2\,\frac{d}{ds}\left(\frac{(s^2+1)^2 - s\,(2(s^2+1))\,(2s)}{(s^2+1)^4}\right)$$

$$= 2\,\frac{d}{ds}\left(\frac{s^2+1-4s^2}{(s^2+1)^3}\right) = 2\,\frac{d}{ds}\left(\frac{1-3s^2}{(s^2+1)^3}\right)$$

$$= 2\,\frac{(-6s)\,(s^2+1)^3 - 3(s^2+1)^2(2s)\,(1-3s^2)}{(s^2+1)^6}$$

$$= \frac{-2\,(6s)\,(s^2+1)^2\,\{(s^2+1)+1-3s^2\}}{(s^2+1)^6}$$

$$= \frac{-12s\,(2-2s^2)}{(s^2+1)^4} = \frac{24s\,(s^2-1)}{(s^2+1)^4}$$

Put s = 1 (From above)

$$\Rightarrow \qquad \int_0^\infty t^3 e^{-1} \sin t\, dt = \frac{24(1)\,(1-1)}{(1+1)^4} = 0.$$

1.3.7 Summary

In this lesson, we have studied about the Laplace Transforms of derivatives and integrals of functions. Moreover, these Laplace Transforms helped us to understand the Laplace Transforms of functions multiplied by t and divided by t. Laplace Transforms of periodic functions have been also studied under this chapter. We tried to understand the concepts easily with the help of some suitable examples.

1.3.8 Key Concepts

Laplace Transforms of derivatives, Laplace Transforms of Integrals, Laplace Transforms of Periodic Functions.

1.3.9 Long Questions

1. Evaluate L ($\sin^2 \alpha\, t \cos \alpha\, t$) for $t \geq 0$.

2. Find L (f(t)) where $f(t) = \begin{cases} \cos t, & 0 < 1 < \pi \\ 0, & \pi < 1 < 2\pi \end{cases}$ if f(t) is periodic with period 2π.

3. Evaluate $L\left(\frac{\cos 4t - \cos 5t}{t}\right)$ for $t > 0$.

1.3.10 Short Questions

1. Show that $\int_0^\infty e^{-3t} t \sin t\, dt = \frac{3}{50}$.

2. Show that $\int_0^\infty te^{-2t} \cos t\, dt = \frac{3}{25}$.

1.3.11 Suggested Readings :

1. A.R. Vasushtha & Dr. R.K. Gupta, Integral Transforms by Krishna Prakashan Media Pvt. Ltd. Meerut.

B.A. PART - III **PAPER-II**

MATHEMATICAL METHODS-I

LESSON NO. 1.4 **Author : Dr. Chanchal**

LAST UPDATED ON JUNE, 2023

INVERSE LAPLACE TRANSFORMS

1.4.1 Objectives

For this lesson, our topics of interest are :

- Inverse Laplace Transforms and its properties.
- Inverse Laplace Transforms of derivatives of integrals.
- Convolution Theorem.

1.4.2 Introduction

If L(f(t)) = F(s) for $t \geq 0$ i.e. Laplace transform of f(t) exists then the function f(t) is called the **Inverse Laplace Transform** of F(s) and is written as $L^{-1}(F(s)) = f(t)$.

i.e. $L^{-1}(F(s))$ is the function whose Laplace Transform is F(s).

For Example : $L(\sin 5t) = \frac{5}{s^2 + 25}$

$$\Rightarrow \quad L^{-1}\left(\frac{5}{s^2 + 25}\right) = \sin 5t$$

Further, we have an important result concerning the existence and uniqueness of inverse Laplace transforms :

Result 1 : If f(t) is piecewise continuous function on each interval $[0, \alpha]$ and is of exponential order for $t > \alpha$.

Also if L(f (t)) = F(s); then prove that the **Inverse Laplace Transform** f (t) is unique.

i.e. If $L^{-1}(F(s)) = f_1(t)$ and $L^{-1}F(s)) = f_2(t)$

Then $f_1(t) = f_2(t)$ for all $t \geq 0$.

