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Lesson No.

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LESSON NO. 2.1 **Author: Dr. Chanchal**

VECTOR DIFFERENTIAL CALCULUS - I

Structure:

2.1.0 Objectives

The prime goal of this section is to study the vector differential and integral calculus. We often call this study as vector analysis or vector field theory. During the study in this particular lesson, our main objectives are

- To discuss the limit, continuity and differentiability of vector functions.
- To study the ordinary and partial differentiation of vector functions.
- To study the integration of vector functions.

2.1.1 Introduction to Vector Functions

Before introducing a vector function, we wish to make the readers familiar with scalar function. A **scalar function** $f(x, y)$ is a function defined at each point in a certain domain D. Its value is real and depends only on the point $P(x, y)$ in space, but not on any particular coordinate system being used. For every point $(x, y) \in D$, f has a real value and we say that a scalar field f is defined in D.

For example : The distance function in 2- D space which defines the distance between the points $P(x, y)$ and $P_0(x_0, y_0)$, given by

$$f(P) = f(x, y) = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

defines a **scalar field**, where domain D is the whole of the 2-D space.

Now, we may define the vector function as:

Vector Function : A function $\vec{f} = \vec{f}(P) = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ defined at each point $P \in D$ is called a vector function and we say that a vector function is defined in D. In cartesian coordinates, we can write

$$\vec{f} = f_1(x, y)\hat{i} + f_2(x, y)\hat{j} + f_3(x, y)\hat{k}$$

If we recall that a curve C in the two dimensional $x - y$ plane van be parameterized by $x = x(t), y = y(t), a \le t \le b$. Then, the position vector of a point P on the curve C can be written as $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$.

Therefore, the position vector of a point on a curve defines a vector function and the vector function may now be defined as

Any vector $\vec{r}$ is said to be a vector function of t if $\vec{r}$ varies continuously with the variation of scalar variable t.

It is written as $\vec{r}$ or $\vec{r}(t)$ or $\vec{r} = \vec{f}(t)$.

For example : The position vector $\vec{r}$ of a particle moving along a curved path is a vector function of time t, where t is a scalar.

2.1.2 Limit and Continuity of Vector Functions

Limit : The vector function $\vec{f}(t)$ has the limit $\vec{l}$ as $t \to a$, if $\vec{f}(t)$ is defined in some neighborhood of a, except possibly at $t = a$, and

$$\underset{t \to a}{lt}\left|\vec{f}(t) - \vec{l}\right| = 0 \quad \text{or} \quad \underset{t \to a}{lt}\,\vec{f}(t) = \vec{l}$$

Continuity : A vector function $\vec{f}(t)$ is said to be continuous at $t = a$, if

(i) $\vec{f}(t)$ is defined in some neighborhood of a,

(ii) $\underset{t \to a}{lt}\,\vec{f}(t)$ exists, and

(iii) $\underset{t \to a}{lt}\,\vec{f}(t) = \vec{f}(a)$

2.1.3 Differentiability of Vector Functions

Let $\vec{r} = \vec{f}(t)$ be a vector function of the scalar variable t,

$$\therefore \vec{r} + \delta\vec{r} = \vec{f}(t + \delta t)$$

$$\Rightarrow \delta\vec{r} = \vec{f}(t+\delta t) - \vec{f}(t)$$

$$\Rightarrow \frac{\delta\vec{r}}{\delta t} = \frac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t}$$

$\therefore \underset{\delta t \to 0}{lt} \dfrac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t}$, if it exists, is called the derivative of vector function $\vec{r}$ with respect to scalar variable t and the vector function $\vec{r} = \vec{f}(t)$ is said to be differentiable. It is denoted by $\dfrac{d\vec{r}}{dt}$ and therefore, we can write

$$\therefore \frac{d\vec{r}}{dt} = \underset{\delta t \to 0}{lt} \frac{(\vec{r} + \delta\vec{r}) - \vec{r}}{\delta t} = \underset{\delta t \to 0}{lt} \frac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t}$$

2.1.3.1 Some Useful Results Concerning Differentiation

i. $\dfrac{d}{dt}\left(\vec{a} + \vec{b}\right) = \dfrac{d\vec{a}}{dt} + \dfrac{d\vec{b}}{dt}$

ii. $\dfrac{d}{dt}\left(\vec{a}.\vec{b}\right) = \vec{a}.\dfrac{d\vec{b}}{dt} + \dfrac{d\vec{a}}{dt}.\vec{b}$

iii. $\dfrac{d}{dt}\left(\vec{a} \times \vec{b}\right) = \vec{a} \times \dfrac{d\vec{b}}{dt} + \dfrac{d\vec{a}}{dt} \times \vec{b}$

iv. $\dfrac{d}{dt}\left(\phi\vec{a}\right) = \phi\dfrac{d\vec{a}}{dt} + \dfrac{d\phi}{dt}\vec{a}$

v. $\dfrac{d}{dt}\left[\vec{a}, \vec{b}, \vec{c}\right] = \left[\dfrac{d\vec{a}}{dt}, \vec{b}, \vec{c}\right] + \left[\vec{a}, \dfrac{d\vec{b}}{dt}, \vec{c}\right] + \left[\vec{a}, \vec{b}, \dfrac{d\vec{c}}{dt}\right]$

vi. $\dfrac{d}{dt}\left(\vec{a} \times \left(\vec{b} \times \vec{c}\right)\right) = \dfrac{d\vec{a}}{dt} \times \left(\vec{b} \times \vec{c}\right) + \vec{a} \times \left(\dfrac{d\vec{b}}{dt} \times \vec{c}\right) + \vec{a} \times \left(\vec{b} \times \dfrac{d\vec{c}}{dt}\right)$

where $\vec{a}, \vec{b}, \vec{c}$ are differentiable vector functions and ϕ is a differentiable scalar function of the same variable t.

Note: The readers may easily prove the above results.

2.1.3.2 Derivative of a Vector Function in Terms of its Components

Let $\vec{r}$ be a vector function of the scalar variable t. Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ where the components x, y, z are functions of t and $\hat{i}, \hat{j}, \hat{k}$ are fixed unit vectors.

$$\therefore \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) = \frac{d}{dt}\left(x\hat{i}\right) + \frac{d}{dt}\left(y\hat{j}\right) + \frac{d}{dt}\left(z\hat{k}\right)$$

$$= \frac{dx}{dt}\hat{i} + x\frac{d\hat{i}}{dt} + \frac{dy}{dt}\hat{j} + y\frac{d\hat{j}}{dt} + \frac{dz}{dt}\hat{k} + z\frac{d\hat{k}}{dt}$$

$$= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

since $\frac{d\hat{i}}{dt} = \vec{0}, \frac{d\hat{j}}{dt} = \vec{0}, \frac{d\hat{k}}{dt} = \vec{0}$ as $\hat{i}, \hat{j}, \hat{k}$ are the constant vectors (a vector is said to be constant if both of its magnitude and direction do not change).

2.1.4 Some Important and Useful Articles

I. The necessary and sufficient condition for the vector function $\vec{f}(t)$ to be constant is $\frac{d\vec{f}}{dt} = \vec{0}$.

II. The necessary and sufficient condition for the vector function $\vec{f}(t)$ to have constant magnitude is $\vec{f}.\frac{d\vec{f}}{dt} = 0$.

III. The necessary and sufficient condition for the vector function $\vec{f}(t)$ to have constant direction is $\vec{f} \times \frac{d\vec{f}}{dt} = \vec{0}$.

Proof : The proof is left for the reader.

2.1.5 Velocity and Acceleration

If the scalar variable t be the time and $\vec{r}$ be the position vector of a moving particle P with respect to the origin O, then $\delta\vec{r}$ is the displacement of the particle in time δt. Thus, $\frac{\delta\vec{r}}{\delta t}$ is the average velocity of the particle during the interval δt. If $\vec{v}$ represents the velocity vector of the particle at P, then

$$\vec{v} = \underset{\delta t \to 0}{lt} \frac{\delta\vec{r}}{\delta t} = \frac{d\vec{r}}{dt}$$

Similarly, if $\vec{a}$ represents the acceleration of the particle at time t, then

$$\vec{a} = \underset{\delta t \to 0}{lt} \frac{\delta\vec{v}}{\delta t} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}$$

2.1.6 Partial Derivatives of Vector Functions

Let $\vec{r} = \vec{f}(x, y, z)$ i.e. $\vec{r}$ is a function of three variables x, y and z.

Then, partial derivative of $\vec{r}$ w.r.t. the variable x is defined as

$$\frac{\partial \vec{r}}{\partial x} = \underset{\delta x \to 0}{lt} \frac{\vec{f}(x+\delta x, y, z) - \vec{f}(x, y, z)}{\delta x}$$ provided this limit exists.

Similarly, $\frac{\partial \vec{r}}{\partial y} = \underset{\delta y \to 0}{lt} \frac{\vec{f}(x, y+\delta y, z) - \vec{f}(x, y, z)}{\delta y}$

and $\frac{\partial \vec{r}}{\partial z} = \underset{\delta z \to 0}{lt} \frac{\vec{f}(x, y, z+\delta z) - \vec{f}(x, y, z)}{\delta z}$

2.1.7 Integration of Vector Functions

Integration is just the reverse process of differentiation. Let $\vec{f}(t)$ and $\vec{F}(t)$ be the two vector functions such that

$$\frac{d}{dt}\left\{\vec{F}(t)\right\} = \vec{f}(t)$$

Then, $\vec{F}(t)$ is called the indefinite integral of $\vec{f}(t)$ w.r.t. t and we write it as

$\int \vec{f}(t)dt = \vec{F}(t) + c$, where c is the integration constant.

Further for the integration, we use the below mentioned useful results:

1. To integrate a vector function, integrate its components.
2. $\int \left(2\vec{r}.\frac{d\vec{r}}{dt}\right)dt = \vec{r}^{\,2} + c$
3. $\int \left(2\frac{d\vec{r}}{dt}.\frac{d^2\vec{r}}{dt^2}\right)dt = \left(\frac{d\vec{r}}{dt}\right)^2 + c$
4. $\int \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right)dt = \vec{r} \times \frac{d\vec{r}}{dt} + \vec{c}$
5. $\int \left(\vec{a} \times \frac{d\vec{r}}{dt}\right)dt = \vec{a} \times \vec{r} + \vec{c}$, where $\vec{a}$ is a constant vector.
6. $\int c\vec{r}dt = c\int \vec{r}dt$

2.1.8 Some Important Examples

Example 1 : If $\vec{r} = (a\cos t)\hat{i} + (a\sin t)\hat{j} + t\hat{k}$, find $\frac{d\vec{r}}{dt}$, $\frac{d^2\vec{r}}{dt^2}$ and $\left|\frac{d^2\vec{r}}{dt^2}\right|$.

