Department of Distance Education

Punjabi University, Patiala

Class : B.A. I (Math) **Semester : 2**
Paper : 6 (Analytic Geometry) **Unit : 2**
Medium : English

Lesson No.

Department website : www.pbidde.org

LESSON NO. 2.1

THE SPHERE

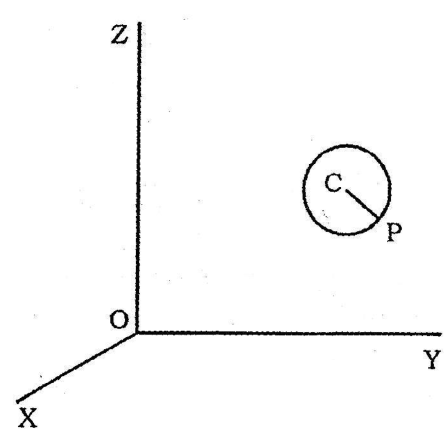
2.1.1 Definition : Sphere :

A sphere is the locus of a point which moves in space such that its distance from a fixed point remain constant.

The fixed point is called the centre and the constant distance is called the radius of the sphere.

2.1.2 Equation of a Sphere :

Let c (α, β, γ) be the centre and r be the radius of the sphere.



Let P (x, y, z) be any point on the sphere.

Join CP, then CP = r

Also $Cp^2 = (x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 \Rightarrow r^2 = (x - \alpha)^2 + (y - \beta)^2 + (z - r)^2$

$\therefore r = \sqrt{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}$

or $(x - \alpha)^2 + (y - \beta)\ 2 + (z - \gamma)^2 = r^2$ (5.1)

(5.1) is the required equation of sphere.

Cor : If the centre of sphere is the origin of the co-ordinates then

$\alpha = \beta = \gamma = 0$

$x^2 + y^2 + z^2 = r^2 \Rightarrow x^2 + y^2 + z^2 - r^2 = 0$ (5.2)

This is the simplest form of the equation to a sphere.

General form of equation of a sphere :

We will prove that the equation

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

represents a sphere. Its centre and radius will also be determined.

The equation

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ (5.3)

can be written as

$(x^2 + 2ux + u^2) + (y^2 + 2vy + v^2) + (z^2 + 2wz + w^2) = (u^2 + v^2 + w^2 - d)$

or $(x + u)^2 + (y + v)^2 + (z + w)^2 = u^2 + v^2 + w^2 - d$ (5.4)

comparing equation (5.4) with (5.1) we get

$\alpha = -u_1, \beta = -u_1, \gamma = -w$ and $r^2 = u^2 + v^2 + w^2 - d$

$\therefore$ (5.3) represents a sphere with centre as (–u, –v, –w)

and radius $= \sqrt{u^2 + v^2 + w^2 - d}$

Note : It may be observed that the equation of a sphere has the following properties:

(i) The co-efficients of x^2, y^2, z^2 are equal.

(ii) xy, yz, xy terms are missing.

(iii) The equation $ax^2 + ay^2 + 2ux + 2vy + 2wy + d$ also represents a circle with

$$\text{centre as} \left(\frac{-u}{a}, \frac{-v}{a}, \frac{-w}{a}\right) \text{and radius } \frac{1}{a}\sqrt{u^2 + v^2 + w^2 - d}$$

2.1.3 Number of Constants in the Equation of a Sphere.

The general equation of sphere is

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

It has four constants namely u, v, w and d. So to determine them, we require at least four conditions each giving rise to a relation between the constants. Now the question arises given four points, is always possible to get a sphere passing through them. The following theorem answers the problem.

Theorem : One and only One sphere passes through four non-coplanar points

(x_i, y_i, z_i) i = 1, 2, 3, 4

Proof :

Since the four points (x_i, y_i, z_i), for i =1, 2, 3, 4 are non-coplanar

$$\therefore \quad \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

of D=o then linear

By the theory of linear equation, the following system of equation in α, β, γ and δ have a unique solution :

$$2x_1\alpha + 2y_2\beta + 2z_1\gamma + \delta = (x_1^2 + y_1^2 + z_1^2)$$

$$2x_2\alpha + 2y_3\beta + 2z_3\gamma + \delta = (x_3^2 + y_3^2 + z_3^2)$$

$$2x_4\alpha + 2y_4\beta + 2z_4\gamma + \delta = (x_4^2 + y_4^2 + z_4^2)$$

Let the unique solution be $\alpha = u$, $\beta = v$, $\gamma = w$, $\delta = d$.

The four points (x_i, y_i, z_i) L = 1, 2, 3, 4 obviously lie on the sphere

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Remarks :

(i) If four point are co-planar, of the on the line, there will be a sphere through them.

(ii) If the four points are co-planar, there can be an infinite number of spheres through them.

Example 1 :

Find the equation of the sphere through the four points

(4, –1, 2); (0, –2, 3); (1, –5, –1); (2, 0, 1)

Solution :

$$\text{Since} \begin{vmatrix} 4 & -1 & 2 & 1 \\ 0 & -2 & 3 & 1 \\ 1 & -5 & -1 & 1 \\ 2 & 0 & 1 & 1 \end{vmatrix} = 0$$

the four points are non-coplanar and an unique sphere passes through these points.

Let equation of the required sphere be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Since it passes through the given four points, we have

$8u - 2v - 4w + d = -21$

$-4v - 6w + d = -13$

$2u - 10v - 2w + d = -27$

$4u + 2w - d = -5$

Solving these equation we get, $u = -2$, $v = 3$, $w = -1$, $d = 5$

$\therefore$ The required equation of the sphere is

$x^2 + y^2 - z^2 - 4x + 6y - 2z + 5 = 0$

Example 2 :

Find the equation of a sphere which passes, through the point (1, 0, 0); (0, 1,0); (0, 0, 1) and has its radius as small as possible.

Solution :

Let the equation of the sphere be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + \alpha = 0$

Since it passes through the three points (1, 0, 0), (0, 1, 0) and (0, 0, 1)

$\therefore$ $1 + 2u + \alpha = 0$, $1 + 2v + \alpha = 0$, $1 + 2w + \alpha = 0$.

$$\therefore\ u = v = w = \frac{\alpha + 1}{2}$$

sub substituting the value of u, v and w in the equation of the sphere we get:

$x^2 + y^2 + z^2 - (1 + \alpha)(x + y + z) + \alpha = 0$ (A)

The radius r of the sphere give in (A) is

$$r^2 = \left(\frac{1+\alpha}{2}\right)^2 + \left(\frac{1+\alpha}{2}\right)^2 + \left(\frac{1+\alpha}{2}\right)^2 - \alpha$$

For r to be minimum $\frac{dr}{d\alpha} = 0$ and $\frac{d^2r}{d\alpha^2}$ is positive

$\therefore$ differentiating with respect to α, we get

$$2r\frac{dr}{d\alpha} = -\frac{3}{2}(1+\alpha) - 1 = 0$$

or $\alpha = -\frac{1}{3}$

also $\frac{d^2r}{d\alpha^2}$ is + ve.

Substituting this value of α in (A), we get the equation of the sphere as

$$x^2+y^2+z^2-\frac{2}{3}(x+y+z)-\frac{1}{3}=0$$

Example 3 :

A plane passes through a fixed point (a,b,c) and cuts the co-ordinates axes in A,B,C. Show that the locus of the centre of the sphere OABC is

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2.$$

Solution :

Let the equation of the plane be

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1$$

Since it passes through (a, b, c)

$$\therefore \frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=1$$

The co-ordinates of the points A, B, C and 0 are obviously (α 0, 0), (0, β, 0) and (0, 0, 0).

$\therefore$ equation of the sphere OABC is

$x^2 + x^2 + z^2 - \alpha x + \beta y - \gamma z = 0$

its centre is given by

$$x=\frac{1}{2}\alpha,\ y=\frac{1}{2}\beta,\ z=\frac{1}{2}\gamma \qquad \ldots (B)$$

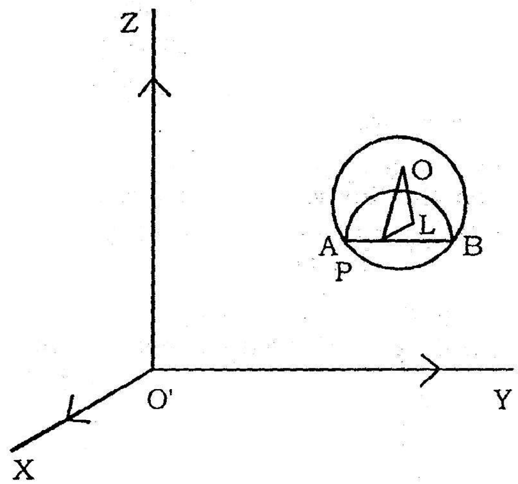
Eliminating α, β, γ from (A) and (B). The locus of the centre is

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2.$$

2.1.4 Plane Section of a Sphere :

To prove that every section of a sphere is a circle.

Let o be the centre of the given sphere and r its radius. Let the given plane cut the sphere in the curve ABC.



Draw OL perpendicular to the plane.

Joing OP and I, P where P is any point on the curve ABC. Since LO is perpendicular to the plane ABC, the angle OLP is a right angle.

$\therefore\ OP^2 = OL^2 + PL^2$

or $\therefore\ PL = \sqrt{OP^2 - OL^2}$

Now 0 is a fixed point and ABC is a fixed plane, therefore OL is the length of the perpendicular from the point to a fixed plane and it thus constant. But OP being the radius of the sphere and is, therefore, constant.

$\therefore$ PL is constant

But P is any point on the curve ABC.

Therefore all points of the curve ABC are equidistant from the point L. Therefore ABC is a circle with centre L and radius as PL centre L & PL = r

Note : 1. If L is length of the perpendicular from the centre of the sphere to the plane of ABC. The radius of the circle ABC is $\sqrt{r^2 - 1^2}$.

2. The centre of the circle is the foot of the perpendicular from the centre of the sphere to the plane.

Definition : Great Circle

Any plane passing through the centre of a sphere cut the surface in a circle which is called a great circle. The radius of the great circle is equal to the radius of the sphere.