1.4.3 Properties of Inverse Laplace Transforms

Theorem 1 (Linearity Property) :

If L(g(t)) = G(s) and L(h(t) = H(s) for any two functions g(t), h(t) ($t \geq 0$) and α, β are any constants then prove

$L^{-1}\{\alpha G(s) + \beta H(s)\} = \alpha L^{-1}(G(s)) + \beta L^{-1}(H(s))$

Proof : Given L(g(t)) = G9s) and L(h(t)) = H(s) for $t \geq 0$

$\therefore \quad \alpha G(s) + \beta H(s) = \alpha L(g(t)) + \beta L(h(t))$

$= L(\alpha g(t) + \beta h(t)$

(Using Theorem 3 i.e. Linear property of Laplace Transform)

By def. of Inverse Laplace Transform, we get

$L^{-1}(\alpha G(s) + \beta H(s)) = \alpha g(t) + \beta h(t) = \alpha L^{-1}(G(s)) + \beta L^{-1}(H(s))$

($\because$ Given implies that $g(t) = L^{-1}(G(s))$ and $h(t) = L^{-1}(H(s))$)

Hence the result.

Theorem 2 : (First Shifting Theorem of Inverse Laplace Transform)

if $L^{-1}(F(s)) = f(t)$ for $t \geq 0$

Prove $L^{-1}(F(s-\alpha) = e^{\alpha t} f(t) = e^{\alpha t} L^{-1}(F(s))$

Proof : Given $L^{-1}(F(s)) = f(t)$

$$\Rightarrow \quad F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

$$F(s-\alpha) = \int_0^\infty e^{-(s-\alpha)t} f(t)\,dt$$

(By def. of Laplace Transform)

$$= \int_0^\infty e^{-st}\{e^{\alpha t} f(t)\}\,dt$$

$$= L(e^{\alpha t} f(t))$$

so by def. of Inverse Laplace Transform we get

$$L^{-1}(F(s-\alpha)) = e^{\alpha t} f(t)$$

$$= e^{\alpha t} L^{-1}(F(s))$$

Hence the result

Note : Changing α to $-\alpha$ in above result.

we get $L^{-1}(F(s+\alpha)) = e^{-\alpha t} L^{-1}(F(s))$.

Theorem 3 : (Second Shifting Theorem of Inverse Laplace Transform)

If $L^{-1}(F(s)) = f(t)$ for $t \geq 0$

then prove $L^{-1}(e^{-\alpha s} F(s)) = g(t)$ where $g(t) = \begin{cases} f(t-\alpha), & t > \alpha \\ 0, & t < \alpha \end{cases}$.

Proof : We have $L^{-1}(F(s)) = f(t)$

$$\Rightarrow \quad F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$$

and given function is $g(t) = \begin{cases} f(t-\alpha), & t > \alpha \\ 0, & t < \alpha \end{cases}$

$$\therefore \quad L(g(t)) = \int_0^\infty e^{-st} g(t)\, dt$$

$$= \int_0^\alpha e^{-st} g(t)\, dt + \int_\alpha^\infty e^{-st} g(t)\, dt$$

$$= \int_0^\alpha e^{-st} (0)\, dt + \int_\alpha^\infty e^{-st} f(t-\alpha)\, dt$$

(By def. of g(t))

$$= 0 + \int_\alpha^\infty e^{-st} f(t-\alpha)\, dt$$

$$= \int_{V=0}^{V=\infty} e^{-s(V+\alpha)} f(V)\, dV \qquad [\text{Put } t-\alpha = V \Rightarrow dt = dV]$$

$$= e^{-\alpha s}\int_0^\infty e^{-sV} f(V)\, dt$$

$$= e^{-\alpha s}\int_0^\infty e^{-st}\, f(t)\, dt \qquad \text{(By changing variable V by t)}$$

$$= e^{-\alpha s} F(s)$$

$\Rightarrow$ $g(t) = L^{-1}(e^{-\alpha s}\, F(s))$

or $L^{-1}(e^{-\alpha s}F(s)) = g(t) = \begin{cases} f(t-\alpha), & f > \alpha \\ 0, & t < \alpha \end{cases}$

Another Form of above Theorem

If $L^{-1}(F(s)) = f(t)$ for $t \geq 0$ and $\alpha \geq 0$ real.