Sol. Here $\vec{r} = (a\cos t)\hat{i} + (a\sin t)\hat{j} + t\hat{k}$

$$\therefore \frac{d\vec{r}}{dt} = (-a\sin t)\hat{i} + (a\cos t)\hat{j} + \hat{k}$$

And $\dfrac{d^2\vec{r}}{dt^2} = (-a\cos t)\hat{i} + (-a\sin t)\hat{j} + 0$

$$\therefore \left|\frac{d^2\vec{r}}{dt^2}\right| = \sqrt{(-a\cos t)^2 + (-a\sin t)^2 + (0)^2} = \sqrt{a^2\cos^2 t + a^2\sin^2 t} = \sqrt{a^2(1)} = \sqrt{a^2} = a$$

Example 2 : A particle moves along the curve $x = t^3 + 1, y = t^2, z = 2t + 5$, where t is time. Find the components of its velocity and acceleration at time $t = 1$ in the direction $\hat{i} + \hat{j} + 3\hat{k}$.

Sol. If $\vec{r}$ is the position vector of any point (x, y, z) on the given curve, then

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = (t^3 + 1)\hat{i} + t^2\hat{j} = (2t + 5)\hat{k}$

$\therefore \vec{v} = \dfrac{d\vec{r}}{dt} = 3t^2\hat{i} + 2t\hat{j} + 2\hat{k}$

and $\vec{a} = \dfrac{d^2\vec{r}}{dt^2} = 6t\hat{i} + 2\hat{j}$

At $t = 1$, $\vec{v} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{a} = 6\hat{i} + 2\hat{j}$

Let $\hat{b}$ be the unit vector in the direction $\hat{i} + \hat{j} + 3\hat{k}$.

$$\therefore \hat{b} = \frac{\hat{i} + \hat{j} + 3\hat{k}}{|\hat{i} + \hat{j} + 3\hat{k}|} = \frac{\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{1+1+9}} = \frac{\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{11}}.$$

Now, component of velocity in the direction of $\hat{i} + \hat{j} + 3\hat{k} = \vec{v}.\hat{b}$

$$= (3\hat{i} + 2\hat{j} + 2\hat{k}).\left(\frac{\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{11}}\right) = \frac{(3)(1) + (2)(1) + (2)(3)}{\sqrt{11}} = \frac{11}{\sqrt{11}} = \sqrt{11}$$

Further, component of acceleration in the direction of $\hat{i} + \hat{j} + 3\hat{k} = \vec{a}.\hat{b}$

$$= (6\hat{i} + 2\hat{j}).\left(\frac{\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{11}}\right) = \frac{(6)(1) + (2)(1) + (0)(3)}{\sqrt{11}} = \frac{8}{\sqrt{11}}$$

Example 3 : If $\vec{r} = t^3\hat{i} + \left(2t^3 - \dfrac{1}{5t^2}\right)\hat{j}$, show that $\vec{r} \times \dfrac{d\vec{r}}{dt} = \hat{k}$.

Sol. Here, $\vec{r} = t^3\hat{i} + \left(2t^3 - \dfrac{1}{5t^2}\right)\hat{j}$

$$\therefore \frac{d\vec{r}}{dt} = 3t^2\hat{i} + \left(6t^2 + \frac{2}{5t^3}\right)\hat{j}$$

$$\therefore \vec{r} \times \frac{d\vec{r}}{dt} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t^3 & 2t^3 - \frac{1}{5t^2} & 0 \\ 3t^2 & 6t^2 + \frac{2}{5t^3} & 0 \end{vmatrix} = (0-0)\hat{i} + (0-0)\hat{j} + \left(6t^5 + \frac{2}{5} - 6t^5 + \frac{3}{5}\right)\hat{k}$$

$$\therefore \vec{r} \times \frac{d\vec{r}}{dt} = \hat{k}.$$

Example 4 : If $\vec{f} = \left(2x^2 y - x^4\right)\hat{i} + \left(e^{xy} - y\sin x\right)\hat{j} + \left(x^2 \cos y\right)\hat{k}$, the find $\frac{\partial \vec{f}}{\partial x}$, $\frac{\partial \vec{f}}{\partial y}$, $\frac{\partial^2 \vec{f}}{\partial x^2}$, $\frac{\partial^2 \vec{f}}{\partial y^2}$. Also, show that $\frac{\partial^2 \vec{f}}{\partial x \partial y} = \frac{\partial^2 \vec{f}}{\partial y \partial x}$.

Sol. Here, $\vec{f} = \left(2x^2 y - x^4\right)\hat{i} + \left(e^{xy} - y\sin x\right)\hat{j} + \left(x^2 \cos y\right)\hat{k}$

$$\therefore \frac{\partial \vec{f}}{\partial x} = \left\{\frac{\partial}{\partial x}\left(2x^2 y - x^4\right)\right\}\hat{i} + \left\{\frac{\partial}{\partial x}\left(e^{xy} - y\sin x\right)\right\}\hat{j} + \left\{\frac{\partial}{\partial x}\left(x^2 \cos y\right)\right\}\hat{k}$$

$$= \left(4xy - 4x^3\right)\hat{i} + \left(ye^{xy} - y\cos x\right)\hat{j} + \left(2x\cos y\right)\hat{k}$$

Now, $$\frac{\partial \vec{f}}{\partial y} = \left\{\frac{\partial}{\partial y}\left(2x^2 y - x^4\right)\right\}\hat{i} + \left\{\frac{\partial}{\partial y}\left(e^{xy} - y\sin x\right)\right\}\hat{j} + \left\{\frac{\partial}{\partial y}\left(x^2 \cos y\right)\right\}\hat{k}$$

$$= \left(2x^2\right)\hat{i} + \left(xe^{xy} - \sin x\right)\hat{j} + \left(-x^2 \sin y\right)\hat{k}$$

Further, $$\frac{\partial^2 \vec{f}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \left(4y - 12x^2\right)\hat{i} + \left(y^2 e^{xy} + y\sin x\right)\hat{j} + \left(2\cos y\right)\hat{k},$$

$$\frac{\partial^2 \vec{f}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \left(x^2 e^{xy}\right)\hat{j} + \left(-x^2 \cos y\right)\hat{k}$$

$$\frac{\partial^2 \vec{f}}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (4x)\hat{i} + \left(xye^{xy} + e^{xy} - \cos x\right)\hat{j} + \left(-2x\sin y\right)\hat{k}$$

and $$\frac{\partial^2 \vec{f}}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (4x)\hat{i} + \left(xye^{xy} + e^{xy} - \cos x\right)\hat{j} + \left(-2x\sin y\right)\hat{k}$$

$$\therefore \frac{\partial^2 \vec{f}}{\partial x \partial y} = \frac{\partial^2 \vec{f}}{\partial y \partial x}$$

Example 5 : If $\vec{A} = x^2yz\hat{i} - 2xz^3\hat{j} + xz^2\hat{k}$ and $\vec{B} = 2z\hat{i} + y\hat{j} - x^2\hat{k}$, then find the value of $\frac{\partial^2}{\partial x \partial y}\left(\vec{A}\times\vec{B}\right)$ at the point (1,0,-2).

Sol. Here, $\vec{A} = x^2yz\hat{i} - 2xz^3\hat{j} + xz^2\hat{k}$ and $\vec{B} = 2z\hat{i} + y\hat{j} - x^2\hat{k}$

$$\therefore \vec{A}\times\vec{B} = \left(x^2yz\hat{i} - 2xz^3\hat{j} + xz^2\hat{k}\right)\times\left(2z\hat{i} + y\hat{j} - x^2\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x^2yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix} = (2x^3z^3 - xyz^2)\hat{i} - (-x^4yz - 2xz^3)\hat{j} + \left(x^2y^2z + 4xz^4\right)\hat{k}$$

$$\therefore \frac{\partial}{\partial y}\left(\vec{A}\times\vec{B}\right) = \left(-xz^2\right)\hat{i} + \left(x^4z\right)\hat{j} + \left(2x^2yz\right)\hat{k}$$

$$\therefore \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial y}\left(\vec{A}\times\vec{B}\right)\right\} = \left(-z^2\right)\hat{i} + \left(4x^3z\right)\hat{j} + \left(4xyz\right)\hat{k}$$

$$\Rightarrow \frac{\partial^2}{\partial x \partial y}\left(\vec{A}\times\vec{B}\right) = \left(-z^2\right)\hat{i} + \left(4x^3z\right)\hat{j} + \left(4xyz\right)\hat{k}$$

At (1,0,-2), $\frac{\partial^2}{\partial x \partial y}\left(\vec{A}\times\vec{B}\right) = \left(-(-2)^2\right)\hat{i} + \left(4(1)^3(-2)\right)\hat{j} + \left(4(1)(0)(-2)\right)\hat{k} = -4\hat{i} - 8\hat{j}$

Example 6 : If $\vec{r} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$, then prove that $\int_1^2 \left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt = -14\hat{i} + 75\hat{j} - 15\hat{k}$.

Sol. Here, $\vec{r} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$

$$\therefore \frac{d\vec{r}}{dt} = 10t\hat{i} + \hat{j} - 3t^2\hat{k}$$

$$\therefore \vec{r}\times\frac{d\vec{r}}{dt} = \left(5t^2\hat{i} + t\hat{j} - t^3\hat{k}\right)\times\left(10t\hat{i} + \hat{j} - 3t^2\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ 10t & 1 & -3t^2 \end{vmatrix} = (-3t^3 + t^3)\hat{i} - (-15t^4 + 10t^4)\hat{j} + (5t^2 - 10t^2)\hat{k}$$

$$\therefore \vec{r}\times\frac{d\vec{r}}{dt} = -2t^3\hat{i} + 5t^4\hat{j} - 5t^2\hat{k} \tag{1}$$

Now, L.H.S. = $\int_1^2 \left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt = \left[\vec{r}\times\frac{d\vec{r}}{dt}\right]_1^2$

$=\left[-2t^3\hat{i}+5t^4\hat{j}-5t^2\hat{k}\right]_1^2$ [using (1)]

$=\left[-2t^3\right]_1^2\hat{i}+\left[5t^4\right]_1^2\hat{j}+\left[-5t^2\right]_1^2\hat{k}$

$=(-16+2)\hat{i}+(80-5)\hat{j}+(-20+5)\hat{k}$

$\therefore \int_1^2\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt=-14\hat{i}+75\hat{j}-15\hat{k}$

2.1.9 Summary

In this lesson, we have studied basically about the limit, continuity and differentiability of vector functions. Through the study, we came to know that the concepts of limit, continuity and differentiability of calculus can easily be used for vector functions. Further, the higher order derivatives and rules of differentiation for vector functions have the same form as in the case of real valued functions.