Any circle other than the great circle in which a plane cuts a sphere is called a small - circle.

Remark :

It is clear that the equation of a sphere and the equation of a plane together represent the equation of a circle. Or we say that the circular section of a sphere has equation.

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

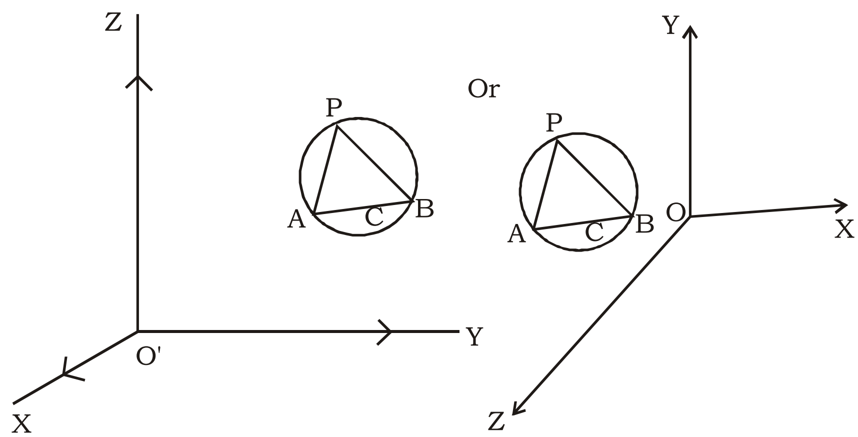
and $lx + my + nz = P$.

2.1.5 Equation of a Sphere on the Line Joining Two Points as a Diameter ;

Let $A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$ be the two given points such at AB is diameter of the sphere.

Let P be any point on the sphere.

Let (x, y, z) be the co-ordinates of P with respect to the co-ordinate axis.



The centre C of the sphere will also be a AB because AB is the diameter of the sphere.

The direction consines of AP and PB are proportional to $(x-x_1)$, $(y-y_1)$, $(z-z_1)$ respectively.

Consider the section of sphere by the plane through AB and P. This will be a great circle as it passes through 0.

Thus APB is right angle (angle in a semi-circle).

$\therefore$ AP is perpendicular to PB

which implies that $(x - x_1)(x - x_2) + (y-y_2) + (z-z_1)(z-z_2) = 0$ ... (5.6)

This is the required equation of the sphere.

2.1.6 Equation of Sphere Through a Given Circle.

Let the equation of the given circle be

$S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

and $u = lx + my + nz - p = 0$ and $u = lx + my + nz = p$

Consider the equation

$S + \lambda u = 0$

i.e. $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d + \lambda (lx + my + nz - p) = 0$ ... (5.7)

It is an equation of second degree in x, y, z in which the coefficient of x^2,y^2 and z^2 are the equal and there are no product terms xy, yz, zx.

Therefore, it represents the equation of a sphere. Also the co-ordinates of any point which satisfy S = 0, U = 0 satisfy the equation

$S + \lambda u = 0$

Hence $S + \lambda u = 0$ is a sphere through the circle

S = 0 and U = 0.

Example 4 :

The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axes in A, B, C. Find the equations of the circum-circle of the triangle ABC. Also find its centre and length of the diameter.

Solution :

Equation of the sphere OABC is

$x^2 + y^2 + z^2 - ax - by - cz = 0$

Hence the equations of the circle ABC are

$x^2 + y^2 + z^2 - ax - by - cz - 0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1=0$.

The line through the centre $\left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right)$ of the sphere perpendicular to the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ is $\frac{x-\frac{1}{2}a}{\frac{1}{a}}=\frac{y-\frac{1}{2}b}{\frac{1}{b}}=\frac{z-\frac{1}{2}c}{\frac{1}{c}}$ ... (ii)

$$\frac{(2x-a)a}{2}=\frac{(2y-b)b}{2}=\frac{(2z-c)c}{2}$$

Any point on (i) is

$$\left(\frac{1}{2}a+\frac{r}{2}\right)\left(\frac{1}{2}b+\frac{r}{b}\right)\left(\frac{1}{2}c+\frac{r}{c}\right)$$

This point will lie in the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ if

$$\frac{1}{a}\left(\frac{1}{2}a+\frac{r}{a}\right)+\frac{1}{b}\left(\frac{1}{2}b+\frac{r}{b}\right)+\frac{1}{c}\left(\frac{1}{2}c+\frac{r}{c}\right)=1$$

i.e. if $r=\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)^{\frac{1}{2}}$ $r=\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}$

or if $r=-\frac{1}{2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}$

$\therefore$ Co-ordinates of the centre are

$$\left(\frac{1}{2}a+\frac{r}{a}\right),\left(\frac{1}{2}b+\frac{r}{b}\right),\left(\frac{1}{2}c+\frac{r}{c}\right) \text{ where } r=-\frac{1}{2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}$$

Radius of the sphere $=\sqrt{\frac{a^2}{4}+\frac{b^2}{4}+\frac{c^2}{4}}=\frac{1}{2}\sqrt{a^2+b^2+c^2}$

length of the perpendicular from the centre $\left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right)$

on the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$

$$\text{is } \frac{\left|\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-1\right|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}=\frac{1}{2\sqrt{a^2+b^2+c^2}}$$

$\therefore$ Radius of the circle

$$\sqrt{\frac{1}{4}(a^2+b^2+c^2)-\frac{1}{4(a^{-2}+b^2+c^{-2})^2}}$$

$\therefore$ diameter = 2 radius $= \sqrt{(a^2+b^2+c^2) - \dfrac{1}{(a^{-2}+b^2+c^{-2})^2}}$

$$\sqrt{\frac{(a^2+b^2)(b^2+c^2)(c^2+a^2)}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}}$$

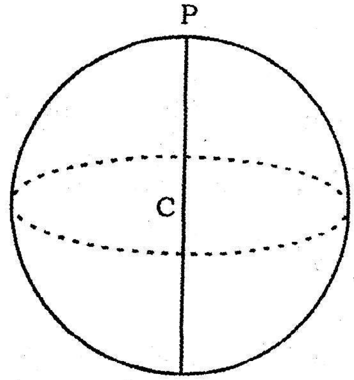
Example 5 : Find the equation of that diameter of the sphere

$x^2 + y^2 + z^2 + 4y - 6z\ 1 = 0$

such that a rotation about it will transfer the point (3, 0, 2) to the point (2,–5,4) along a great circle of the sphere. Also find the angle through which the sphere must be rotated.

Solution :

C (0, –2, 3) is the centre of the given sphere and A (3, 0, 2) and B (2, –5, 4) be the two given points.



Figure

The diameter PQ about which the sphere must be rotated to transfer the point A to the point B is perpendicular to the plane ABC.

Any plane through the point C is

$ax + b(y + 2) + (z - 3) = 0$

It will pass through A and B if

$3a + 2b - c = 0$

and $2a - 3b + c = 0$

or $\dfrac{a}{-1} = \dfrac{b}{-5} = \dfrac{x}{-13}$

$\therefore$ equation of the plane ABC is

$x + 5(y + 2) + 13(z - 3) = 0$

Now the line PQ passes through the point C and is perpendicular to the plane ABC. Therefore its equations are

$$\frac{x}{1} = \frac{y+2}{5} = \frac{z-3}{13} \Rightarrow \frac{x}{1} = \frac{y+2}{5}$$

$$\frac{x}{1} = \frac{z-3}{13}$$

Let Q be the angle through which that sphere is rotated for shifting the point A to the point B. Then $<ACB = \theta$

Direction ratios of AC and BC are $(3, 2, -1)$ and $(2, -3, 1)$

$$\therefore \text{Cos}\theta = \frac{3.2 + 2(-3) + (-1)x(1)}{\sqrt{9+4+1} \times \sqrt{9+4+1}} = -\frac{1}{14} \text{ or } \theta = \text{Cos}^{-1}\left(-\frac{1}{14}\right)$$

Example 6 :

A sphere S has (1, 2, 3), (3, 2, 1) as the opposite ends of a diameter. Find the equation of the sphere through the intersection of S and the plane $x + 2y + 3z + 1$ whose centre lies on the plane $x = 0$.

Solution :

The equation of S is

$(x - 1)(x - 3) + (y - 2)(y - 2) + (z - 3)(z - 1) = 0$

or $x^2 + y^2 + z^2 - 4x - 4y - 4z + 10 = 0$ ... (i)

Any sphere through the intersection of (i) and the plane

$x + 2y + 3z - 1 = 0$ is

$x^2 + y^2 + z^2 - 4x - 4y - 4z + 10 = \lambda(z + 2y + 3z - 1) = 0$ ... (ii)

Centre of (ii) $\frac{1}{2}(4-\lambda), \frac{1}{2}(4-2\lambda), \frac{1}{2}(4-3\lambda)$

It will lie on the plane $x = 0$, if $\lambda = 4$

Substituting $\lambda = 4$ in (ii) we get the equation of the required sphere as

$x^2 + y^2 + z^2 + 4y + 8z + 6 = 0$

Example 7 :

Prove that the circles

$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0, 5y + 6z + 1 = 0$ ... (i)

and $x^2 + y^2 + z^2 - 3x + 4y + 5z - 6 = 0, x + 2y - 7z = 0$ ... (ii)

lie on the same sphere. Find the centre and radius.

Solution :

Any sphere through (i) is

$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 + \lambda(5y + 6z + 1) = 0$ ... (iii)

Any sphere through (ii) is

$x^2 + y^2 + z^2 + 3x - 4y + 5z - 6 + \mu (x + 2y - 7z) = 3$ ... (iv)

The circles (i) & (ii) will lie on the same sphere if we can find and such that (iii) & (iv) represent the same sphere.