Prove $L^{-1}(e^{-\alpha s} F(s)) = f(t-\alpha)\; h\,(t-\alpha)$

where $h(t-\alpha) = \begin{cases} 1, & t > \alpha \\ 0, & t < \alpha \end{cases}$ or $h(t) = \begin{cases} t, & t > 0 \\ 0, & t < 0 \end{cases}$

is known as Unit Step Function or Heavyside's unit step function.

Theorem 4 : (Change of Scale Property)

If $L^{-1}(F(s)) = f(t)$ for $t \geq 0$

Prove (i) $L^{-1}\left(F\left(\frac{s}{\alpha}\right)\right) = \alpha f\,(\alpha t)$

(ii) $L^{-1}(F(\alpha\, s)) = \frac{1}{\alpha} f\left(\frac{t}{\alpha}\right)$ where $\alpha > 0$.

Proof : We have $L^{-1}(F(s)) = f(t)$ for $t \geq 0$

$\Rightarrow$ $F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$

(i) $\therefore$ $L\,(f(\alpha t) = \int_0^\infty e^{-st} f(\alpha t)\, dt$ Put $\alpha\, t = V \Rightarrow \alpha\, dt = dV$

$$= \int_{V=0}^{V=\infty} e^{-s\frac{V}{\alpha}} f(V)\, \frac{dV}{\alpha}$$

$$= \frac{1}{\alpha}\int_0^\infty e^{-\left(\frac{s}{\alpha}\right)V} dV$$

$$= \frac{1}{\alpha}\int_0^\infty e^{-\frac{s}{\alpha}t} f(t)\, dt$$

$$= \frac{1}{\alpha} F\left(\frac{s}{\alpha}\right) \qquad \text{(Change the variable V by t)}$$

$$\Rightarrow \qquad f(\alpha\, t) = L^{-1}\left(\frac{1}{\alpha} F\left(\frac{s}{\alpha}\right)\right)$$

$$= \frac{1}{\alpha} L^{-1}\left(F\left(\frac{s}{\alpha}\right)\right)$$

$$\Rightarrow \qquad L^{-1}\left(F\left(\frac{s}{\alpha}\right)\right) = \alpha f(\alpha t).$$

Now, the reader can easily prove the (ii) part.

1.4.4 Inverse Laplace Transforms of Derivatives and Integrals

For the concerned topic, we have the following results :

Result 2 : Inverse Laplace Transform of Derivatives

(i) If $L^{-1}(F(s)) = f(t)$ for $t \geq 0$

Prove $L^{-1}(F'(s)) = -t\, f(t)$.

(ii) Generalisation : If $L^{-1}(F(s)) = f(t)$, for $t \geq 0$

Prove $L^{-1}(F^n(s)) = (-1)^n t^n f(t)$.

Result 3 : (i) If $L^{-1}(F(s)) = f(t)$ for $t \geq 0$ and $f(0) = 0$.

Then prove $L^{-1}(s\, F(s)) = f'(t)$

(ii) Generalisation : If $L^{-1}(F(s)) = f(t)$ for $t \geq 0$

and $f(0) = f'(0) = f''(0) = \ldots\ldots = f^{n-1}(0) = 0$.

Then prove $L^{-1}(s^n F(s)) = f^n(t) = \dfrac{d^n f(t)}{dt^n}$.

Result 4 : Inverse Laplace Transform of Integrals

(i) If $L^{-1}(F(s)) = f(t)$ for $t \geq 0$

Then prove $L^{-1}\left(\int_s^{\infty} F(s)\,ds\right) = \frac{f(t)}{t}$

(ii) If $L^{-1}(F(s)) = f(t)$ for $t \geq 0$

Then prove $L^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(t)\,dt$.