2.1.10 Self Check Exercise

1. If $\vec{a}=(\sin\theta)\hat{i}+(\cos\theta)\hat{j}+\theta\hat{k}$, $\vec{b}=(\cos\theta)\hat{i}-(\sin\theta)\hat{j}-3\hat{k}$ and $\vec{c}=2\hat{i}+3\hat{j}-3\hat{k}$. Find $\frac{d}{d\theta}\left\{\vec{a}\times\left(\vec{b}\times\vec{c}\right)\right\}$ at $\theta=\frac{\pi}{2}$.
2. If $\vec{r}=(\cos nt)\hat{i}+(\sin nt)\hat{j}$, where n is a constant, show that $\vec{r}\times\frac{d\vec{r}}{dt}=n\hat{k}$.
3. If $\vec{r}$ is a unit vector, then prove that $\left|\vec{r}\times\frac{d\vec{r}}{dt}\right|=\left|\frac{d\vec{r}}{dt}\right|$.
4. A particle moves along a curve whose parametric equations are $x=e^{-t}, y=a\cos 3t, z=b\sin 3t$, where t is time. Find the components of its velocity and acceleration at time $t=0$.
5. If $\vec{a}=xyz\hat{i}+xz^2\hat{j}-y^3\hat{k}$ and $\vec{b}=x^3\hat{i}-xyz\hat{j}-x^2z\hat{k}$, then sow that $\frac{\partial^2\vec{A}}{\partial y^2}\times\frac{\partial^2\vec{B}}{\partial x^2}$ at the point (1,1,0) is $-36\hat{j}$.
6. Evaluate $\int_0^1\left\{t\hat{i}+(t^2-2t)\hat{j}+(5t^2+3t^3)\hat{k}\right\}dt$
7. If $\vec{r}.d\vec{r}=0$, show that r =constant.

2.1.11 Suggested Readings

1. RK Jain, SRK Lyenger — Advanced Engineering Mathematics
2. JR Sharma — Advanced Calculus

LESSON NO. 2.2 **Author : Dr. Chanchal**

VECTOR DIFFERENTIAL CALCULUS – II

Structure :

2.2.0 Objectives

The prime objective of this lesson is to study the operators such as gradiant, divergence and curl, on scalar and vector point functions. Further the physical interpretation of these operators is also elaborated under the same.

2.2.1 Introduction to Point Functions

Point Function : A variable quantity whose value at any point in a region of space depends upon the position of the point, is called a point function.

Point function are of two types :

(i) Scalar Point Function (ii) Vector Point Function

Def : Scalar Point Function :

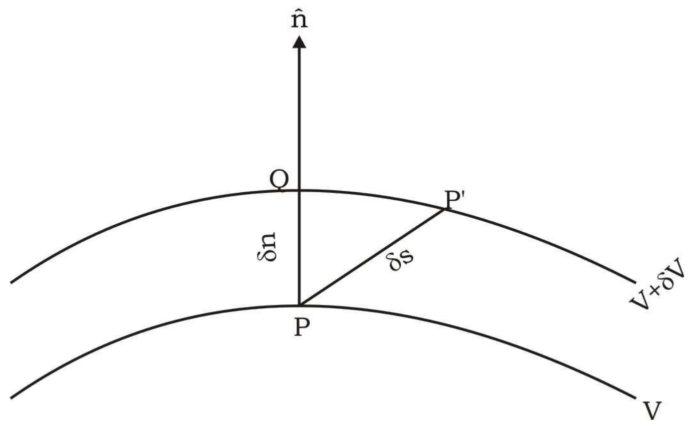
A function f (x, y, z) is called a scalar point function if it associates a scalar with every point in region R of a space. Region R is called scalar field. The temperature distribution in a heated body, density of a body and potential due to gravity are examples of scalar point functions.

Def : Vector Point Function :

If a function $\vec{f}(x,y,z)$ defines a vector at every point of the region R of a space then $\vec{f}(x,y,z)$ is called a vector point function and R is called a vector field.

Def : Level Surface :

Let V (x, y, z) be a scalar point function over a certain region. All those points which satisfy an equation of the type V(x, y, z) = c constitute a family of surfaces in three dimensional space. The surfaces of this family are called level surfaces. The value of the function at every point of a level surface is the same.



2.2.2 Gradient of a Scalar Point Function and its Physical Interpretation

Consider the level surfaces of the function through P, P' with values, V, V + δV respectively. Let Q be the point at which the second surface is cut by the normal at P to the first, and let δn be the length PQ. Then the limiting value of $\frac{\delta V}{\delta n}$ as $\delta n \to 0$ is the directional derivative of V in the direction normal to the level surface at P, and is written as $\frac{\delta V}{\delta n}$. If then $\hat{n}$ is the unit vector normal to the level surface at P and having the sense from P to Q, then the vector $\frac{\delta V}{\delta n}\hat{n}$ is called the gradient of the function V and is denoted by grad V or ΔV.

In rectangular coordinates, $\Delta V = \frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)V$

Note : (i) $V = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ is a vector operator, Its operation on a scalar function is a vector.

(ii) Δ can be operated on a scalar point function only and not on a vector point function.

2.2.3 Some Important and Useful Articles

Art 1 : Prove that grad V is a vector normal to the surface V(x, y, z) = c, where c is constant.

Proof : The equation of level surface is V(x, y, z) = c

Let $\vec{r}$ be the position vector of any point P (x, y, z) on given surface.

$\therefore \quad \vec{r} = x\,\hat{i} + y\hat{j} + z\hat{k}$

Again let $\vec{r} + d\vec{r}$ be the position vector of any neighbouring point

Q (x + dx, y + dy, z + dz) on given surface.

$\therefore \quad \vec{r} + d\vec{r} = (x + dx)\,\hat{i} + (y + dy)\,\hat{j} + (z + dz)\,\hat{k}$

$\therefore \quad \overrightarrow{PQ} = \vec{r} + d\vec{r} - \vec{r} = d\vec{r} \Rightarrow d\vec{r} = (dx)\,\hat{i} + (dy)\,\hat{j} + (dz)\,\hat{k}$

As Q $\to$ P, the line PQ becomes tangent at P to the level surface.

$\therefore \quad d\vec{r} = (dx)\hat{i} + (dy)\,\hat{j} + (dz)\,\hat{k}$

lies in the tangent plane to the surface at P.

$\therefore \quad d\vec{r}.\nabla V = \left\{(dx)\hat{i} + (dy)\hat{j} + (dz)\hat{k}\right\}.\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right)$

$= \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = dV$

= 0 as V is constant.

$\therefore \quad \nabla V$ is perpendicular to $d\vec{r}$

But $d\vec{r}$ lies in the tangent plane to the surface at P

$\therefore \quad \nabla V$ is normal to the surface at P.

Art 2 : Show that $dV = d\vec{r}\,.\,\nabla V$

Proof : Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\therefore \qquad d\vec{r}.\nabla V = \left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right).\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right) = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

$$\therefore \qquad d\vec{r}\,.\,\nabla V = dV\,.$$

Art 3 : If u and v are two scalar point functions, then

(i) $\nabla\,(u+v) = \nabla\, u + \nabla\, v$ (ii) $\nabla\,(uv) = u\,\nabla\, v + u\,\nabla\, v$

(iii) $\nabla\left(\frac{u}{v}\right) = \frac{v\nabla u - u\nabla u}{u^2}$

Proof : (i) L.H.S. = $\nabla\,(u + v) = \hat{i}\frac{\partial}{\partial x}(u+v) + \hat{j}\frac{\partial}{\partial y}(u+v) + \hat{k}\frac{\partial}{\partial z}(u+v)$

$$= \hat{i}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) + \hat{j}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) + \hat{k}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial z}\right)$$

$$= \left(\hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z}\right) + \left(\hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z}\right)$$

$$= \nabla u + \nabla v = \text{R.H.S.}$$

(ii) L.H.S. = $\nabla(uv) = \hat{i}\frac{\partial}{\partial x}(uv) + \hat{j}\frac{\partial}{\partial y}(uv) + \hat{k}\frac{\partial}{\partial z}(uv)$

$$= \hat{i}\left(u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}\right) + \hat{j}\left(u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}\right) + \hat{k}\left(u\frac{\partial v}{\partial z} + v\frac{\partial u}{\partial z}\right)$$

$$= u\left(\hat{i}\frac{\partial v}{\partial x} + \hat{j}\frac{\partial v}{\partial y} + \hat{k}\frac{\partial v}{\partial z}\right) + v\left(\hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z}\right) = u\,\nabla v + u\nabla u$$

= R.H.S.

(iii) L.H.S. = $\nabla\left(\frac{u}{v}\right) = \hat{i}\frac{\partial}{\partial x}\left(\frac{u}{v}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{u}{v}\right) + k\frac{\partial}{\partial z}\left(\frac{u}{v}\right)$

$$= \hat{i}\left(\frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}\right) + \hat{j}\left(\frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2}\right) + \hat{k}\left(\frac{v\frac{\partial u}{\partial z} - u\frac{\partial v}{\partial z}}{v^2}\right)$$

$$= \frac{v\left(\hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z}\right) - u\left(\hat{i}\frac{\partial v}{\partial x} + \hat{j}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial v}{\partial z}\right)}{v^2} = \frac{v\,\nabla u - u\nabla v}{v^2}$$

= R.H.S.

2.2.4 Extreme Value and its Evaluation

A maximum or a minimum value of a function is called an extreme value.