Equating the coefficient of like terms in (iii) & (iv), we get

$-2 = -3 - \mu$ ... (v)

$3 + 5\lambda = -4 + 2\mu$ ... (vi)

$4 + 6\lambda = 5 - 7\mu$ ... (vii)

$5 + \lambda = -6$ ... (viii)

$\Rightarrow$ $\lambda = -6 - 5$

$\lambda = -11$

From (v) & (viii) = −1, = 1

There values also satisfy (vi) & (vii)

Hence the two circle lie on the same sphere. The equation of the sphere is obtained by substituting for in (iiii) or for μ in (iv)

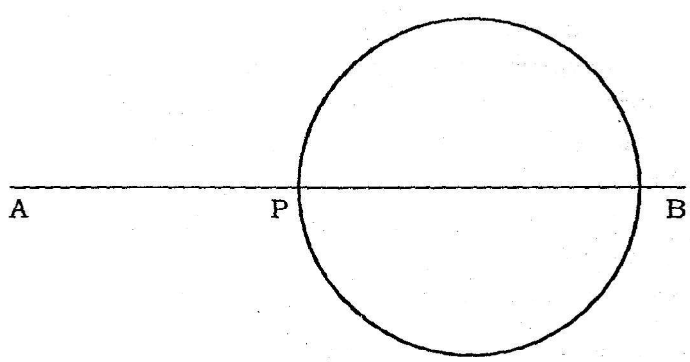
Substituting the μ in (vi) we get

$x^2 + y^2 + z^2 - 2x - 2z = 0$ as the equation of the required sphere, its centre (1, 1,1) and radius = 3.

2.1.7 Intersection of a Straight Line and a Sphere.

Let the equation of the sphere be

$S = x^2 + y^2 + z^2 - 2ux - 2vy - wz + = 0$ ... (5.8)



Figure

and the equation of the straight line through a point A with co-ordinates (x_1,y_1,z_1) be

$$\frac{x - x_1}{1} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \text{ (say)} \quad \ldots (5.9)$$

Here l, m, n are the direction cosines of the line.

Any point on this line is

$(x_1 + lr, y_1 + mr, z_1 + nr)$

If it lies on the sphere (5.8) then

$(x^2 + lr)^2 + (y_1 + mr)^2 + (z_1 + nr)^2 + 2u(x_1 + lr)^2 + 2v(y_1 + mr)^2 + 2w(z_1 + nr) + d = 0$

or $r^2 (l^2 + m^2 + n^2) + 2r\, [1(x_1 + u) + m(y_1 + v) + n\,(z_1 + w)] + x_1^2 + y_1^2 + z_1^2$

$+ 2ux_1 + 2vy_1 + 2wz_1 + d_1 = 0$

or $r^2 + 2r\, [1(x_1 + u) + m(y_1 + v) + n\,(z_1 + w) + S_1 = 0$

where $S_1 = x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d$

Now 5.10 is a quadratic equation in r and therefore given two value of r, corresponding to each of which we have a point common to the line and the sphere.

Hence any straight line through $A(x_1 + y_1 + z)$ meets the sphere in two points which are and different, or real and coincident, or imaginary in accordance with the nature of the roots of equation 5.10 (iii).

Cor. 1 :

Let r_1, and r_2 be the roots of the equation

$r^2 + 2r\, [l(x_1 + u) + m(y_1 + v) + n\,(z_1 + w)] + S_1 = 0$

then $r_1 r_2 = S_1$, which is independent of l, m, n.

Thus if any straight line through a fixed point A meets the sphere in the points P and Q. then the rectangle contained by the segments A. AQ is constant i.e. AP.PQ = S_1 = constant.

This constant is called the power of the point A with respect to the sphere. Definition 5.10 : Power of a point.

Power of a point with respect to a sphere is the expression obtained by substituting the co-ordinates of the point in the left hand member of the equation of the sphere when the right : hand member is zero and when the co-efficients of x^2, y^2, z^2 are each unity.

Cor. 2 : Position of a point relative to a sphere.

If the point A is outside the sphere then AP and AQ are of the same signs, therefore AP.AQ = S_1 is positive.

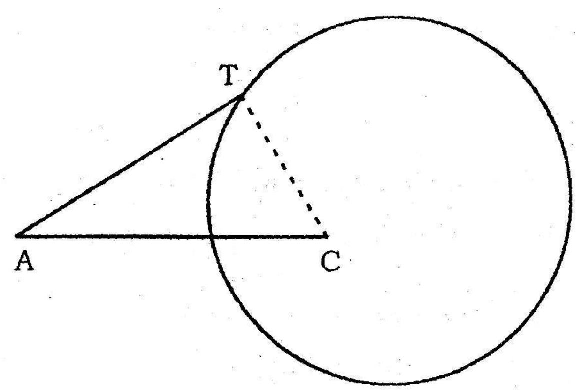
Hence we consider that the points (x_1, y_1, z_1) is inside, on or outside the sphere S=0 according as S > =, or > 0.

Cor. 3 : Length of the tangent from a point.

If the point A is outside the sphere and two points P and Q coincide at T, then AT becomes a tangent to the sphere and we get.

At. At = S_1 or AT = $\sqrt{S_1}$

$(AT)^2 = S_1$ or $AT = \sqrt{S_1}$



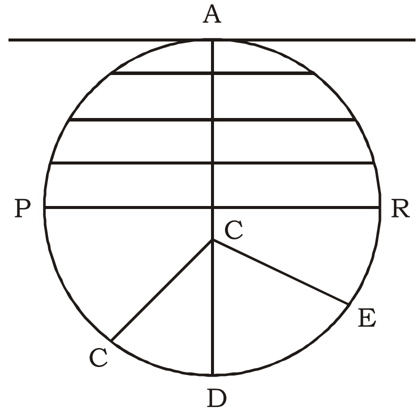
Figure

Hence the square of the length of the tangent from an external points is equal to the power of the point with respect to the sphere.

The point T is called the point of contact of the tangent.

2.1.8 Tangent Plane at a Point

Let A be any point on a sphere with centre at C. Join Ac. Let PQ be any chord of the sphere perpendicular to AC. Let Ac meet PQ at M. Then M is the middle point of PQ. Let PQ move parallel to itself such that M remains on CA. The limiting position of PQ as M reaches A is called the tangent line to the sphere at A.



This implies that the tangent line is perpendicular to CA. Since we are dealing in three dimesional space therefore, there can lie infinitely may tangent lines at A.

The locus of the tangent lines at A is a plane and is called the tangent plane of the sphere at A. It is obvious that A is perpendicular of the tangent plane at A.

To find the equation of the tangent plane at the point $p(x_1, y_1, z_1)$ to the sphere.

$S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + = 0$ ... (5.11)

Any line through (x_1, y_1, z_1) is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \qquad \text{... (5.12)}$$

It meets the sphere in points whose distances from P are given the equation.

$x^2 + 2r\,[l(x_1 + u) + m(y_1 + v) + n\,(z_1 + w)]\ S_1 = 0$ ... (5.13)

The line (5.12) will be a tangent line to be sphere (5.11) at (x_1, y_1, z_1) if both the roots of the equation (5.13) are zero.

The conditions for there are

(1) $S_1 = 0$

(2) $l(x_1 + u) + m(yl + v) + n\,(zl + w) = 0$

$S_1 = 0$ implies that point P lies on the sphere.

Condition (2) gives a relation between (l, m, n) when the 5.12 touches the sphere.

To obtain the locus of the line (11.12) for different values of l, m, n we eliminate l, m, n between (2) are 5.12. This gives :

$(x - x_1)\,(x_1 - u) + (y - y_1)\,(y_1 - v) + (z - z_1)\,(z_1 - w) = 0$

or $xx_1 + yy_1 + zz_1 + ux + vy + wz = x_1^2 + y_1^2 + z_1^2 + ux_1 + vy_1 + wz_1$

Adding $ux_1 + vy_1 + wz_1 + d$ to both sides we get

$xx_1 + yy_1 + zz_1 + u(x - x_1) + v(y + y_1) + w\,(z + z_1) + d$

$x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0 \ \Theta\ S_1 = 0$

Hence the locus of the line is

$xx_1 + yy_1 + zz_1 + u(x - x_1) + v(y + y_1) + w\,(z + z_1) + d = 0$ ... (5.14)

which is the required equation of the tangent place at (x_1, y_1, z_1) to the sphere S.

The point (x_1, y_1, z_1) is called the point of contact to the tangent plane.

Note : It may be noted that the equation of the tangent plance can be written down at once by changing

2x to (x + xl), 2y to (y + yl)

and 2 z to $(z + z_1)$

in the equation of the sphere.

Cor. 1 :

The tangent plane at any point is perpendicular to the radius through the point.

Equation of the tangent plane at (x_1, y_1, z_1) to the sphere.

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$

or $(x_1 + u)x + (y_1 + v)y + (z_1 + w)z + ux_1 + vy_1 + wz_1 + d = 0$

The direction ratios of the normal to the tangent plane are

$[x_1 + u, y_1 + v, z_1 + w]$

Also the direction ratios of the lime joining the centre $(-u, -v, -w)$ to the point (x_1, y_1, z_1) are $(x_1 + u, y_1 + v, z_1 + w)$.

Therefore the normal to the tangent plane is parallel to the radius through the point.

Hence the tangent plane at any point is perpendicular to the radius through the point.

Thus the equation to the tangent plane at any point of sphere can be obtained by using the fact that it is plane passing through the given point are perpendicular to the line joining the point to the centre of the sphere.

Condition to Tangency

To find the condition that the plane $lx + my + nz = p$ touches the sphere.

$S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Hence we have to find the condition under which $lx + my + nz = p$ will be tangent plane to the sphere S at some point of it.

The plane $lx + my + nz = p$ will be a tangent plane to the sphere S if the length of the perpendicular from the centre of the sphere $(-u, -v, -w)$ to the plane is equal to the radius of the sphere i.e. if

$$\left[\frac{-lu + my - nw - 9}{\sqrt{l^2 + m^2 + n^2}}\right] = \sqrt{u^2 + v^2 + w^2 - d}$$

$$\Rightarrow [-lu + my - nw - 9] = (\sqrt{u^2 + v^2 + w^2 - d})(\sqrt{l^2 + m^2 + n^2})$$

of if $(lu + my + nw + p)^2 = (u^2 + v^2 + w^2 - d)(l^2 + m^2 + n^2)$.