1.4.5 Convolution Theorem

To find Inverse Laplace Transform of the product of two functions, whose inverse Laplace Transforms are known or can be easily evaluated, the **Convolution Theorem** will help us, which is stated as

Theorem 5 : If $L^{-1}(F(s)) = f(t)$ and $L^{-1}(G(s)) = g(t)$ then prove

$$L^{-1}(F(s)\,G(s)) = \int_0^t f(z)\,g(t-z)\,dz \text{ for } t \geq 0$$

Note : The integral on R.H.S is known as convolution of f and g and denoted as f = g

Proof : To prove $L^{-1}(F(s)\,G(s)) = \int_0^t f(z)\,g(t-z)\,dz = f * g$

We have to show $L(f * g) = F(s)\,G(s)$.
By def. of Laplace transform,

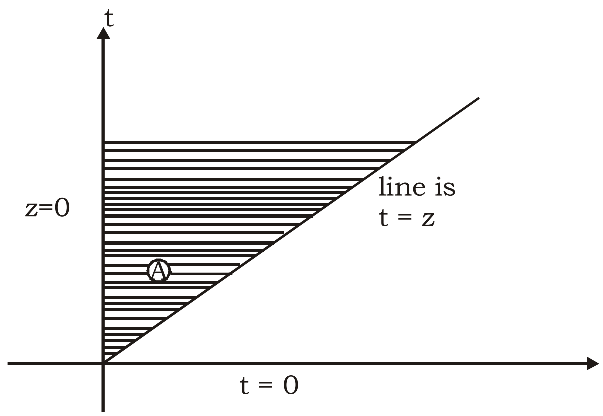
$$L(f * g) = \int_0^{\infty} e^{-st}(f * g)(t)\,dt$$

$$= \int_0^{\infty} e^{-st}\left(\int_0^t f(z)\,g(t-z)\,dz\right)dt$$

$$= \int_0^{\infty}\int_0^{1} e^{-st} f(z)\,g(t-z)\,dz\,dt$$

$$= \iint_A e^{-st} f(z)\,g(t-z)\,dz\,dt$$

where A is the shaded region



Put z = q and t – z = r

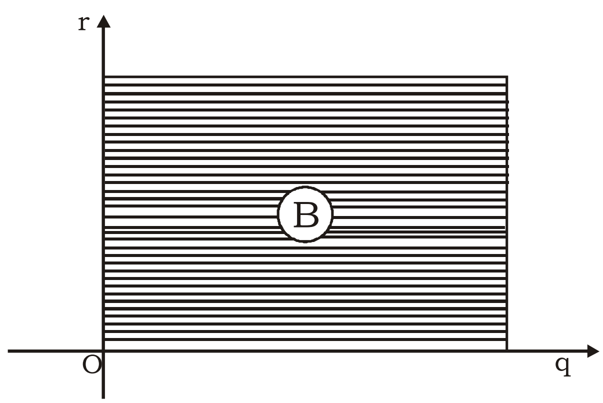
$\Rightarrow$ z = q and t = q + r

$$\therefore \quad \frac{\partial(t,z)}{\partial(q,r)} = \begin{vmatrix} \frac{\partial z}{\partial q} & \frac{\partial z}{\partial r} \\ \frac{\partial t}{\partial q} & \frac{\partial t}{\partial r} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

Hence dz dt = (1) dq dr = dq dr

So, the new variables transform region A into region

$B = \{(q, r) \mid q \geq 0, r \geq 0\}$



$$\therefore \quad L(f * g) = \iint_B e^{-s(q+r)} f(q)\, g(r)\, dq\, dr$$

$$= \int_0^\infty \int_0^\infty e^{-sq} e^{-sr} f(q)\, g(r)\, dq\, dr$$

$$= \left(\int_0^\infty e^{-sq} f(q)\, dq \right) \left(\int_0^\infty e^{-sr} g(r)\, dr \right)$$

Change variable q, r by t (By Property of double integrals)

$$= \left(\int_0^\infty e^{-st} f(t)\, dt \right) \left(\int_0^\infty e^{-st} g(t)\, dt \right)$$

$= L\,(f(t)\; L\,(g(t)) = F\,(s)\; G\,(s).$

$(\because\; L^{-1}(F(s)) = f(t) \text{ and } L^{-1}(G\,(s)) = g\,(t))$

Hence the Result.