Saddle Point. If a function f has neither max. nor minimum at a joint (x_1, y_1), then function is said to have a saddle point at (x_1, y_1)

Lagrange Multiplier

In this method, we want to find the points (x, y) that give the extrema (maximum or minimum value) of a function f(x, y) subject to the constraint g(x, y) = d, where d is a constant. This will occur only when the gradients Δf and Δg (directional derivatives) are orthogonal to the given curve [surface] g(x, y) = d. Thus Δf and Δg are parallel; and hence there must be a constant λ such $\Delta f = \lambda\,\Delta g$. Here λ is called a Lagrange multiplier. The condition $\Delta f = \lambda \Delta g$ together with the original constraint yield three equations in the unknowns x, y and λ :

$$f_x(x, y) = \lambda g_x(x, y),\ f_y(x, y) = \lambda g_y(x, y),\ g(x, y) = d$$

Solutions of the system for x and y give the coordinates for the extrema of f(x, y) subject to the constraint g(x, y) = d.

2.2.5 Divergence of a Vector Point Function

Let $\vec{F}$ be any differentiable vector function. Then divergence of $\vec{F}$, written as div $\vec{F}$, is defined as

$$\text{div}\,\vec{F} = \nabla\,.\,\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\,\vec{F} = \hat{i}\,.\,\frac{\partial \vec{F}}{\partial x} + \hat{j}\,.\,\frac{\partial \vec{F}}{\partial y} + \hat{k}\,.\,\frac{\partial \vec{F}}{\partial z}$$

Note 1. Divergence is always of a vector point function and $\nabla\,.\,\vec{F}$ is a scalar point function.

2. Let $\vec{F} = F_1\,\hat{i} + F_2\hat{j} + F_3\hat{k}$

Now div $\vec{F} = \nabla . \vec{F} = \hat{i} . \frac{\partial \vec{F}}{\partial x} + \hat{j} , \frac{\partial \vec{F}}{\partial y} + \hat{k} . \frac{\partial \vec{F}}{\partial z}$

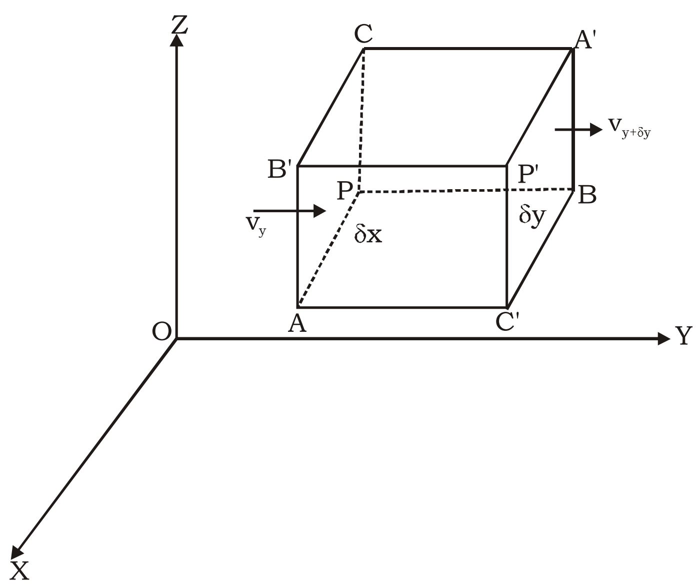
$$= \hat{i} . \left(\frac{\partial F_1}{\partial x} \hat{i} + \frac{\partial F_2}{\partial x} \hat{j} + \frac{\partial F_3}{\partial x} \hat{k} \right) + \hat{j} . \left(\frac{\partial F_1}{\partial y} \hat{i} + \frac{\partial F_2}{\partial y} \hat{j} + \frac{\partial F_3}{\partial y} \hat{k} \right) + \hat{k} . \left(\frac{\partial F_1}{\partial z} \hat{i} + \frac{\partial F_2}{\partial z} \hat{j} + \frac{\partial F_3}{\partial z} \hat{k} \right)$$

$\therefore$ div $\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

3. A vector $\vec{F}$ is said to be solenoidal iff div $\vec{F}$ = 0.

2.2.5.1 Physical Interpretation

Draw a small parallelopiped with edges δx, δy, δz parallel to the axes in the mass of fluid, with one of the corners at P(x, y, z). Let $\vec{v} = u_x \hat{i} + v_y \hat{j} + v_z \hat{k}$ be the velocity of fluid at P



Amount of fluid entering the face PB' in unit time = v_yδz δx.

Amount of fluid leaving the face P'B in unit time = $v_{y+\delta y} \delta z\, \delta x = \left(v_y + \frac{\partial v_y}{\partial y} \delta y \right) \delta z\, \delta x$ nearly.

$[\because$ of Taylor's Theorem$]$

$\therefore$ net loss of amount of fluid due to flow across these two faces per unit volume

$$= \frac{\partial v_y}{\partial y} \delta x \, \delta y \, \delta x$$

$\therefore$ total loss of amount of fluid due to flow along all the faces per unit volume

$$= \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) \delta x \, \delta y \, \delta x$$

$\therefore$ rate of loss of fluid per unit volume $= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \text{div } \vec{v}$

$\therefore$ div $\vec{v}$ is the rate at which fluid is issuing at a point per unit volume.

2.2.6 Curl of a Vector Point Function

Let $\vec{F}$ be any differentiable vector function. Then curl $\vec{F}$ is defined as

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \times \vec{F} = \hat{i} \times \frac{\partial \vec{F}}{\partial x} + \hat{j} \times \frac{\partial \vec{F}}{\partial y} + \hat{k} \times \frac{\partial \vec{F}}{\partial z}$$

Note 1. Curl $\vec{F}$ is also known as rotational $\vec{F}$ or rot $\vec{F}$.

2. Curl is always of vector point function and $\nabla \times \vec{F}$ is a vector.

3. Let $\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$

$\therefore$ $\text{curl } \vec{F} = \nabla \times \vec{F} = \hat{i} \times \frac{\partial \vec{F}}{\partial x} + \hat{j} \times \frac{\partial \vec{F}}{\partial y} + \hat{k} \times \frac{\partial \vec{F}}{\partial z}$

$$= \hat{i} \times \left(\frac{\partial F_1}{\partial x} \hat{i} + \frac{\partial F_2}{\partial x} \hat{j} + \frac{\partial F_3}{\partial x} \hat{k} \right) + \hat{j} \times \left(\frac{\partial F_1}{\partial y} \hat{i} + \frac{\partial F_2}{\partial y} \hat{j} + \frac{\partial F_3}{\partial y} \hat{k} \right) + \hat{k} \times \left(\frac{\partial F_1}{\partial z} \hat{i} + \frac{\partial F_2}{\partial z} \hat{j} + \frac{\partial F_3}{\partial z} \hat{k} \right)$$

$$= \left(\frac{\partial F_2}{\partial x} \hat{k} - \frac{\partial F_3}{\partial x} \hat{j} \right) + \left(-\frac{\partial F_1}{\partial y} \hat{k} + \frac{\partial F_3}{\partial y} \hat{i} \right) + \left(\frac{\partial F_1}{\partial z} \hat{j} - \frac{\partial F_2}{\partial z} \hat{i} \right)$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

4. A vector $\vec{F}$ is said to be irrotational vector iff curl $\vec{F} = \vec{0}$.

2.2.6.1 Physical Interpretation

If $\vec{v}$ is the velocity of a particle of a rigid body whose angular velocity is $\vec{A}$, then curl $\vec{v} = 2\vec{A}$.

Proof : Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of the particle whose velocity is $\vec{v}$

$\therefore$ $\vec{v} = \vec{A} \times \vec{r}$

$$\therefore \text{ curl } \vec{v} = \text{curl}\left(\vec{A} \times \vec{r}\right) = \Sigma \hat{i} \times \frac{\partial}{\partial x}\left(\vec{A} \times \vec{r}\right)$$

$$= \Sigma \hat{i} \times \left(\vec{A} \times \frac{\partial \vec{r}}{\partial x}\right) \qquad \left[\because \vec{A} \text{ is a constant vector}\right]$$

$$= \Sigma\, \hat{i} \times \left(\vec{A} \times \hat{i}\right) = \Sigma\left(\hat{i}\,.\,\hat{i}\right)\vec{A} - \Sigma\left(\hat{i}\,.\,\vec{A}\right)\hat{i} = 3\,\vec{A} - \vec{A}$$

$\therefore$ curl $\vec{v} = 2\vec{A}$.

2.2.7 The Laplacian Operator and Harmonic Function

Laplacian Operator : The operator ∇^2 i.e., $\nabla.\nabla$ is defined as $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ and is called be Laplacian operator.

Note : 1. ∇^2 is a scalar operator where as ∇ is a vector operator.

2. If F is a scalar point function, then $\nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}$ and if $\vec{F}$ is a vector point function, then $\nabla^2 \vec{F} = \frac{\partial^2 \vec{F}}{\partial x^2} + \frac{\partial^2 \vec{F}}{\partial y^2} + \frac{\partial^2 \vec{F}}{\partial z^2}$.

3. ∇^2 F=0 is called Laplacian equation.

Harmonic Function : A function satisfying Laplace's equation is Harmonic Function.

2.2.8 Some Important Examples

Example 1 : Prove that $\vec{a}.v\left(\vec{b}.\nabla\frac{1}{r}\right)=\frac{3\vec{a}.\vec{r}\ \vec{b}.\vec{r}}{r^5}-\frac{\vec{a}.\vec{b}}{r^3}$ where $\vec{a}$ and $\vec{b}$ are constants.

Proof : Consider $\nabla\left(\frac{1}{x}\right)=\nabla(r^{-1})\ =(-1)\,r^{-3}\vec{r}\ \{\because \nabla r^n = nr^{n-2}\vec{r}\}$

$$\therefore \qquad \vec{b}\,.\,\nabla\left(\frac{1}{r}\right)=-\frac{\vec{b}\,.\vec{r}}{r^3}$$

$$\therefore \qquad \nabla\left(\vec{b}\,.\nabla\frac{1}{r}\right)=-\nabla\left(\frac{\vec{b}\,.\,\vec{r}}{r^3}\right)=-\left[\frac{1}{r^3}\,\nabla\left(\vec{b}.\vec{r}\right)+\vec{b}\,.\,\vec{r}\,\nabla\left(r^{-3}\right)\right]$$

$$=-\left[\frac{1}{r^3}\,\vec{b}+\vec{b}\,.\,\vec{r}\,(-3)\,r^{-5}\vec{r}\right]$$

$$\therefore \qquad \nabla\left(\vec{b}\,.\,\nabla\frac{1}{r}\right)=-\frac{1}{r^3}\vec{b}+\frac{3}{r^5}\,\vec{b}\,.\,\vec{r}\,\vec{r}$$

$$\therefore \qquad \vec{a}\,.\,\nabla\left(\vec{b}\,.\,\nabla\,\frac{1}{r}\right)=-\frac{1}{r^3}\,\vec{a}\,.\,\vec{b}+\frac{3}{r^5}\,\vec{a}\,.\,\vec{r}\,\vec{b}\,.\vec{r}$$

$$=\frac{3\vec{a}\,.\,\vec{r}\ \vec{b}.\vec{r}}{r^5}-\frac{\vec{a}.\vec{b}}{r^3}\,.$$

Example 2 : Minimise the function f(x, y) = x^2 + $2y^2$ subject to the constaint g(x,y) = 2x + y = 9.