This is the required condition.

Locus of the middle points of a system of parallel chords.

Let $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ be the sphere.

Let the chords be parallel to line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

Let (α, β, γ) be middle point of one of the chords.

Then its equation is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

Any point in it is,

(α + lr, β + mr, γ + nr)

This point will lie on the sphere if

$(\alpha + lr)^2 + (\beta + mr)^2 + (\gamma + nr)^2 + 2u (\alpha + lr) + 2v (\beta + mr) + 2w(\gamma + nr) + d = 0$

or if r2 $(l^2 + m^2 + n^2) + 2r [l(u + \alpha) m (v + \beta) + n (w + \gamma)]$

$+ \alpha^2 + \beta^2 + \gamma^2 + 2ua + 2v\beta + 2w\gamma + d = 0$ ... (5.16)

Since (α, β, γ) is the middle point of chord (9.15) the two values of r must be equal in magnitude and opposite in sign.

$\therefore$ Sum of the roots of (9.16) is zero.

i.e. l(u + z) + m(v + x) +n (w + y) = 0

Hence the locus of (α, β, γ) is

l(x + u) +m(y + v) + (z + w) = 0 ... (5.17)

This is a plane through the centre of the sphere and perpendicular to the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

Example 8 :

Show that the plane 4x + 9y + 14z – 64 = 0 touches the sphere $3(x^2 + y^2 + x^2)$ – 2x – 3y – 4z – 22 = 0.

Find also the point of contact

Solution :

Equation of the sphere in the standard form is

$$x^2+y^2+z^2-\frac{2}{3}x-y-\frac{4}{3}z-\frac{22}{3}=0$$

its centre is $\left(\frac{1}{3},\frac{1}{2},\frac{2}{3}\right)$

and radius $=\sqrt{\frac{1}{9}+\frac{1}{4}+\frac{4}{9}+\frac{22}{3}}=\frac{1}{6}\sqrt{293}=\frac{\sqrt{293}}{6}$

length of the perpendicular from the centre to the plane is

$$\frac{\frac{4}{3}+\frac{9}{2}+\frac{28}{3}-64}{\sqrt{16+81+196}}=\frac{\sqrt{293}}{6}=\text{radius of the sphere.}$$

Hence the given plane is a tangent plane to the sphere. Now equation of any line through the $\left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right)$ and perpendicular to the plane.

$4x + 9y + 14z - 64 = 0$ is

$$\frac{x-\frac{1}{2}}{4} = \frac{y-\frac{1}{2}}{9} = \frac{z-\frac{2}{3}}{14} = r\,(\text{Say}) \Rightarrow x - \frac{1}{2} = 4r.$$

Any point on this has co-ordinates

$$\left(4r + \frac{1}{3}, 9r + \frac{1}{2}, 14r + \frac{2}{3}\right)$$

It will lie on the plane $4x + 9y + 14z = 64 = 0$, if

$$4\left(4r + \frac{1}{3}\right) + 9\left(9r + \frac{1}{2}\right) + 14\left(14 + \frac{2}{3}\right) - 64 = 0$$

$$\text{or } r = \frac{1}{8}$$

$\therefore$ the point of contact is

$$\left(4 \times \frac{1}{4} + \frac{1}{3}, 9 \times \frac{1}{2}, 14 \times \frac{1}{6} + \frac{2}{3}\right)$$

i.e. (1, 2, 3).

Example 9 :

Find the equation of the sphere of radius r which touches the three co-ordinates axis.

Solution :

Let the equation of the sphere be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ ... (i)

Its radius will be r if $u^2 + v^2 + w^2 - d^2 = r^2$ ... (ii)

It cuts the X-axis where $(y = 0, z = 0)$

$x^2 + 2ux + d = 0.$

Since the X-axis is to touch the sphere, therefore

$u^2 - d = 0$ or $u^2 = d$

Similarly $v^2 = w^2 = d$

Substituting in (ii) we get,

$$u^2 = v^2 = w^2 = d = \frac{1}{2}r$$

Hence the equations of the sphere are

$$x^2 + y^2 + z^2 \pm \sqrt{2}rx \pm \sqrt{2}ry \pm \sqrt{2}rz + \frac{1}{2}r^2 = 0$$

$$\Rightarrow 2x^2 + 2y^2 + 2z^2 + \sqrt{2}rx \pm \sqrt{2}ry \pm \sqrt{2}rz + r^2 = 0$$

The number of sphere is eight.

Example 10 :

Find the equations of the sphere which pass through the circle

$x^2 + y^2 + z^2 = 9$, $x + z = I$ ... (i)

and touch the plane

$2x + 2y + z = 5$ ... (ii)

Any sphere through the circle (i) is

$x2 + y2 + z2 - 9 - 2K(x + y + z - 1) = 0$ ... (iii)

centre of this sphere is (k, k, k) and its radius

$$= \sqrt{3K^2 + 9 + 2K}$$

The sphere (iii) will touch the plane (ii) if

$$\frac{(2K + 2K + K - 5)}{3} = \pm\sqrt{3K^2 + 9 - 2K}$$

or $(5K - 5)^2 = 9(3K - 2K + 9)$

or $K^2 + 16K + 28 = 0 \;\therefore\; K = -2, -14$

Hence the equations of the sphere are

$x^2 + y^2 + z^2 - 9 + 4(x + y + z - 1) = 0$

$\Rightarrow x^2 + y^2 + z^2 + 4x + 4y + yz - 13 = 0$

and $x^2 + y^2 + z^2 - 9 + 28(x + y + z - 1) = 0$

$\Rightarrow x^2 + y + z^2 + 28x + 28y + 28z - 37 = 0$

or $x^2 + y^2 + z^2 - 5x + 4y + 4z - 13 = 0$

$\Rightarrow x^2 + y^2 + z^2 - 5x + 4y + 4z - 13 = 0$

and $x^2 + y^2 + z^2 - 28x + 28y + 28z - 37 = 0.$

2.1.9 Intersection of Two Circles

Let the equation of the two circle be

$S_1 = x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$ ... (5.12)

and $S_1 = x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$ ... (5.13)

The centres of thee sphere are $(-u_1, v_1, -wl)$ and $(-u_2, -v_2, -w_2)$

and radii $\sqrt{u_1^2+v_1^2+w_1^2-d_1}$ and $\sqrt{u_2^2+v_1^2+w_2^2-d_2}$

The two sphere will intersect each other only if the distance between their centres is less than the sum of their radii

i.e. if $\sqrt{(u_1+u_2)^2+(v_1-v_2)^2+(w_1-w_2)^2} < \sqrt{u_1^2+v_1^2+w_1^2-d_1}+\sqrt{u_2^2+v_2^2+w_2^2}$ (5.20)

(Which is the required condition that the two sphere S_1 and S_2 intersect Now subtituting S_2 from S_1 we get.

$S_1 - S_2 = 2(u_1 - u_2)\,x + 2\,(v_1 - v_2)\,y + 2\,(w_1 - w_2)z + d_1 - d_2 = 0$ ... (5.21)

(5.21) is a first degree equation in x, y, a. Therefore it represents a plane. Also it is satisfied by the co-ordinate of point which satisfy $S_1 = 0$ and $S_2 = 0$.

Therefore is a plane through the common points of the two spheres i.e. it is a plane passing through the curve of intersection of the spheres.

It may be noted that $S_1 - S_2 = 0$ represents a plane only if the two sphere are in their standard from i.e. the co-efficients of x^2, y^2, z^2 in both the sphere are equal to unity.

$\therefore$ The two spheres will intersect in a plane if condition (5.20) is satisfied and the co-efficients of x^2, y^2, z^2 in both sphere are equal to one.

It may be noted that the curve of intersection of the two spheres is the same as the curve of intersection of the plane $S_1 - S_2 = 0$ and either of the sphere.

Hence the curve of intersection of two sphere if they intersect is a circle.

2.1.10 Radical Plane :

The locus of a point whose powers with respect to the two sphere are equal is a plane called the radical plane of the two spheres.

Equation of the Radical Plane :

Let and equation of the two spheres be

$S_1 = x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$

$S_2 = x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_1 = 0$

Let P (x_1, y_1, z_1) be any point on the radical plane, then by definition.

Power of P with respect to S_1 = Power of P with respect to S_2.

$$\therefore x_1^2+y_1^2+z_1^2+2u_1x_1+2v_1y_1+2w_1z_1+d_1$$

$$=x_2^2+y_2^2+z_2^2+2u_2x_1+2v_2y_1+2w_2z_1+d_2=0$$

or $2(u_1 - u_2)\,x_1 + 2(v_1 - v_2)\,y_1 + 2(w_1 - w_2)\,z_1 + d_1 - d_2 = 0$

Hence the locus of (x_1, y_1, z_1) is

$2(u_1 - u_2)x + 2(v_1 - v_2)\,y + 2(w_1 - w_2)\,z + d_1 - d_2 = 0$

Which is evidently a plane.

It may be observed that the radical plane can be written as $S_1 = S_2$ or $S_1 = S_2 = 0$.

Cor 1 : The radical plane of two sphere is perpendicular to the line joining their centres. The direction ratios of the normal to the radical plane are

$2(u_1 - u_2) : 2(v_1 - v_2) : 2(w_1 - w_2)$ or $(u_1 - u_2), (v_1 - v_2), (w_1 - w_2)$

Also the direction ratios of the line joining the centres of the two spheres are $(u_1 - u_2), (v_1 - v_2), (w_1 - w_2)$.

Thus the normal to the radical plane is parallel to the joining the centres of the spheres.

Hence the radical plane of the two spheres is perpendicular to the line joning the centres.

Cor. 2 :

If the two sphere intersect, their radical plane is the plane of intersection, i.e. the circle of intersection lies in the radical plane.

(Try Yourself)

Example 11 :

Find the locus of a point where powers with respect to two given spheres are in a constant ratio.