Remarks : 1. $f*g = g*f \left(L^{-1}(F(s)\, G(s) = \int_0^t f(z) g(t-z)\, dz = \int_0^t g(z) f(t-z)\, dz \right)$

2. $f*(g+h) = f*g + f*h$

1.4.6 Some Important Examples

Example 1 : Show that

(i) $L^{-1}\left(\dfrac{s}{(s^2+\alpha^2)^2} \right) = \dfrac{1}{2\alpha}\, t \sin \alpha t$

(ii) $L^{-1}\left(\dfrac{s^2-\alpha^2}{(s^2+\alpha^2)^2} \right) = t \cos \alpha t$

Sol. We know $L\,(e^{i\alpha t}) = \dfrac{1}{s - i\alpha}$

$\Rightarrow$ $L\,(te^{i\alpha t}) = (-1) \dfrac{d}{ds} \left(\dfrac{1}{s - i\alpha} \right)$

$$= \frac{1}{(s - i\alpha)^2}$$

$\Rightarrow \qquad L\,(t\,(\cos \alpha t + i \sin \alpha t) = \dfrac{(s+i\alpha)^2}{((s-i\alpha)\,(s+i\alpha))^2}$

$\Rightarrow \qquad L\,(t \cos \alpha t) + iL\,(t \sin \alpha t) = \dfrac{(s^2-\alpha^2)+(2i\,\alpha)s}{(s^2+\alpha^2)^2}$

weget $L\,(t\cos\alpha t) = \dfrac{s^2-\alpha^2}{(s^2+\alpha^2)^2}$ and $L(t \sin \alpha\, t) = \dfrac{2\alpha s}{(s^2+\alpha^2)^2}$

$\therefore$ By def. of Inverse Laplace Transform

we get $t\cos\alpha t = L^{-1}\left(\dfrac{s^2-\alpha^2}{(s^2+\alpha^2)^2}\right)$

and $t \sin \alpha\, t = L^{-1}\left(\dfrac{2\alpha s}{(s^2+\alpha^2)^2}\right) = 2\,\alpha\, L^{-1}\left(\dfrac{s}{(s^2+\alpha^2)^2}\right)$

$\Rightarrow \qquad L^{-1}\left(\dfrac{s^2-\alpha^2}{(s^2+\alpha^2)^2}\right) = t\cos\alpha t$

$\Rightarrow \qquad L^{-1}\left(\dfrac{s}{(s^2+\alpha^2)^2}\right) = \dfrac{t\sin\alpha t}{2\alpha}.$

Example 2 : Find Inverse Laplace Transform of following functions

(i) $\dfrac{s^2}{(s-\alpha)^3}$ (ii) $L^{-1}\left(\dfrac{8s+3}{s^2e^{8s}}\right)$

Sol. (i) $L^{-1}\dfrac{s^2}{(s-\alpha)^3} = L^{-1}\left(\dfrac{((s-\alpha)+\alpha)^2}{(s-\alpha)^3}\right)$