Proof : Given : f(x, y) = $x^2 + 2y^2$, g(x, y) = 2x + y – 9 = 0

Now, using the condition that $\nabla f = \lambda\nabla g$ and the constaint, we get

$$\left(\hat{i}\,\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\,\frac{\partial}{\partial z}\right)(x^2+2y^2)=\lambda\left(\hat{i}\,\frac{\partial}{\partial x}+\hat{j}\,\frac{\partial}{\partial y}+\hat{k}\,\frac{\partial}{\partial z}\right)(2x+y-9)$$

or $\left(2x\hat{i}+4y\hat{j}\right)=\lambda\,(2\hat{i}+\hat{j})$

$2x = 2\lambda$ i.e. $x = \lambda$ … (1)

$4y = \lambda$ … (2)

and $2x + y = 9$ … (3)

Eliminating λ from (1) and (2), we get

$$x = 4y$$

From (3) and (4), we get

$$8y + y = 9 \text{ or } 9y = 9 \Rightarrow y = 1$$

$\therefore$ From (4), $x = 4$

$\therefore$ $f(4, 1) = 16 + 2 = 18$

Hence, 18 is the minimum value of f.

Example 3 : Determine the constant 'a' so that the vector $\vec{F} = (x + 3y)\,\hat{i} + (y - 2z)\,\hat{j} + (x + az)\,\hat{k}$ is solenodial.

Sol. $\vec{F} = (x + 3y)\,\hat{i} + (y - 2z)\,\hat{j} + (x + az)\,\hat{k}$

$\therefore$ $\text{div}\,\vec{F} = \nabla \,.\, \vec{F}$

$$= \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right) . \left((x + 3y)\,\hat{i} + y - 2z\hat{j} + (x + az)\,\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(x + az)$$

$$= 1 + 1 + a = a + z$$

Now, since div. $\vec{F}$ is solenoidal

$\therefore$ div $\vec{F}$ $= 0 \Rightarrow a + 2 = 0$

$$\Rightarrow a = -2.$$

Example 4 : Show that the vector $(\sin y + z)\hat{i} + (x\cos y - z)\hat{j} + (x - y)\hat{k}$ is irrotational.

Proof : Here, $\vec{F} = (\sin y + z)\,\hat{i} + (x\cos y - z)\,\hat{j} + (x - y)\hat{k}$

$$\therefore \quad \text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sin y + z & x\cos y - z & x - y \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(x - y) - \frac{\partial}{\partial z}\left(x\cos y - z\right)\right]\hat{i} - \left[\frac{\partial}{\partial x}(x - y) - \frac{\partial}{\partial z}(\sin y + z)\right]$$

$$\hat{j} + \left[\frac{\partial}{\partial x}(x\cos y - z) - \frac{\partial}{\partial y}(\sin y + z)\right]\hat{k}$$

$$= (-1+1)\,\hat{i} - (1-1)\,\hat{j} + (\cos y - \cos y)\,\hat{k}$$

$$= \vec{O} - \vec{O} + \vec{O} = \vec{O}$$

$\therefore$ $\vec{F}$ is irrotational.

Example 5 : Prove that div $\hat{r} = \dfrac{2}{r}$.

Sol. L.H.S. $= \text{div}\,\hat{r} = \text{div}\left(\dfrac{\vec{r}}{r}\right) = \nabla\,.\left(r^{-1}\vec{r}\right)$

$$= \nabla\, r^{-1}.\,\vec{r} + r^{-1}\text{div}\,\vec{r}\ \left\{\because \text{div}\,(u\vec{v}) = \nabla u\,.\,\vec{v} + u\,\text{div}\,\vec{v}\right\}$$

$$(-1)\,r^{-3}\vec{r}\,.\,\vec{r} + r^{-1}(3)\ \ \left\{\because \text{div}\,\vec{r} = 3\right\}$$

$$= -\frac{1}{r} + \frac{3}{r} = \frac{2}{r}$$

= R.H.S.

2.2.9 Some Useful Formuale

1. $\text{div}\left(\vec{u}+\vec{v}\right) = \text{div}\,\vec{u} + \text{div}\,\vec{v}$
2. $\text{curl}\left(\vec{u}+\vec{v}\right) = \text{curl}\,\vec{u} + \text{curl}\,\vec{v}$
3. $\text{div}\left(\vec{u}\vec{v}\right) = \nabla u.\vec{v} + u\,\text{div}\,\vec{v}$
4. $\text{curl}\left(u\vec{v}\right) = \nabla u \times \vec{v} + u\,\text{curl}\,\vec{v}$
5. $\text{div}\left(\vec{u}\times\vec{v}\right) = \vec{v}\,.\,\text{curl}\,\vec{u} - \vec{u}\,.\,\text{curl}\,\vec{v}$
6. $\text{curl}\left(\vec{u}\times\vec{v}\right) = \vec{v}\,.\,\nabla\vec{u} - \vec{u}\,.\,\nabla\vec{v} + \vec{u}\text{div}\,\vec{v} - \vec{v}\,\text{div}\,\vec{u}$
7. $\text{grad}\left(\vec{u}.\vec{v}\right) = \vec{v}\,.\,\nabla\vec{u} + \vec{u}\,.\,\nabla\vec{v} + \vec{v}\times\text{curl}\,\vec{u} + \vec{u}\times\text{curl}\,\vec{v}$
8. $\text{div grad}\,(v) = \nabla^2 v$
9. $\text{curl grad}\,v = \vec{o}$
10. $\text{div curl}\,\vec{v} = o$

11. grad div $\vec{v}$ = curl curl $\vec{v} + \nabla^2\vec{v}$

12. curl curl $\vec{v}$ = grad div $\vec{v} - \nabla^2\vec{v}$.

2.2.10 Self Check Exercise

1. Find div $\vec{f}$ and curl $\vec{F}$ where $\vec{f} = xy^2\,\hat{i} + 2x^2yz\,\hat{j} - 3yz^2\vec{k}$

2. Prove that curl $\vec{f} = \vec{O}$ where $\vec{f} = z\hat{i} + x\hat{j} + y\hat{k}$.

3. Show that the vector $\vec{F} = \dfrac{-y\vec{i} + x\vec{j}}{x^2 + y^2}$ is irrotational.

4. Prove that div $\left\{\dfrac{f(r)}{r}\vec{r}\right\} = \dfrac{1}{r^2}\dfrac{d}{dr}\left[r^2 f(r)\right]$.

5. Find the value of $\nabla \cdot \left(\dfrac{\vec{r}}{r}\right)$, where $\vec{r}$ and r have their usual meanings.

   Hence prove that $\nabla\left[\nabla \cdot \left(\dfrac{\vec{r}}{r}\right)\right] = \dfrac{-2}{r^3}\vec{r}$.

2.2.11 Suggested Readings

R. K. Jain, SRK Lyengar : Advanced Engineering Mathematics (Narosa Publications)

LESSON NO. 2.3 Author : Dr. Chanchal

THEOREMS OF GAUSS, GREEN AND STOKES

Structure :

2.3.0 Objectives

The prime goal of this lesson is to study the proof and applications of Gauss divergence theorem, Green's theorem and stoke's theorem.

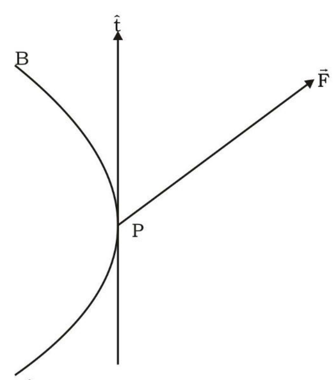
2.3.1 Introduction

As we have already studied about the double and triple integrals in lesson no. 4 and 5. So, now we can easily understand the basic concepts of line, surface and volume integrals, which are summarized below :

I. Tangential Line Integral :

The tangential line integral of a vector function $\vec{F}$ along a curve C from A to B is the definite integral of the scalar resolute of $\vec{F}$ in the direction of the tangent to the curve measured from a fixed point in the sense A to B, and the limits of integration being the values of s corresponding to the points A and B.

If $\hat{t}$ is the unit tangent at the point P and $\vec{F}$ is the value of the function here, then tangential line integral



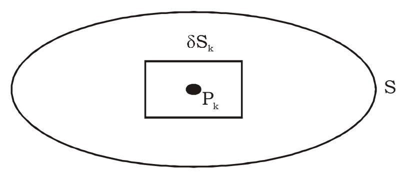
$$= \int_A^B \vec{F}.\hat{t}\,ds = \int_A^B \vec{F}.\frac{d\vec{r}}{ds}ds = \int_A^B \vec{F}.d\vec{r}$$

$$= \int_A^B \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right).\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$= \int_A^B \left(F_1dx + F_2dy + F_3dz\right).$$

II. Surface Integral :

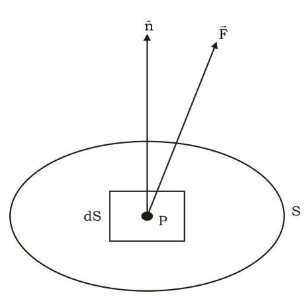
Any integral which is to be evaluated over a surface over a surface is called a surface integral.



Let f (x, y, z) be a single valued function defined over a surface S of finite area. Subdivide the area S into n elements of areas δS_1, δS_2,...., δS_n. In each part δS, we choose an arbitrary point P_k (x_k y_k, z_k). Form the sum $\sum_{k=1}^{n} f\left(P_k\right)\delta S_k$. Take the limit of this sum as $n \to \infty$ in such a way that the largest of the areas δS_k approaches zero. This limit if it exists, is called the surface integral of f(x, y, z) over S and is denoted by $\iint_S f(x,y,z)\,dS$ or $\int_S f\,dS$.