Solution :

Let the given sphere be

$$x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0 \qquad \dots (i)$$

and $$x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_1 = 0 \qquad \dots (ii)$$

Let P(x, y, z) be the moving point let its power with respect to (i) is k times its power with respect to (ii), then

$$x_1^2 + y_1^2 + z_1^2 + 2u_1x_1 + 2v_1y_1 + 2w_1z_1 + d_1$$

$$= k(x_1^2 + y_1^2 + z_1^2 + 2u_2x_1 + 2v_2y_1 + 2w_2z_1 + d_2)$$

or $$(1-k)\ (x_1^2 + y_1^2 + z_1^2) + 2\,(u_1 - ku_2)\ x_1 + 2(v_1 - kv_2)\ y_1 + 2\ (w_1 - kw_2)\ z_1 + d_1 - kd_2 = 0$$

Hence the locus of P is

$$(1 - k)\ (x^2 + y^2 + z^2) + 2\ (u_1 - ku_2)\ x + 2\ (v_1 - kv_2)\ y + 2\ (w_1 - kw_2)\ z_1 + d_1 - kd_2 = 0$$

or

$$x^2 + y^2 + z^2 + \frac{2}{1-k}(u_1 - ku_2)\ x + \frac{2}{1-k}\ (v_1 + kv_2)\ y + \frac{2}{1-k}(w_1 - kw_2)\ z + \frac{d_1 - kd_2}{1-k} = 0$$

which again is a sphere

2.1.11 Self Check Exercise

1. Find the equation of the sphere circumscribing the ttrahedron whose locus are $\frac{y}{b} + \frac{z}{c} = 0, \frac{z}{c} + \frac{x}{a} = 0, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + 1$

2. A sphere throw the origin has its center in the positive octant and cuts the planes x = 0, y = 0 in circles of radius a, b,c respectively show that its equation is

$$x^2+y^2+z^2-2\sqrt{\frac{b^2+c^2-a^2}{2}}x-2\sqrt{\frac{c^2+a^2-b^2}{2}}y$$

$$-2\sqrt{\frac{a^2+b^2-c^2}{2}}\,z=0$$

3. A sphere of constant radius k passes through the origin and meets the axes in A,B,C prove that the centroid of triangle ABC lies on the sphere $9(x^2+y^2+z^2)=4k^2$.
4. Show that the points (5, 0, 2), (2, –6, 0), (7, 3–, 8) (4, –9, 6) are the concyclic.
5. Find the condition that the circles
$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 = lx + my + nz - p = 1$ to be replaced by and $x^2 + y^2 + z^2 + 2u^1x + 2v^1y + 2w^1z + d^1 = 0\ l^1x + m^1y + n^1z + n^1z - p^1$ lie on the same sphere.
6. Prove that the some of the squares of the intercepts made by a given sphere on any mutually pertpendicular lines through a fixed points is constant.
7. Find the equations of the tangent planes to the spheres
$x^2 + y^2 + z^2 - 2x + 4y + 6z - 10 = 0$.

Which pass through the line $\frac{x+3}{14}=\frac{y+1}{3}=\frac{z-5}{4}$

8. Find equation of the sphere which touches the sphere
$x^2 + y^2 + z^2 - 3x + 2y + 4z - 43 = 0$ internelly a (3, 2, 4) and has a radius equal to 3.
9. Show that the spheres
$x^2 + y^2 + z^2 - 4x - 7y - 10z + 34 = 0$

and $x^2 + y^2 + z^2 + 2x + 2y + 2z - 143 = 0$ touch internally. Find the co-ordinates of their points of contact.

10. Find the equation of the sphere which pass through the points (4, 1, 0), (2, –3, 4), (1, 0, 0) and touches the planes 2x + 2y + – z – 11 = 0.
11. Find the locus of the centeres of spheres of constant radius which pass through a given point and touch a given line.
12. Prove that the centers of the spheres which touch the lines y = mx, z = c, y = –mx, z = –c lie on the surface $mxy + cz(l - m^2) = 0$.
13. Tengent plane at any point of the sphere
$x^2 + y^2 + z^2 - r^2 = 0$

meets the co-ordinate axes at A, B, C. Show that the locus of the point of intersection of planes drawn parallel to the co-ordinate planes through A,B,C in the surface

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{r^2}$$

14. Find the equation of the plane which touches each of the circles $x = 0, y^2 + z^2 = a^2, y = 0, z^2 + x^2 = b^2, = z = 0, x^2 + y^2 = c^2$. How many much planes are three ?

15. Show that the locus of points from which three mutually perpendicular planes can be drawn to touch the elliped $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z=0$ is $x^2 + y^2 + z^2 = a^2 + b^2$.

16. Find the equation of the radical line of the spheres
$x^2 + y^2 + z^2 + 2x + 2y + 2z + 2 = 0$
$x^2 + y^2 + z^2 + 4y = 0$
$x^2 + y^2 + z^2 + 3x - 2y + 8z + 6 = 0$

17. Write down the condition when sphere S and the plane may have
(i) more than one common point.
(ii) a unique common point.
(iii) no common point.

18. When are the two spheres $s_1 = 0$ and $s_2 = 0$ are orthongal where $s = 0$ is $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

19. Define the radical plane of two spheres.

20. Define co-axial spheres and limting point.

Reference Book

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LESSON NO. 2.2

CONE-I

2.2.1 : The Cone

A cone is a surface generated by a straight line which passes through a fixed point and satisfy one other condition; for instance it may intersect a given curve, touch a given surface, or make a constant angle with a line through a given point.

The straight line is called generator of the cone and the given point is called its vertex.

If the straight line intersects a given curve (surface) then that curve (surface) is called the guiding curve (surface) of the cone.

2.2.2 : Homogeneous Equation

Art. :- The $f(x, y, z) = 0$ is said to be homogeneous if $f(rx, ry, rz)=0$ for all values of r.

Proof. :- The equation

$$f(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

is homogeneous, because if x, y, z are changed to rx, ry, rz the equation becomes.

$f(rx, ry, rz) = a(rx)^2 + b(ry)^2 + c(rz)^2 + 2f(ry)(rz)$

$+ 2g(rg)(rx) + 2h(rx)(ry)$

$= r^2(ax^2 + by^2 + cz^2 + 2fyz + 2gzxb + 2hxy) = r^2 0 = 0$

$\therefore f(x, y, z) = 0 = f(rx, ry, rz) = 0$

$\therefore f(x, y, z) = 0$ is homogeneous in x, y, z.

2.2.3 Equation of the Cone with Vertex at the Origin.

Here, It will be proved that the equation of a cone with vertex at the origin is homogeneous whatever be the guiding curve or guiding surface.

Proof : Let the equation of the cone with vertex at the origin be

$$f(x, y, z) = 0.$$

Let P (x_1, y_1, z_1) be any point on the cone, then

$f(x_1, y_1, z_1) = 0$. ... (6 . 1)

Now equation of the line OP is $\frac{x}{x^1} = \frac{y}{y^1} - \frac{z}{z^1} \Rightarrow \frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1}$

Any point on OP is (rx_1, ry_1, rz_1)

Since OP is a generator of the cone $\therefore$ it completely lies on the cone $\therefore$ Every point liest on the cone

$\therefore$ the point $f(rx_1, ry_1, rz_1) = 0$

From (6.1) & (6.2) it follows that. ... (6.2)

$f(x, y, z) = 0$ is homogeneous equation.

Converse :

In the converse part, we will prove that every homogeneous equation of degree $n \geq 2$, represents a cone with vertex at the origin or it repersents through the origin.

Proof : Let $f(x, y, z) = 0$ be a homogeneous equation of degree n (≤ 2) in x, y, z. Suppose that $f(x_1, y_1, z)$, cannot be decomposed into n linear factors.

Since $f(x, y, z) = 0$ is homogeneous

$\therefore f(rx, ry, rz) = 0$ for all r.

In particular $f(0, 0, 0) = 0$ which implios that the origin lies on the surface repersented by $f(x, y, z) = 0$.

$f(rx, ry, rz) = 0$ for all values of r.

implies that every point on the line through the lies on the surface $f(x, y, z)=0$.

Hence we can say that the surface is generated by the lines through the origin.

Intersection of this surface with any plane can be regarded as a guiding curve, the given equation represents a cone with vertex at the origin.

Art. : If $f(x, y, z) = 0$ can be resolved into n linear factors in x, y, z, then it will represent n planes through the origin.

Proof : The equation $ax^2+by^2+2hxy+2fyz + 2gzx = 0$

will represent a pair of planes through the origin if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ b & f & c \end{vmatrix} = 0$$

$a(bc - f^2) - h(hc - bg) + g(h - b^2) = 0$

$abc - af^2 - h^2c - bgh + Ihf - b^2 g = 0$

$abc - bgh + ghf - af^2 - ch^2 - gb^2 = 0$

We can consider the two planes through the origin as a degenerate cone.

$\therefore$ We can say that a homegeneous equation of degree 2 in x, y, z represents a equation of the cone and conversely.

Proof :

Let the equation of the cone with vertex at the origin be represented by the homegeneous equation.

$f(x, y, z) = 0$

Let (l, m, z) be the direction ratios of any generators of the cone.

Equation of the generators is

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

Any point on the generators is (lr, mr, nr)

Since the point lies on the cone

$\therefore f(lr, mr, nr) = 0$

But $f(x, y, z) = 0$ is homogeneous

$\therefore f(l, m, n) = 0.$

$\therefore f(l, m, n) = 0$

Conversely :

Let $f(x, y, z) = 0$ be the equation of the cone with vertex at the origin.

Let $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ be equation of a line through the origin such that $f(l, m,n)=0$.

Now we are to prove that $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is a generators of the cone.

For this, it is sufficent to prove that every point on $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ lies on the cone.

Now any point on $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is (lr, mr, nr)=0 for any real number r.

Since, $f(x, y, z) = 0$ is homegeneous in x, y, z and $f(l, m, n) = 0$ therefore, $f(lr, mr, nr) = 0$.

which implies $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is a generator of the cone $f(x, y, z) = 0$

Hence we can say that the direction ratios of a generator of a cone satisfy the equation of the cone.