$= e^{\alpha t}L^{-1}\left(\dfrac{(s+\alpha)^2}{s^3}\right)$

$= e^{\alpha t}\ L^{-1}\left(\dfrac{s^2+2\alpha s+\alpha^2}{s^3}\right)$

$$= e^{\alpha t} L^{-1}\left(\frac{1}{s} + 2\alpha\frac{1}{s^2} + \alpha^2\frac{1}{s^3}\right)$$

$$= e^{\alpha t}\left(L^{-1}\left(\frac{1}{s}\right) + 2\,\alpha L^{-1}\left(\frac{1}{s^2}\right) + \alpha^2 L^{-1}\left(\frac{1}{s^3}\right)\right)$$

$$= e^{\alpha t}\left(1 + 2\alpha t + \alpha^2\frac{t^2}{2}\right)$$

(ii) Here $L^{-1}\left(\frac{8s+3}{s^2}\right) = L^{-1}\left(\frac{8}{s} + \frac{3}{s^2}\right)$

$$= 8L^{-1}\left(\frac{1}{s}\right) + 3L^{-1}\left(\frac{1}{s^2}\right)$$

= 8 (1) + 3t = 3t + 8 = f(t), say

∴ Using Second shifting Theorem.

Example 3 : Evaluate : $L^{-1}\left(\frac{3s}{9s^2+27}\right)$

Sol. Take $F(s) = \frac{s}{s^2+27}$

⇒ $L^{-1}\left(\frac{s}{s^2+27}\right) = \cos(3\sqrt{3}\,t) = f(t)$ (say)

∴ By change of scale Property

$$L^{-1}\left(\frac{3s}{9s^2+27}\right) = L^{-1}\left(\frac{3s}{(3s)^2+27}\right)$$

$$= L^{-1}(F(3s)) = \frac{1}{3} f\left(\frac{t}{3}\right) = \frac{1}{3}\cos\left(3\sqrt{3}\,\frac{t}{3}\right)$$

$$= \frac{1}{3}\cos\left(\sqrt{3}\,t\right).$$

Example 4 : Evaluate : (i) $L^{-1}\left(4\tan^{-1}\frac{2}{s}\right)$ (ii) $L^{-1}\left(\frac{s}{(s^2+\alpha^2)^2}\right)$

Sol. (i) Let $F(s) = 4\tan^{-1}\frac{2}{s}$

$\Rightarrow$ $$F'(s) = \frac{4\frac{d}{ds}\left(\frac{2}{s}\right)}{1+\left(\frac{2}{s}\right)^2} = \frac{4s^2\left(-\frac{2}{s^2}\right)}{s^2+4}$$

$$= \frac{-8}{s^2+4}$$

$\therefore$ $$L^{-1}(F'(s)) = L^{-1}\left(\frac{-8}{s^2+4}\right)$$

$$= (-4)\,L^{-1}\left(\frac{2}{s^2+2^2}\right) = -4\sin 2t.$$

$\Rightarrow$ $-tf(t) = -4\sin 2t$ (Using Result 2)

$\Rightarrow$ $$f(t) = \frac{4\sin 2t}{t}$$

$\Rightarrow$ $$L^{-1}(F(s)) = \frac{4\sin 2t}{t}.$$

(ii) Firstly find a function whose derivative is $\frac{s}{(s^2+\alpha^2)^2}$

Here $$\int\frac{s}{(s^2+\alpha^2)^2} = \frac{1}{2}\int(s^2-\alpha^2)^{-2}(2\,s)\,ds$$

$$= \frac{1}{2}\,\frac{(s^2+\alpha^2)^{-2+1}}{-2+1}$$

$$= -\frac{1}{2(s^2+\alpha^2)} = F(s) \qquad \text{(Say)}$$

$\therefore$ $$\frac{d}{ds}\left(-\frac{1}{2\,(s^2+\alpha^2)}\right) = \frac{s}{(s^2+\alpha^2)^2}$$

Now $L^{-1}(F(s)) = L^{-1}\left(-\frac{1}{2(s^2+\alpha^2)}\right)$

$$= -\frac{1}{2\alpha} L^{-1}\left(\frac{\alpha}{s^2+\alpha^2}\right)$$

$$= -\frac{1}{2\alpha}\sin \alpha t$$

= f(t) (say)