Remarks :

1. For any continuous vector point function $\vec{F}(x,y,z)$ defined at any point P of surface S, the flux of $\vec{F}$ over S is given by $\int \vec{F}.\hat{n}\,dS = \iint_S \vec{F}.\overrightarrow{dS}$



where $\vec{F}\cdot\hat{n}$ is the normal component of $\vec{F}$ at P with $\hat{n}$ being the unit vector in the direction of outward normal to the surface S at P.

2. If $\hat{n}$ makes angles α, β, γ with the co-ordinate axes and $\vec{F}\cdot F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$,

then $\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\beta)\,dS$

$$= \iint F_1 dydz + F_2 dzdx + F_3 dxdy$$

wher dydz, dzdx, dxdy are the orthozonal projections of S on the co-ordinate planes.

3. A vector point function is said to be solenoidal in a region if its flux across every closed surface in the region is zero.

III. Volume Integral :

Let V be the volume bounded by the surface S. Let f(x, y, z) be a single valued function of position defined over V. Subdivide the volume V into n elements of volumes δV_1, δV_2,...., δV_n. In each part δV_k, choose an arbitrary point $P_k(x_k, y_k, z_k)$. Form the sum $\sum_{k=1}^{n} f(P_k)\,\delta V_k$. Take the limit of the sum in such a ways that the largest of the volumes $\delta V_k \to 0$. This limit, if it exists, is called the volume integral of f over V and is denoted by $\iiint_V f\,dV$ or $\int_V f\,dV$.

If we divide the volume V into small cuboids by drawing lines parallel to the three co-ordinate axis, then

dV = dx dy dz and

$\therefore$ volume integral $= \iiint_V f\,dx\,dy\,dz$.

If $\vec{F}$ is a vector point function, then volume integral $= \iiint_V \vec{F}\,dV$

Let $\vec{F} = F_1\hat{i} + F_2\,\hat{j} + F_3\hat{k}$, then

volume integral $= \iiint_V F_1 dx\,dy\,dz + \iiint_V F_2\,dx\,dy\,dz + \iiint_V F_3\,dx\,dy\,dz$.

2.3.2 Gauss's Divergence Theorem

Statement : If $\vec{F}$ is a continuously differentiable vector point function and S is a closed surface enclosing a region V. Then $\int_S \vec{F}\cdot\hat{n}\,dS = \int_V \operatorname{div}\vec{F}\,dv$ where $\hat{n}$ is the unit outward drawn normal vector.

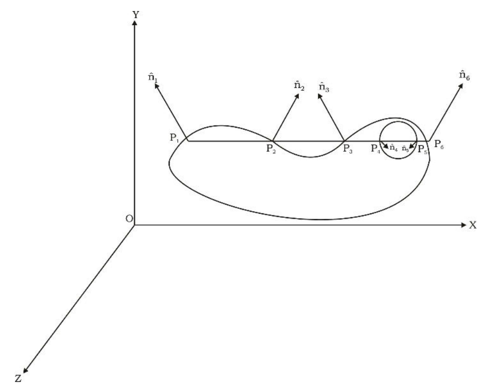
OR

The normal surface integral of $\vec{F}$ over the boundary of a closed region is equal to the space integral of divergence of $\vec{F}$ taken throughout the enclosed space.

Proof : Take rectangular axes parallel to the unit vector $\hat{i}\,.\,\hat{j}\,.\,\hat{k}$ and let $\vec{F} = U\hat{i} + V\hat{j} + W\hat{k}$ where U, V, W are components of $\vec{F}$ along the axes. Now we have to prove that

$$\int_S \left(U\hat{i} + V\hat{j} + W\hat{k}\right).\,\hat{n}\,dS = \iiint \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}\right) dx\,dy\,dz$$

where dx dy dz is the volume element dv. For fixed values of y, z take the rectangular prism parallel to x-axis bounded by the planes y, y + dy, z, z + dz, the area of its normal section being dy dz. Such a prism cuts the boundary an even number of times at points P_1, P_2,........., P_{2n}, since the boundary surface is closed. If a point moves along the prism in the direction of x-increasing, it enters the region at P_1, P_3,....., P_{2n-1} and leaves it at the points P_2, P_4,...., P_{2n}.



Let $I = \iiint \frac{\partial U}{\partial x}\,dx\,dy\,dz = \iint \left(-U_1 + U_2 - U_3 + \ldots - U_{2n-1} + U_{2n}\right) dy\,dz$

where U_r is the value of U at the point P_r

Let dS, be the area of the element of the boundary intercepted by the prism at the point P_r.

Now dy dz is the area of projection of this element on the yz-plane, we have

$$dy\,dz = -\hat{i}.\hat{n}_r\,dS_r \text{ if r is odd}$$

$$= \hat{i}.\hat{n}_r\,dS_r \text{ if r is even}$$

because the angle which the vector $\hat{n}_r$ makes with $\hat{i}$ is acute or obtuse according as r is even or odd.

$$I = \int \hat{i}\,.\left(U_1\hat{n}_1 dS_1 + U_2\hat{n}_2\,dS_2 + \ldots + U_{2n}\hat{n}_{2n} dS_{2n}\right)$$

$\therefore \quad \int_V \frac{\partial U}{\partial x} dv = \int_S U\hat{i} \, . \, \hat{n} \, dS$

Similarly, $\int_V \frac{\partial V}{\partial y} dv = \int_S V\hat{j} \, . \, \hat{n} \, dS$

and $\int_V \frac{\partial W}{\partial z} dv = \int_S W \, \hat{k} \, . \, \hat{n} \, dS$

Adding $\int_V \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} \right) dv = \int_S \left(U \, \hat{i} + V \, \hat{j} + W\hat{k} \right) . \, \hat{n} \, dS$

$\Rightarrow \quad \int_V \text{div} \, \vec{F} \, dV = \int \vec{F} \, . \, \hat{n} \, dS$

If $\quad \vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and

α, β, γ be the angles which outward drawn unit normal $\hat{n}$ makes with positive directions of axes.

Then, Divergence Theorem can be written as

$$\iiint \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx \, dy \, dz = \iint \left(F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma \right) dS$$

or $$\iiint \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx \, dy \, dz = \iint \left(F_1 dydz + F_2 \, dzdx + F_3 dxdy \right) \cdot$$

2.3.3 Green's Theorem

Firstly, we prove the Green's theorem in space.

Statement : If U and V are two continuously differentiable scalar point functions within a region bounded by a closed surface S such that ∇U and ∇V are also continuously differentiable. Then $\int \left(U\nabla^2 V - V\nabla^2 U \right) dv = \int \left(U \, \nabla V - V\nabla U \right) . \, \hat{n} \, dS$

Proof : By Gauss's Theorem

$$\int \vec{F} \, . \, \hat{n} \, dS = \int \text{div} \, \vec{F} \, dv$$

Let $\quad \vec{F} = U \, \Delta V$

Then $\quad \text{div} \, \vec{F} = \text{div} \left(U\nabla V \right) = \nabla V.\nabla V + U\nabla.\nabla V = \nabla U.\nabla V + U\nabla^2 V$

$\therefore \quad \int U\nabla V.\hat{n} \, dS = \int \nabla U \, . \, \nabla V \, dv + \int U\nabla^2 V \, dv \qquad \ldots (1)$

Similarly $\int V\nabla \, U.\hat{n} \, dS = \int \nabla U \, . \, \nabla U \, dv + \int \nabla\nabla^2 U \, dv \qquad \ldots (2)$

Subtracting (2) from (1)

$$\int (U\,\nabla V - V\nabla U)\,.\,\hat{n}\,dS = \int \left(U\nabla^2 V - V\nabla^2 U\right) dv$$

$$\Rightarrow \qquad \int \left(U\nabla^2 V - V\nabla^2 U\right) dv = \int (U\,\nabla V - V\nabla U)\,.\,\hat{n}\,dS\,.$$

Now, we prove the Green's theorem in plane.

Statement : Let R be a closed bounded region in the xy-plane whose boundary C consists of finitrely many smooth curves. Let M and N be continuous functions of x and y having continuous partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in R . Then

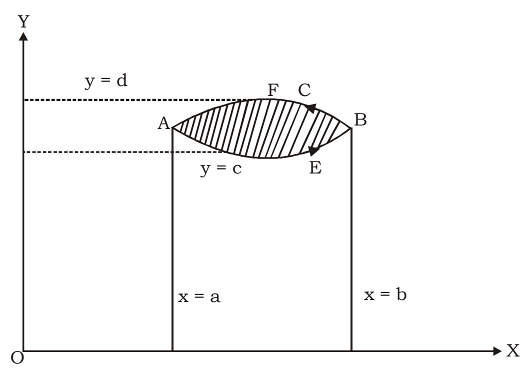
$$\iint\limits_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint\limits_C (M\,dx + N\,dy)$$

the line integral being taken along the entire boundary C of R such that R is on the left as one advances in the direction of integration.