Example 1

Find the equation of the cone with vertex at the origin and which passes through the curve

$ax^2 + by^2 + cz^2 = 1$, $lx + my + nz = p$.

Solution :

The equation of the cone is obtained by making the equation $ax^2 + by^2 + cz^2 = 1$, homegenous with the help of the equation $lx+my+nz=p$

Now $lx + my + nz = p$ or $\frac{lx+my+nz}{p} = 1$

$\therefore$ making $ax^2 + by^2 + cz^2 = 1$ homogeneous, we get

$$ax^2 + by^2 + cz^2 = \left(\frac{lx+my+nz}{p}\right)^2 \Rightarrow ax^2 + by^2 + cz^2 = \frac{(lx+my+nz)^2}{p^2}$$

or $p^2 (ax^2 + by^2 + cz^2) - (lx + my + cz)^2 = 0$

which is the required equation of the cone.

Example 2

The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meets the co-ordinate axes in A, B, C. Find the equation of the cone generated by the lines drawn from O to meet the circle A, B, C.

Solution :

The co-ordinates of the points A, B, C are obviously A(a, o, o), B(o, b, o), C(o, o, c).

Equation of the sphere O A B C is

$x^2 + y^2 + z^2 - ax - by - cz = 0$. [Proved in the previous lesson].

$\therefore$ Equation of the circle A B C is represented by the equation of the sphere i.e.

$x^2 + y^2 + z^2 - ax - by - cz = 0$ and the equation of the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

The equation of the cone is obtained by making the equation of the sphere homegenous with the help of the equation of the plane.

Thus the equation of the required cone is

$$x^3 + y^2 + z^2 - ax\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) - by\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) - cz\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 0$$

$$\text{or } yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$

Example 3

Show that the cone of the second degree can be found to pass through any two sets of rectangular axes through the same origin.

Solution :

Equation of a cone of the second degree with vertex at the origin is given by $ax^2 + by^2 + 2fyz + 2gzx + 2hxy = 0$.

Equation of any line through the origin with direction cosines, l, m, n, is

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

This line will lie on the cone if

al2 + bm2 + cn2 + 2 fmn + 2gnl + 2hlm = 0.

It means that the cone will pass through the x-axis if it satisfies the direction ratios of x-axis i.e., (l, 0, 0). if.

a.1 + b.0 + c.0 + 2.f.0.0 + 2g0.l + 2h 1.0 = 0

or if a = 0

Similarly the cone will pass through he y-axis and z-axis if b = o, and c = o.

Therefore the equation of the cone passing through one set of rectangular axis is fyz + gzx + hxy = 0 ... (i)

Let the direction ratios of the second set of axis through the same origin be

$l_1, m_1, n_1, \quad l_2, m_2, n_2, \quad l_3, m_3, n_3$

The cone (i) will pass through the first two of these new axes if

$fm_1 n_1 + gn_1l_1 + hl_1 m_1 = 0$ ... (ii)

and $fm_2 n_2 + gn_2l_2 + hl_2m_2 = 0$ ... (iii)

adding (ii) & (iii) we get.

$fm_1 n_1 + m_1n_2 + m_3n_3 = 0, n_1l_1 + n_2l_2 + n_3l_3 = 0$

But we know that $m_1n_1 + m_1n_2+m_3n_3 = 0, n_1l_1 + n_2l_2 + n_3l_3 = 0$

and $l_1m_1 + l_2m_2 + l_3 m_3 = 0$

$\therefore f(-m_3 n_3) + g (-n_3l_3) + h (-l_3m_3) = 0$

or $fm_3n_3 + gn_3l_3 + hl_3m_3 = 0$

Which shows that the cone (i) passes through the 3rd axis of the second set of axes.

2.2.4

Quadric Cone. A cone which is cut by an arbitrary straight line (other than a generator) in two and only two points is called a quadric cone.

It may be understood that the equation of the quadric cone will always be of second in x, y, z and any plane section of this will be a conic section which is taken as the guiding curve of the cone.

2.2.4.1 Equation of a Quadric Cone

To find the equation of the cone with vertex at (α, β, γ) and guiding curve the conic $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0, z = 0$.

Equation of any line through α, β, γ with direction ratios l, m, n is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \ldots (6.3)$$

Line (6.3) will be meet the plane z = 0, at

$$\left(a - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$$

This point will lie on the given conic if

$$a\left(a - \frac{l\gamma}{n}\right)^2 + b\left(\beta - \frac{mr}{n}\right)^2 + 2h\left(\alpha = \frac{l\gamma}{n}\right)\left(\beta - \frac{m\gamma}{n}\right) + 2g\left(\alpha - \frac{l\gamma}{n}\right) + 2f\left(\beta - \frac{m\gamma}{n}\right) + c = 0$$

...... (6.4)

(6.4) gives the condition that the line (3.3) will intersect the given conic, Eliminating l, m, n between (12.3) # (12.4) we get.

$$+2g\left(\alpha - \frac{x-\alpha}{z-\gamma}.\gamma\right)^2 + b\left(\beta - \frac{y-\beta}{z-\gamma}.\alpha\right) + 2h\left(\alpha - \frac{x-\alpha}{z-\gamma}.\gamma\right)\left(\beta - \frac{y-\beta}{z-\gamma}.\gamma\right)$$

$$+2g\left(\alpha - \frac{x-\alpha}{z-\gamma}.\gamma\right)^2 + 2f\left(\beta - \frac{y-\beta}{z-\gamma}.\gamma\right) + c = 0$$

or $a(\alpha z - \gamma x)^2 + b(\beta z - \gamma y)^2 + 2h(\alpha z - \gamma x)(\beta z - \gamma y)$

$+ 2g(az - \gamma x)(z-\gamma) + 2f(\beta z - \gamma y)(z-\gamma) + c(z-\gamma)^2 = 0$ ... (6.5)

This is the required equation of the cone.

We can write the equation (6.5) in the convenient form by collecting coefficients of z^2, $z\gamma$, and r^2.

co-efficient of $z^2 = a\alpha^2 + b\beta^2 + 2h\alpha\beta + 2g\alpha + 2f\beta + c$.

co-efficient of $-2z\gamma + x(a\alpha + h\beta + g) + y(h\alpha + b\beta + f)(g\alpha + f\beta + c)$

co-efficient of $\gamma^2 = ax^2 + by^2 + 2hxy + 2gx + 2fyxa$.

If we write the given conic as ie $r^2 = ax^2 + bj^2 + 2gx + [2hxj + 2fyxa]$

f(x, y) = 0, and z=0. then the equation (6.5) of the cone can be written as

$$z^2 f(a, b) - z\gamma\left(z\frac{\partial f}{\partial a} + y\frac{\partial f}{\partial a} + f\frac{\partial f}{\partial a}\right) + g^2 f\,(x, y) = 0$$

where t is an auxiliary varible which is introduced to make f (α, β) homogeneous in α, β, t and it replaced by unity after differentiation.

2.2.4.2 Condition for the General Equation of Second Degree to Represent a Cone.

Let the general equation of second degree, viz

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$ ... (6.6)

represent a cone with vertex at (α, β, γ)

Shifting the origin to (α, β, γ) the equation (6.6) gets transformed to

$a\,(x+\alpha)^2 + b\,(x+\beta)^2 + c(x+\gamma)^2 + 2\,f\,(y+\beta)\,(z+\gamma)$

$+2g(z+\gamma)\,(x+\alpha)\,2h\,(x+\alpha)\,(y+\beta)$

$+a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\,\beta\gamma\alpha + 2h\alpha\beta$

$+2u\alpha + 2v\beta + 2\,w\gamma + d = 0.$

Since this represents a cone with vertex at the origin, so that should be homogeneous in x, y, z. Therefore the coefficients o x, y, z and the constant term should be zero i.e.

$a\alpha + h\beta + g\gamma + u = 0$ ... (6.7)

$\alpha h + b\beta + f\,\gamma + v = 0$ ... (6.8)

$g\alpha + f\beta + c\gamma + w = 0$ ... (6.9)

and

$a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2h\alpha\beta 2g\gamma\alpha + 2u\alpha + 2v\beta + 2w\gamma + d = 0$

or $\alpha\,(a\alpha + h\beta + g\gamma + u) + \beta\,(h\alpha + b\beta + f\gamma + 2u)$

$+ \gamma\,(g\alpha + f\beta + c\gamma + w) + u\alpha + v\beta + w\gamma + d = 0$ ... (6.10)

$\therefore$ Eliminating α, β, γ between (6.7), (6.8), (6.9) & (6.10) the required condition is

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0 \qquad \text{... (6.11)}$$

If condition (6.11) is satisfied then the co-ordinates of the vertex of the cone can be obtained by solving any three of the equations (6.7) to (6.10)

Note : It may be noted that if we write the equation (6.6) as F (x, y, z and make it homogeous with the help of an auxiliary variable t. Then the equations (6.7) to (6.10) can be written as.

$$\frac{\partial F}{\partial \alpha} = \frac{\partial F}{\partial \beta} = \frac{\partial F}{\partial \gamma} = \frac{6F}{\partial t} = 0$$

where F is F(x, y, z, t)

and t is replaced by unity after differentiation.

Example 4 :

Find the equation of the cone whose vertax is (α, β, γ) and base

(i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0, z = 0$

(ii) $y^2 = 4ax, z = 0$

Solution :

Equation of the line through (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad \dots (i)$$

It meets z = 0 where.

$$x = \frac{\alpha n - \gamma \alpha}{n}, y = \frac{\beta n - \gamma m}{n}, z = 0$$

The points must lie of the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\therefore \frac{(\alpha m - \gamma l)^2}{a^2} + \frac{\beta n - \gamma m}{b^2} = n^2 \Rightarrow \frac{(am - \gamma l)^2}{a^2} = n^2 - \frac{(Bn - \gamma m)}{b^2}$$

eliminating (l, m, n) between (i) & (ii) we get.

$$\frac{(az - \gamma x)^2}{a^2} = \frac{(\beta z - \gamma y)^2}{b^2} = (z - \gamma)^2$$

This the required equation of the cone.