Using $L^{-1}(F'(s)) = -t\, f(t)$ (By Result 2)

we get $L^{-1}\left(\frac{s}{(s^2+\alpha^2)^2}\right) = -t\left(-\frac{1}{2\alpha}\sin \alpha t\right)$

$$= \frac{1}{2\alpha} t \sin \alpha\, t.$$

Example 5 : Evaluate : $L^{-1}\left(2\log \frac{s^2+\beta^2}{s^2+\alpha^2}\right)$

Sol. Let $F(s) = 2\log \frac{s^2+\beta^2}{s^2+\alpha^2}$

$= 2\,(\log (s^2+\beta^2)) - \log (s^2+\alpha^2))$

$\Rightarrow$ $F'(s) = \frac{2}{s^2+\beta^2}(2s) - 2\,.\,\frac{1}{s^2+\alpha^2}(2s)$

$\Rightarrow$ $F'(s) = 4\left(\frac{s}{s^2+\beta^2} - \frac{s}{s^2+\alpha^2}\right)$

$\Rightarrow$ $L^{-1}(F'(s)) = 4\left(L^{-1}\left(\frac{s}{s^2+\beta^2}\right) - L^{-1}\left(\frac{s}{s^2+\alpha^2}\right)\right)$

$= 4\,(\cos \beta t - \cos \alpha t)$

$\Rightarrow$ $-t\, f(t) = 4\,(\cos \beta t - \cos \alpha t)$ (Using Result 2)

$\Rightarrow$ $f(t) = \frac{4\,(\cos \alpha t - \cos \beta t)}{t}.$

Example 6 : Find inverse Laplace transform of $\frac{1}{s^3(s+1)}$

Sol. We know $L^{-1}\left(\frac{1}{s+1}\right) = e^{-t}$

$\therefore$ By using property of Inverse Laplace Transform repeatedly Result 4

We have $L^{-1}\left(\frac{1}{s^3(s+1)}\right) = \int_0^t \left(\int_0^t \left(\int_0^t e^{-t}dt\right)dt\right)dt$

$$= \int_0^t \left(\int_0^t \left(\frac{e^{-t}}{-1}\right)_0^t dt\right)dt = \int_0^t \left(\int_0^t -(e^{-t}-1)\,dt\right)dt$$

$$= \int_0^t \left(e^{-1}+t\right)_0^t dt = \int_0^t \left\{(e^{-t}+t)-(e^0+0)\right\} dt$$

$$= \int_0^t (e^{-t}+t-1)\,dt = \left(\frac{e^{-t}}{-1}+\frac{t^2}{2}-t\right)_0^t$$

$$= \left(-e^{-t}+\frac{t^2}{2}-t\right)-(-e^0+0-0) = -e^{-t}+\frac{t^2}{2}-t+1\cdot$$

Example 7 : Find inverse Laplace Transform of following functions $\frac{1}{(s+\alpha)(s+\beta)}$

Sol. Let $\frac{1}{s+\alpha} = F(s)$ and $\frac{1}{s+\beta} = G(s)$

So that given function = F (s) G (s)

We know $L^{-1}(F(s)) = L^{-1}\left(\frac{1}{s+\alpha}\right) = e^{-\alpha t} = f(t)$ say

and $L^{-1}(G(s)) = L^{-1}\left(\frac{1}{s+\beta}\right) = e^{-\beta t} = g(t)$ say

Now using convolution Theorem

$$L^{-1}\left(\frac{1}{(s+\alpha)(s+\beta)}\right) = L^{-1}\ (F(s)\,G(s))$$

$$= \int_0^t f(z)\,g(t-z)\,dz = \int_0^t e^{-\alpha z} e^{-\beta\,(t-z)} dz$$

$$= \int_0^t e^{-\beta t+(\beta-\alpha)\,z}\ dz = e^{-\beta t}\int_0^t e^{(\beta-\alpha)\,z}\ dz$$

$$= e^{-\beta t}\left(\frac{e^{(\beta-\alpha)\,z}}{\beta-\alpha}\right)_0^t$$

$$= \frac{e^{-\beta}(e^{(\beta-\alpha)} - e^0)}{\beta-\alpha}$$

$$= \frac{e^{-\alpha t} - e^{-\beta t}}{\beta-\alpha}.$$

Example 8 : Apply convolution Theorem to show that

$$\int_0^t z e^{-t-8z} dz = \frac{e^{-t}}{64}\,(1-(1+8t)\,e^{-8t})$$

Sol. Given integral can be written as

$$\int_0^t e^{-9z-t+z} z\ dz = \int_0^t e^{-9z} z e^{(-t+z)} dz = \int_0^t f(z)\,g(t-z)\,dz$$