Proof : First of all, we prove the theorem for a special region R bounded by a closed curve C and having the property that any straight line parallel to any one of the coordinate axes and intersecting R has only one segment (or a single point) in common with R. This means that R can be represented in both of the forms

$a \le x \le b,\ f(x) \le y \le g(x)$

and $c \le y \le d,\ p(y) \le x \le q(y)$



In the figure, the equations of the curves AEB and BFA are y = f(x) and y = g(x) respectively. Similarly the equations of the curves FAE and EBF are x = p(y) and x =q(y) respectively.

$$\text{Now} \iint\limits_R \frac{\partial M}{\partial y}\,dx\,dy = \int_{x=a}^{b}\left[\int_{y=f(x)}^{y=g(x)} \frac{\partial M}{\partial y}\,dy\right] dx = \int_{x=a}^{b} \left[M(x,y)\right]_{y=f(x)}^{y=g(x)} dx$$

$$= \int_{x=a}^{b} \left[M\{x, g(x)\} - M\{x, f(x)\}\right] dx$$

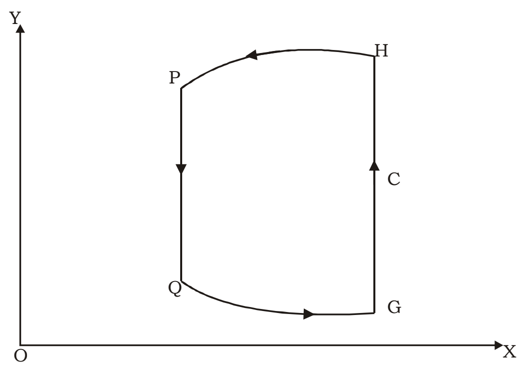
$$= -\int_a^b M[x, f(x)]\,dx - \int_b^a M[x, g(x)]\,dx$$

$$= -\oint_C M(x, y)\,dx$$

[Since y = f (x) is the curve AEB and y = g(x) is the curve BFA]

If portions of C are segments parallel to y-axis such as GH and PQ in the figure, then result proved above is not affected. The line integral $\int M\,dx$ over GH is zero as on GH, x = constant

$\Rightarrow \quad dx = 0.$



Similarly the line integral over PQ is zero. The equations of QG and HP are y=f(x) and y = g(x) respectively. Hence we have

$$-\iint_R \frac{\partial M}{\partial y}\,dx\,dy = \oint_C M(x, y)\,dx$$

$$\text{Again } \iint_R \frac{\partial N}{\partial x}\,dx\,dy = \int_{y=c}^{d}\left[\int_{x=p(y)}^{q(y)} \frac{\partial N}{\partial x}\,dx\right]dy = \int_{y=c}^{d}[N(x, y)]_{x=p(y)}^{q(y)}\,dy$$

$$= \int_{y=c}^{d}[N\{q(y), y\} - N\{p(y), y\}]\,dy$$

$$= \int_{y=c}^{d}[N\{q(y), y\}]\,dy + \int_d^c [N\{p(y), y\}]\,dy = \oint_C N(x, y)\,dy$$

$$\therefore \quad \iint_R \frac{\partial N}{\partial x}\,dx\,dy = \oint_C N(x, y)\,dy \qquad \ldots (2)$$

Adding (1) and (2), we get,

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \oint_C (M\, dx + N\, dy).$$

Note : Vector form of Green's Theorem in a Plane

Let $\vec{F} = M\,\hat{i} + N\,\hat{j}$

Now $\vec{r} = x\hat{i} + y\hat{j}$

$\Rightarrow$ $d\vec{r} = (dx)\,\hat{i} + (dy)\,\hat{j}$

$\therefore$ $M\,dx + N\,dy = (M\hat{i} + N\hat{j})(\hat{i}\,dx + \hat{j}\,dy) = \vec{F}\,.\,d\,\vec{r}$

Also $\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & O \end{vmatrix} = -\frac{\partial N}{\partial z}\,\hat{i} + \frac{\partial M}{\partial z}\,\hat{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}$

$\therefore$ $\operatorname{curl}\vec{F}.\hat{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

$\therefore$ Green's theorem in plane can be written as $\iint_R \operatorname{curl}\vec{F}\,.\,\hat{k}\,dR = \oint_C \vec{F}\,.\,d\vec{r}$

where Dr = dx dy and $\hat{k}$ is a unit vector perpendicular to the xy-plane.

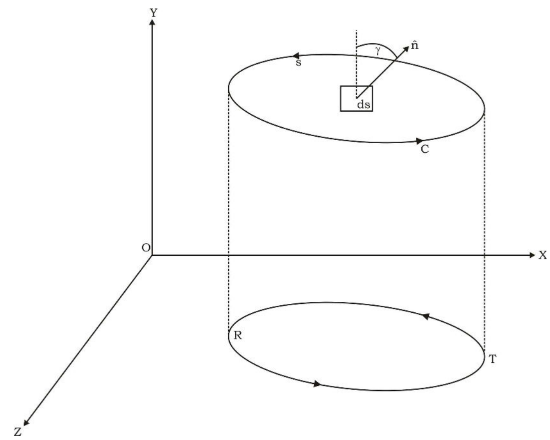
2.3.4 Stoke's Theorem

If $\vec{F}$ is any continuously differentiable vector point function and S is the surface bounded by a curve C, then

$$\oint_C \vec{F}\,.\,d\vec{r} = \int_S \operatorname{curl}\vec{F}\,.\,\vec{n}\,dS$$

where unit normal vector $\vec{n}$ at any point of S is drawn in the sense in which a right handed screw would move when rotated in the sense of description of C.

Proof : Let S be a surface which is such that its projections on the x y, y z and z x planes are regions bounded by simple closed curves. Suppose S can be represented simultaneously in z = f (x, y), y = g(x, z), x = h (z, y) where f, g, h are continuous functions and have continuous first derivaties.



Consider the integral $\iint\limits_S \left[\Delta \times \left(F_1\ \hat{i}\right)\right] . \vec{n}\ dS$ where $\vec{F} = F_1\ \hat{i} + F_2\ \hat{j} + F_3\ \hat{k}$

$$\text{Now } \nabla \times \left(F_1\ \vec{i}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & 0 & 0 \end{vmatrix} = \frac{\partial F_1}{\partial z}\ \hat{j} - \frac{\partial F_1}{\partial y}\ \hat{k}$$

$$\therefore \qquad \left[\nabla \times \left(F_1\ \hat{i}\right)\right] . \vec{n} = \frac{\partial F_1}{\partial z}\ \hat{j} . \vec{n} - \frac{\partial F_1}{\partial y}\ \hat{k} . \vec{n} = \frac{\partial F_1}{\partial z} \cos \beta - \frac{\partial F_1}{\partial y} \cos \gamma$$

where α, β, γ are the angles which outward drawn $\vec{n}$ makes with the positive direction of x, y, z axes.

$$\therefore \qquad \iint\limits_S \left[\nabla \times \left(F_1\ \hat{i}\right)\right] . \vec{n}\ dS = \iint\limits_S \left(\frac{\partial F_1}{\partial z} \cos \beta - \frac{\partial F_1}{\partial y} \cos \gamma\right) dS$$

We shall prove that

$$\iint\limits_S \left(\frac{\partial F_1}{\partial z} \cos \beta - \frac{\partial F_1}{\partial y} \cos \gamma\right) dS = \oint\limits_C F_1\ dx$$

Let R be the orthogonal projection of S on the xy-plane and let τ be its boundary which is oriented as shown in the figure. Using the representation z = f(x, y) of S,

$$\oint\limits_C F_1(x, y, z)\, dx = \oint\limits_\tau F_1\ \left[x, y\ f(x, y)\right] dx$$

$$= \oint_{\tau} \{F_1[x, y, f(x, y)]\, dx + 0\, dy\}$$

$$= -\iint_R \frac{\partial F_1}{\partial y}\, dx\, dy \qquad [\because \text{ of Green's Theorem in plane}]$$

But $\frac{\partial}{\partial y}\{F_1\,[x, y, f(x, y)\} = \frac{\partial}{\partial y}\{F_1\,(x, y, z)\} + \frac{\partial}{\partial z}\{F_1(x, y, z)\}\,\frac{\partial f}{\partial y}\ [\because z = f\,(x, y)]$

$$\therefore \qquad \oint_C F_1\,(x, y, z)\, dx = -\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y}\right) dx\, dy \qquad \ldots (2)$$

Now the equation z = f(x, y) of the surface S can be written as

$$\phi\,(x, y, z) = z - f\,(x, y) = 0$$

$$\therefore \qquad \text{grad}\ \phi - \frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k}$$

Let $|\,\nabla\phi\,| = a$

Since grad ϕ is normal to S and $\vec{n}$, grad ϕ both are in positive direction of z-axis.

$$\therefore \qquad \vec{n} = \frac{\text{grad}\ \phi}{a}$$

or
$$(\cos\alpha)\,\hat{i} + (\cos\beta)\,\hat{j} + (\cos\gamma)\,\hat{k} = -\frac{1}{a}\frac{\partial f}{\partial x}\hat{i} - \frac{1}{a}\frac{\partial f}{\partial y}\hat{j} + \frac{1}{a}\hat{j}$$

$$\therefore \qquad \cos\alpha = -\frac{1}{a}\frac{\partial f}{\partial x},\ \cos\beta = -\frac{1}{a}\frac{\partial f}{\partial y},\ \cos\gamma = \frac{1}{a}$$

Now $dS = \frac{1}{\cos\gamma}\, dx\, dy = a\, dx\, dy$

$$\therefore \qquad \iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) dS = \iint_R \left[\frac{\partial F_1}{\partial z}\,.\left(-\frac{1}{a}\frac{\partial f}{\partial y}\right) - \frac{\partial F_1}{\partial y}\,.\,\frac{1}{a}\right].\, a\, dx\, dy$$

$$= -\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y}\right) dx\, dy$$

$$= \oint_C F_1\,(x, y, z)\, dx \qquad [\because \text{of (2)}]$$

$$\therefore \qquad \iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) dS = \oint_C F_1\,(x, y, z)\, dx$$

$$\therefore \quad \oint_C F_1 dx = \iint_S \left[\nabla \times \left(F_1\, \hat{i}\right)\right] . \vec{n}\, dS \qquad \dots (3)$$

Similarly, we have,

$$\oint_C F_2 dy = \iint_S \left[\nabla \times \left(F_2\, \hat{j}\right)\right] . \vec{n}\, dS \qquad \dots (4)$$

and $$\oint_C F_3 dz = \iint_S \left[\nabla \times \left(F_3\, \hat{k}\right)\right] . \vec{n}\, dS \qquad \dots (5)$$

Adding (3), (4), (5), we get,

$$\oint_C \left(F_1 dx + F_2 dy + F_3 dz\right) = \iint_S \left[\nabla \times \left(F_1\, \hat{i} + F_2\, \hat{j} + F_3 \hat{k}\right)\right] . \vec{n}\, dS$$

$$\therefore \quad \oint_C \vec{F} . d\vec{r} = \iint_S \left(\nabla \times \vec{F}\right) . \vec{n}\, dS \quad \text{or} \oint_C \vec{F} . d\,\vec{r} = \int_s \text{curl}\, \vec{F} . \vec{n}\, dS$$

If the surface S does not satisfy the restrictions imposed, even then the Stoke's Theorem will be true provided S can be divided into surface S_1, S_2,...., S_p with boundaries C_1, C_2,...., C_p which do satisfy restrictions. Stoke's Theorem holds for each such surface. The sum of the surface integrals over S_1, S_2,...., S_p will give us surface integral over S while the sum of the integrals over C_1, C_2,..., C_p will give us line integral over C.