(ii) Subsititing for x and y in the equation $y^2 = 4ax$, we get

$(\beta n - \gamma m)^2 = 4an\ (\alpha n - \gamma l)$ ... (ii)

Eliminating l, m, n (i) & (iii), we get

$(\theta z - \gamma y)^2 = 4a\ (z - \gamma)\ (\alpha z - \gamma x)^2$

which the required equation the cone.

Example 5 :

The section of a cone whose vertex is P and guiding curve the ellips $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

z = 0, by the plane x = 0 is a rectangular hyperbola. Show that the locus of P is $\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$.

Solution :

Let the co-ordinates of P be (α, β, γ) then the equation of the cone from Example 13.4 is

$$\frac{(\alpha z-\gamma x)^2}{a^2}+\frac{(\beta z-\gamma y)}{b^2}=z(z-\gamma)^2.$$

It meets the plane x = 0, where

$$\frac{\alpha^2 z^2}{a^2}+\frac{(\beta-y\gamma)^2}{b^2}=(z-\gamma)^2=0$$

This conic will be a rectangular hypherbola is the sum of the coefficients of x^2 and y^2 is zero.

i.e. if $\frac{\alpha^2}{a^2}+\frac{\gamma^2}{b^2}+\frac{\beta^2}{b^2}-1=0$

∴ the locus of P (α, β, γ) is

$$\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1 \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{b^2}=1$$

Example 6 :

A cone whose base circle is

$x^2 + y^2 + 2gx + 2fy = 0, z = 0$

passes through a fixed point (0, 0, c). If the section of cone by y=0, be a rectangular hyperbola prove that the vertex lies on a fixed circle.

Solution :

Let the co-ordinates of the vertex be (α, β, γ)

Any line through (α, β, γ) is

$$\frac{x-\alpha}{l}+\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

It meets z = 0, where, $x=\frac{\alpha n-\gamma l}{n}, y=\frac{\beta n-\gamma m}{n}$ … (i)

It will lie on the conic $x^2 + y^2 + 3gx + 2fy = 0$ if

$(\alpha n-\gamma l)^2+(\beta z-\gamma m)^2+2gn(\alpha n-\gamma l)+2fn(\beta n-\gamma m)=0$ … (ii)

Eliminating l, m, n between (i) & (ii), we get the cone as

$(\gamma z - \gamma x)^2 + (\beta z - \gamma y)^2 + 2g (z - \gamma) (\alpha z - \gamma x) + 2f (z - \gamma) (\beta z - \gamma y) = 0$

This cone will pass through (0, 0, c) if

$\alpha^2 c + \beta^2 c + 2g(c - \gamma) \alpha + 2f (c - \gamma) \beta = 0$

The cone meets the plane y = 0, where ... (iii)

$(\alpha z - \gamma x)^2 + \beta^2 z^2 + 2g (z - \gamma) (\alpha z + \gamma x) + 2f (z - \gamma) \beta z = 0$

This will be a rectangular hypherbola if sum of the co-effcients of x^2 and z^2 is zero i.e. if

$\gamma^2 + \alpha^2 + \beta^2 + 2ga + 2f\beta = 0.$ ... (iv)

Hence from (iii) & (iv) The locus of (α, β, γ) are

$c(x^2 + y^2) + 2g (c - z)x + 2f (c - z) y = 0$ ... (v)

and $x^2 + y^2 + z^2 + 2gx + 2fy = 0$

Now (v)–c (vi) gives.

$- cz^2 - 2gxz - fyz = 0$

or $2gx + 2fy + cz = 0$. which is a plane

Hence the locus of (α, β, γ) is.

$x^2 + y^2 + z^2 + 2gx + 2fy = 0 = 2gx + 2fy + cz$

which is a fixed circle

Example 7

Two cones pass through the curves

$y = 0,\ z^2 = 4ax,\ x = 0,\ z^2 = 4$ by

and have a common vertex. Show that the plane z = 0 cuts them in two conics which meet in four concyclic points, the vertex lies on the surface

$$z^2\left(\frac{x}{a} + \frac{y}{b}\right) = 4x^2 = y^2$$

Solution :

Let the common vertex be (α, β, γ)

Any line through (α, β, γ) is

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} \qquad \text{... (i)}$$

It meets y = 0,

where $x = \dfrac{m\alpha - l\beta}{m},\ z = \dfrac{m\gamma - n\beta}{n}$

it lies on $z^2 = 4ax$

Therefore $x = \frac{(m\gamma - n\beta)^2}{m^2} = 4a\,\frac{(m\alpha - l\beta)}{m}$

or $(m\gamma = n\beta) = 4am\ (ma - l\beta)$ ... (ii)

eliminating (l, m, n) between (i) and (ii) we get.

$(\gamma y - \beta z)^2 = 4a(y - \beta)\ (\alpha y - \beta x)$. ... (iii)

This is a cone through the curve z2 = 4ax, y = 0.

Similarly we can find the cone through the curve $z^2 = 4$ by, x = 0.

which is

$(\gamma x - \alpha z)^2 = 4b\ (x - \alpha)\ (\beta x - \alpha y)$. ... (iv)

cones (iii) & (iv) meet z = 0, where

$\gamma^2 y^2 = 4a\ (y - \beta)\ (\alpha y - \beta x)$ ... (v)

and $\gamma^2 x^2 = 4b\ (x - \alpha)\ (\beta x - \alpha y)$ ... (vi)

Any conic through the intersection of (v) & (vi) is

$\gamma^2 x^2 - 4a(y-\beta)\ (\alpha y - \beta x) + k\ [\gamma^2 x^2 - 4b\ (x-\alpha)\ (\beta x - \alpha y)] = 0$

This will be a circle if co-efficient of x^2 = co-efficient of xy = 0.

i.e. if $k(\gamma^2 - 4b\beta) = \gamma^2 - 4a\alpha$

and $4\alpha\beta + 4kba = 0$.

eliminating k between these two equations, we get.

$$-\frac{\alpha\beta}{b\alpha}\gamma^2 - 4b\beta = \gamma^2 - 4a\alpha$$

or $$\gamma^2\left(\frac{\alpha}{a} + \frac{\beta}{b}\right) = 4\alpha^2 + \beta^2$$

or the locus of (α, β, γ) is

$$z^2\left(\frac{x}{a} + \frac{y}{b}\right) = 4\ (x^2 + y^2)$$

Example 8

Prove that the equation $7x^2 + 2y^2 + 2z^2 - 10zx + 10xy + 2x - 2y + 2z - 17 = 0$ represents cone. Find its vertex.

Solution

Here a + 7, b+2, c+2, f+0, g=–5, h=s, f=0, v=–1, w=1, w=1, d=17

$$\therefore \quad \begin{vmatrix} 7 & 5 & -5 & 13 \\ 5 & 2 & 0 & -1 \\ -5 & 0 & 2 & 1 \\ 13 & -1 & 1 & -17 \end{vmatrix}$$

$\therefore$ the given equation represents a cone.

To find the vertex we solve the equations.

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$$

or $\quad 14x-10x-10y+26=0$

$4y + 10x - 2 = 0$

$4z - 10x + 2 = 0$

Solving for (x, y, z) we set.

$x=1, y=2, z=2.$

Hence the vertex is (1./3, 2)

Definition 2.2.5 : Circular Cone

A circular cone is the surface generated by a line passing through a fixed point, and the touching a given circle.

Definition 2.2.6 Elliptic Cone

An elliptic cone is the surface generated by a line passing through a fixed point and touching a given ellipse.

2.2.7 Self Check Exercise

1. Show that the lines drawn through the point (α, β, γ) whose direction ratios satisfy the relation
$al^2 + bm^2 + cn^2 = 0$
generate the cone
$a(x-\alpha)2 + b(y-\beta)^2 + c(z-\gamma)^2 = 0$
2. Find the equation of the cone with vertex at the origin and passing through the circle given by
$x^2+y^2+z^2+x-2y+3z-4=0, x-y+z=2$
3. Find the equation of the cone with vertex at the origin and the guiding curve
$x^2+y^2+z^2+x-2y+3z-4=0, x-y+z=2$
4. Find the equation of the cone with vertex at the origin and the guiding curve
$x^2+y^2+z^2+x+y+z+1=0$
$x^2+y^2+z^2+2x+2y+2z+3=0$

5. Planes through OX and OY include an angle a; show that their line of intersection lies on the cone
$z^2 (x^2+y^2+z^2)=x^2y^2 \tan^2 a$.
6. Find the equation of the cone with vertex (3, 4, 5) and base the conic
$3y^2+4z^2 = 16, 2x+z=0$
7. Prove that the equation
$x^2-2y^2+3z^2+5yz-6zx-4xy+8x-19y-2z-20=0$
represents a cone. Find its vertex
8. Find the equation of the cone with vertex (1, 2, 3) and guiding curve the circle.
$x^2+y^2+z^3=4, x+y+z=1$
9. Find the condition that
$ax^2 + by^2 + cz^2 + 2ux-2vy + 2wz + d = 0$
may represent a cone.

LESSON NO. 2.3

CONE -II

2.3.1 : Right Circular Cone

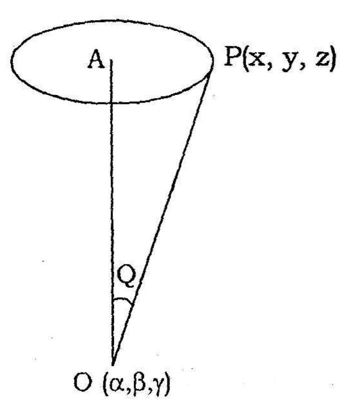
Right Circular cone is a surface generated by a straight line which passes through a fixed point and makes a constant angle with a fixed line through the fixed point.

The fixed point is called the vertex and the fixed line the axis of the cone. The fixed angle is called the semivertical angle of the cone.

To find the equation of the right circular cone with vertex at (α, β, γ) and the axis the line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

and having its semivertical angle θ.



Let O (α, β, γ) be the vertex of the cone and OA the axis.