$= L^{-1}\,(F\ (s)\ G\ (s))$ where $f\ (z) = ze^{-9z}$

$g\ (z) = e^{-z}$

$= L^{-1}\left(\frac{1}{(s+9)^2}\,\frac{1}{s+1}\right)$ For this step reason is

$$= L^{-1}\left(\frac{1}{(s+1)\,(s+9)^2}\right)$$

$$= e^{-t}\ L^{-1}\left(\frac{1}{s(s+8)^2}\right)$$

$$\left[\because L^{-1}\left(\frac{1}{(s+8)^2}\right) = e^{-8t}\,L^{-1}\left(\frac{1}{s^2}\right) = e^{-8t}\ t\right.$$

$$= e^{-t}\left(-\frac{1}{8}(te^{-8t}-0)+\frac{1}{8}\left(\frac{e^{-8t}}{-8}\right)_0^t\right)$$

$$= e^{-t}\left(-\frac{1}{8}te^{-8t}-\frac{1}{6}\left(e^{-8t}-1\right)\right)$$

$$= \frac{e^{-t}}{64}(-(8t+1)e^{-8t}+1) = \frac{e^{-t}}{64}(1-(1+8t)e^{-8t})$$

Hence the result.

1.4.7 Summary

In this lesson, we have studied the inverse Laplace Transforms and its important properties similar to the properties of Laplace Transforms. Result are stated to find the inverse Laplace Transforms of derivatives and integrals. Moreover, we have stated and prove the convolution theorem related to the evaluation of inverse Laplace Transforms of product of two functions.

1.4.8 Key Concepts

Inverse Laplace Transforms, Linearity Property, First Shifting Theorem, Second Shifting Theorem, Change of Scale Property, Convolution Theorem.

1.4.9 Long Questions

1. Evaluate (i) $L^{-1}\left\{\frac{1}{(s-1)^5\,(s+2)}\right\}$ (ii) $L^{-1}\left\{\frac{30}{\left(\frac{s}{50}\right)^2-50}\right\}$

2. Evaluate : (i) $L^{-1}\left(\frac{s^2-\pi^2}{(s^2+\pi^2)^2}\right)$ (ii) $\frac{13}{s\,(s^2+169)}$ (iii) $L^{-1}\left(\frac{1}{s\,(\alpha^2 s^2+\beta^2)}\right)$

3. Prove 2 * 2 * 2 * 2 * .... * 2 (k times) = $\frac{2^k t^{k-1}}{\underline{|k-1}}$

4. Use Convolution Theorem to show that

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx = \frac{\overline{|m}\,\overline{|n}}{\overline{|m+n}}\ (m, n > 0)$$

(i.e. relation between Beta and Gamma Function)

Hence evaluate

$$\int_0^t \sinh z \cosh(t-z)\,dz = \frac{1}{2}\, t \sin ht$$

1.4.10 Short Questions

1. Evaluate (i) $L^{-1}\left(\frac{1}{s}\sin\frac{1}{s}\right)$ (ii) $L^{-1}\left(\frac{2s}{s^4+s^2+1}\right)$

1.4.11 Suggested Readings :

1. A.R. Vasushtha & Dr. R.K. Gupta, Integral Transforms by Krishna Prakashan Media Pvt. Ltd. Meerut.

Mandatory Student Feedback Form

https://forms.gle/KS5CLhvpwrpgjwN98

Note: Students, kindly click this google form link, and fill this feedback form once.