2.3.5 Some Important Examples

Example 1 : Find the work done when a force $\vec{F} = (x^2 - y^2 + x)\,\hat{i} - (2xy + y)\,\hat{j}$ moves a particle in xy-plane from (0, 0) to (1, 1) along the parabola $y^2 = x$.

Sol. Let C denote the arc of the parabola $y^2 = x$ from the point (0, 0) to the point (1, 1). The parametric equations of the parabola $y^2 = x$ can be taken as $x = t^2$, $y = t$. At the point (0, 0), t = 0 and at the point (1, 1), t = 1.

Now $\vec{F} = (x^2 - y^2 + x)\,\hat{i} - (2xy + y)\,\hat{j}, d\vec{r} = (dx)\,\hat{i} + (dy)\,\hat{j}$

$$\therefore \quad \text{work done} = \int_C \vec{F} . d\vec{r} = \int_C \left[(x^2 - y^2 + x)\, dx - (2xy + y)\, dy\right]$$

$$= \int_0 \left[\left(x^2 - y^2 + x\right) \frac{dx}{dt} - (2xy + y) \frac{dy}{dt}\right] dt$$

$$= \int_0^1 \left[\left(t^4 - t^2 + t^2\right)(2t) - \left(2t^3 + t\right)(1)\right] dt$$

$$= \int_0^1 \left(2t^5 - 2t^3 - t\right) dt = \left[\frac{2t^6}{6} - \frac{2t^4}{4} - \frac{t^2}{2}\right]_0^1$$

$$= \left[\left(\frac{1}{3} - \frac{1}{2} - \frac{1}{2}\right) - (0 - 0 - 0)\right] = -\frac{2}{3}.$$

Example 2 : Verify Gauss Divergence Theorem for the vector $\vec{A} = x\hat{i} + y\hat{j} + z\hat{k}$ over the region bounded by $x^2 + y^2 + z^2 = a^2$

Sol. Let S denote the entire surface of the sphere $x^2 + y + z^2 = a^2$ and V be the volume of the sphre

We have to prove

$$\iint_S \vec{A} \cdot \vec{n}\, ds = \iiint_V \nabla \cdot \vec{A}\, dV \qquad \ldots (1)$$

R.H.S. of (1) = $\iiint_V \nabla \vec{A}\, dV$

$$= \iiint_V \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot \left(x\hat{i} + y\hat{j} + z\hat{k} \right) dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \right] dV$$

$$= \iiint_V (1+1+1)\, dV = \iiint_V 3dV = 3\iiint_V dV = 3 \times \text{ (volume of sphere with radius a)}$$

$$= 3.\frac{4}{3}\pi a^3 = 4\pi a^3$$

For the surface S, let $\phi(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0$

$\therefore \quad \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right)(x^2 + y^2 + z^2 - a^2) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\therefore \quad \vec{n} = \text{unit normal to S} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{2(x\hat{i} + y\hat{j} + z\hat{k})}{\sqrt{4x^2 + 4y^2 + 4z^2}}$

$$= \frac{2(x\hat{i} + y\hat{j} + z\hat{k})}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a} \qquad \left[\because x^2 + y^2 + z^2 = a^2\right]$$

L.H.S. of (1) $= \iint_S \vec{A} \cdot \vec{n}\, dS = \iint_S \left(x\hat{i} + y\hat{j} + z\hat{k} \right) \cdot \left(\dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{a} \right) dS$

$$= \frac{1}{a}\iint_S (x^2 + y^2 + z^2)\, dS = \frac{1}{a}\iint_S a^2\, dS \qquad \left[\because x^2 + y^2 + z^2 = a^2\right]$$

$$= \frac{a^2}{a}\iint_S dS = a \times \text{ (surface areaof the sphere with radius a)}$$

$= a \times 4\pi a^2 = 4\pi a^3$

$\therefore \quad$ L.H.S. of(1) = R.H.S. of (1)

Hence Gauss divergence Theorem is verified.

Example 3 : Verify Green's Theorem in the plane for $\oint_C \left[(x^2 - xy^3)\, dx + \left(y^2 - 2xy\right) dy \right]$

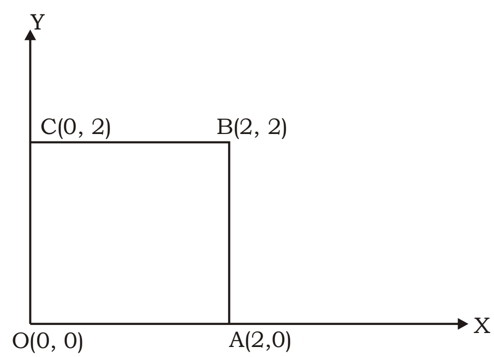
where C is the square with vertices (0, 0), (2, 0), (2, 2), (0, 2).

Sol. By Green's Theorem in plane, $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy = \oint_C \left(M\, dx + N\, dy \right)$

We have to verity this result.
Now $M = x^2 - xy^3$, $N = y^2 - 2xy$

$\therefore \quad \frac{\partial M}{\partial y} = -3xy^2, \frac{\partial N}{\partial x} = -2y$

Curve C is the square shown in the figure.



$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy = \int_{y=0}^{2} \int_{x=0}^{2} (-2y + 3xy^2)\, dx\, dy$$

$$= \int_{y=0}^{2} \left[-2xy + \frac{3x^2}{2} y^2 \right]_{x=0}^{2} dy = \int_0^2 \left(-4y + 6y^2 \right) dy$$

$$= \left[-2y^2 + 2y^3 \right]_0^2 = \left(-8 + 16 \right) - 0 = 8$$

Now along OA, y = 0 $\Rightarrow$ dy = 0, x varies from 0 to 2
along AB, x = 2 $\Rightarrow$ dx = 0, y varies from 0 to 2
along BC, y = 2 $\Rightarrow$ dy = 0, x varies from 2 to 0
along CO, x = 0, $\Rightarrow$ dx = 0, y varies from 2 to 0

$$\therefore \quad \oint_C \left(M\, dx + N\, dy \right) = \int_{OA} \left(M\, dx + N\, dy \right) + \int_{AB} \left(M\, dx + N\, dy \right)$$

$$+ \int_{BC} \left(M\, dx + N\, dy \right) + \int_{CO} \left(M\, dx + N\, dy \right)$$

$$= \int_0^2 x^2 dx + \int_0^2 \left(y^2 - 4y\right) dy + \int_2^0 (x^2 - 8x)\, dx + \int_2^0 y^2\, dy$$

$$= \left[\frac{x^3}{3}\right]_0^2 + \left[\frac{y^3}{3} - 2y^2\right]_0^2 + \left[\frac{x^3}{3} - 4x^2\right]_2^0 + \left[\frac{y^3}{3}\right]_2^0$$

$$= \left(\frac{8}{3} - 0\right) + \left(\frac{8}{3} - 8\right) + (0 - 0) - \left(\frac{8}{3} - 16\right) + \left(0 - \frac{8}{3}\right)$$

$$= \frac{8}{3} + \frac{8}{3} - 8 - \frac{8}{3} + 16 - \frac{8}{3} = 8$$

Hence Green's theorem is verified.

Example 4 : Verify Stoke's Theorem for $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$ where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C its boundary.

Sol. The boundary C of S is a circle in the xy-plane having radius as unity and centre at origin. Therefore, the equations of curve C are $x^2 + y^2 = 1$, $z = 0$. The parametric equations of C are $x = \cos t$, $y = \sin t$, $z = 0$, $0 \le t \le 2\pi$.

Now $\oint_C \vec{F} \cdot d\vec{r} = \oint_C (y\hat{i} + z\hat{j} + x\hat{k}) \cdot \{(dx)\hat{i} + (dy)\hat{j} + (dz)\,\hat{k}\}$

$$= \oint_C (y\, dx + z\, dy + x\, dz) = \oint_C y\, dx \qquad [\because z = 0 \text{ on } C, dz = 0]$$

$$= \int_0^{2\pi} \sin t \frac{dx}{dt}\, dt = -\int_0^{2\pi} \sin^2 t\, dt = -\frac{1}{2}\int_0^{2\pi} \left(1 - \cos 2t\right) dt = -\frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= -\frac{1}{2}\left[\left(2\pi - \frac{1}{2}\sin 4\pi\right) - \left(0 - \frac{1}{2}\sin 0\right)\right] = -\frac{1}{2}\left[(2\pi - 0) - (0 - 0)\right] = -\pi$$

$$\therefore \quad \oint_C \vec{F} \cdot d\vec{r} = -\pi \qquad \ldots (1)$$

$$\text{Now curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\hat{i} - \hat{j} - \hat{k}$$

Let S_1 be the plane region bounded by the circle C

$$\therefore \quad \iint_S \text{curl } \vec{F} \cdot \vec{n}\, dS = \iint_{S_1} \text{curl } \vec{F} \cdot \vec{k}\, dS = \iint_{S_1} \left(-\hat{i} - \hat{j} - \hat{k}\right) \cdot \hat{k}\, dS$$

$$= \iint_{S_1} (-1)\, dS = -S_1 = -\text{(area of circle of radius 1)}$$

$$= -\pi\,(1)^2 = -\pi$$

$$\therefore \quad \int_S \text{curl}\, \vec{F} \,.\, \vec{n}\, dS = -\pi$$

From (1) and (2), the theorem is verified.

2.3.6 Self Check Exercise

1. If $\vec{F} = (3x^2 + 6y)\,\vec{i} - 14yz\,\vec{j} + 20xz^2\vec{k}$, then evaluate $\oint_C \vec{F} \,.\, d\,\vec{r}$ from (0, 0, 0) to (1, 1, 1) along the path $x = t$, $y = t^2$, $z = t^3$.
2. Evaluate $\iint_S (ax^2 + by^2 + cz^2)\, dS$ over the sphere $x^2 + y^2 + z^2 = 1$.
3. Verify Green's Theorem in the plane for $\oint_C \left[(3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy\right]$ where C is the rectangle with vertices (0, 0), $(\pi, 0)$. $\left(\pi, \frac{\pi}{2}\right)$ and $\left(0, \frac{\pi}{2}\right)$.
4. Verify Stoke's Theorem for a vector field defined by $\vec{F} = (x^2 - y^2)\,\vec{i} + 2xy\,\vec{j}$ in the rectangular region in the xy-plane bounded by the lines $x = 0$, $x = a$, $y = 0$, $y = b$.

2.3.7 Suggested Readings

R. K. Jain, SRK Lyengar : Advanced Engineering Mathematics (Naresa Publications)