Let P (x, y, z) be any point on the cone. Then by definition the angle

A O P = θ

Direction ratios of O P are

$(x - \alpha, y - \beta, z - \gamma)$

The direction radios of O A are l, m, n.

$$\therefore \cos\theta = \frac{l(x-\alpha)+m(y-\beta)+n\,(z-\gamma)}{\sqrt{l^2+m^2+n^2}+x\,\sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}}$$

Squaring both sides, we get the equation of the cone as

$$[l(x-\alpha) + m(y-\beta) + n(z-\gamma)]^2 + n\ (z-\gamma)]^2 = (l^2+m^2+n^2)\ [(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2] \cos^2\theta \quad \dots (1)$$

Cor.I if $\alpha = \beta = \gamma = O$ i.e. if the vertex is at the origin.

Then $(lx + my + nz)^2 = (l^2 + m^2 + n^2)\ (x^2 + y^2 + z^2) \cos^2\theta$ ... (2)

This is a homogeneous equation in (x, y, z) and therefore represents a cone with vertex at the origin.

Equation (2) can also be written as

$(x + my - nz)^2 = (l^2 + m^2 + n^2)\ (x^2 + y^2 + z^2)\ (1\text{-}\mathrm{Sin}^2\theta)$

or $(l^2 + m^2 + n^2)\ (x^2 + y^2 + z^2)\ (lx - my + nz)^2 = (l^2 - m^2 + n^2)\ (x^2+y^2+z^2)\ \mathrm{Sin}^2\theta$

or $(ny - mz)^2 + (lz - nx)^2 + (mx + ly)^2 = (l^2 + m^2 + n^2)\ (x^2 + y^2 + z^2)\ \mathrm{Sin}^2\theta$

(By Lagrange's identity)

Cor 2. If we take the z-axis as the axis of the cone.

Then obviously l=m=0, n=1 and we get the equation of the cone from (2) as

$z^2 = (x^2 + y^2 + z^2) \cos^2\theta$

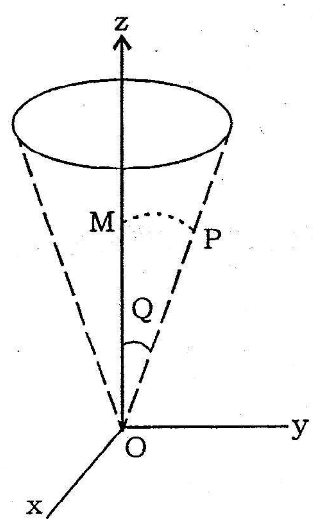
or $z^2 (1 - \cos^2\theta) = (x^2 + y^2) \cos^2\theta$

or $z^2 \tan^2\theta = x^2 + y^2$ ... (3)

This is the standard from of a right circular cone with vertex at the origin and the z-axis as its axis.

Form (3) of the right circular cone can be obtained independently also.

Let P (x_1, y_1, z_1) be any point on the cone. Draw P M perpendicular to the axis O Z of the cone.



Co-ordinates of M are $(0, 0, z_1)$

$\therefore$ $MP^2 = (x_1-0)^2 + (y_1-0)^2 + (z_1-z_1)\,2 = x_1{}^2 + y_1{}^2$

$$\tan\theta = \frac{MP}{OM} = \frac{\sqrt{x_1^2 + y_1^2}}{z_1} \Rightarrow \tan^2\theta = \frac{x_1^2 + y_1^2}{z_1^2}$$

or $z_1{}^2 \tan^2\theta = x_1{}^2 + y_1{}^2$

Hence the locus of (x_1, y_1, z_1) is

$z^2 \tan^2\theta = x^2 + y^2$

which is same as (3)

Theorem : Section of a circular Cone by any plane perpendicular, to the axis is a circle.

Proof : Without loss of generality, we can assume that the vertex of the cone is at the origin and its axis is the z axis. Therefore, the equation of the cone is

$z^2 \tan^2\theta = x^2 + y^2$

Any plane perpendicular to the axis of the cone has equation Z = C whese C is a constant.

Therefore the intersection of the cone (4) & the plane Z = C has equation

$c^2 \tan^2\theta = x^2 + y^2$, and $z = c$

or $c^2 + c^2 \tan^2\theta = z^2 + x^2 + y^2$ and $z = c$

or $(1 + \tan^2\theta)\, c^2 = x^2 + y^2 + z^2$ and $z = c$

or $\sec^2\theta\, c^2 = x^2 + y^2 + z^2$ and $z = c$

$x^2 + y^2 + z^2 = c^2 \sec^2\theta$ is sphere with centre at (0, 0, 0) and radius $c \sec\theta$. Its section by the plane $z = c$ will be a circle.

2.3.2 Condition for Circular Cone :

Condition that the general homogeneous equation of second degree in x, y, z should represent a circular cone.

Let the general homogeneous equation of second degree in x,y, z be

$\alpha x^2 + by^2 + cz^2 + 2f\,yz + 2g\,zx + 2h\,xy = 0.$ ... (6)

Suppose it represents a circular cone with vertex at the origin. Let the direction ratins of its axis be (l, m, n) and its semivertical angle θ be given by

$\cos^2\theta = [l^2, m^2, n^2]^{-1}$

Therefore, from (2) the equation of the cone is

$x^3 + y^2 + z^3 - (lx + my + nz)^2 = 0.$ ... (7)

This must be identical with (6).

comparing (6) & (7) we get.

$$\frac{1-l^2}{a} = \frac{1-m^2}{b} = \frac{1-n^2}{c} = \frac{-mn}{f} = \frac{-nl}{g} = \frac{-lm}{h} = \lambda \text{ (say)}$$

$\therefore\ 1 - l^2 = a\gamma,$, $-mn = f\gamma.$

$\therefore\ 1 - m^2 = b\gamma,$, $-nl = g\gamma.$

$\therefore\ 1 - n^2 = c\gamma,$, $-lm = h\gamma.$

Therefore, $(-nl)\ (-lm) = g\gamma \,.\, h\gamma.$

or $l^2mn = gh\gamma^2$

or $-l^2f\gamma = gh\gamma^2$ or $l^2 = -\dfrac{gh}{f}\gamma \Rightarrow fl^2 + ghr = 0$

Therefore, $1 + \dfrac{gh}{f}\gamma - a\gamma$

or $\dfrac{1}{\gamma} = \dfrac{af - gh}{f}$

Similarly $\dfrac{1}{\gamma} = \dfrac{bg - hf}{g} = \dfrac{ch - fg}{h}$

$$\therefore \frac{1}{\gamma} = \frac{af - gh}{f} = \frac{bg - hf}{g} = \frac{ch - fg}{h}$$

which is the required condition provided non of f, g, h is zero.

If f = 0, then mn = 0 so that m–0 or n=0.

which implies that h = 0 or g = 0

It g = 0 and h = 0 then l=0; and then the condition becomes.

$f^3 = (a-b)\ (a-c)$

if f, g and h all are zero then two of a, b, c must be equal.

Example 1

Find the equation of the circular cone which passes through the joint (1, 1, 2) and has its vertex at the origin and axis the line

$$\frac{x}{2} = \frac{y}{-4} = \frac{z}{3}$$

Solution

Direction ratios of the generator through (1, 1, 2) are (1, 1, 2). The semi vertical angle θ is given by

$$\cos\theta = \frac{1.2 + 1.\quad(-4) + 2.3}{\sqrt{1+1+4} \times \sqrt{4+16+9}} = \frac{4}{\sqrt{6 \times 29}}$$

∴ the equation of the cone is

$(lx + my + nz)^2 = (4 + 16 + 9)\ (x^2 + y^2 + z^2) \cos^2\theta$

or $(2x - 4y + 3z)^2 = 8\ (x^3 + y^2 + z^2)$

Example 2

Show that the cone generated by rotating the

$$\text{line } \frac{x}{l} = \frac{y}{m} = \frac{z}{n} \qquad \ldots \text{(i)}$$

$$\text{about the line } \frac{x}{a} = \frac{y}{b} = \frac{z}{c} \qquad \ldots \text{(ii)}$$

as axis is $(l^2 + m^2 + n^2)\ (ax + by + cz)^2 = (al + bm + cn)^2\ (x^2+y^2+z^2)$

Solution :

Let θ be the semivertical angle of the cone

$$\text{then } \cos\theta = \frac{al + bm + cn}{\sqrt{a^2 + b^2 + c^2} \times \sqrt{l^2 + m^2 + n^2}} \Rightarrow \frac{\cos\theta\,(ol + bm + cn)}{\sqrt{a^2 + b^2 + c^2}\ \sqrt{l^2 + m^2 + n^2}}$$

the equation of the cone is

$$(ax+by+cz)^2 = (a^2+b^2+c^2)\ (x^2+y^2+z^2)\ .\ \frac{(al + bm + cn)^2}{(a^2 + b^2 + c^2)\,(l^2 + m^2 + n^2)}$$

or $(l^2 + m^2 + n^2)\ (a+x+by+cz)^2 = (al+bm+cn)^2\ (x^2+y^2+z^3)$

Example 3

Show that the equation

$33x^2 + 13y^2 - 95z^2 \times 144yz - 96zx - 48xy = 0$

represents a right circular cone whose axis is the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{6}$

Find also its semi vertical angle θ.

Solution

Since the equation (i) is a homogeneous equation of degree two in x, y, z therefore, it represents a cone with vertex at the origin. Let the direction ratios of its axis be l, m, n and the semivertical angle θ be given by $\cos^2\theta = (l^2 + m^2 + n^2)^{-1}$.

The equation (i) must be identical with the equation

$$x^2 + y^2 + z^2 - (lx + my + nz) \qquad \ldots \text{(ii)}$$

Comparing the co-efficents of (i) and (ii) we get

$$\frac{1-l^2}{33} = \frac{1-m^2}{13} = \frac{1-n^2}{-95} = \frac{mn}{72} = \frac{nl}{48} = \frac{lm}{24} \qquad \ldots \text{(iii)}$$

From the first three members of (iii)

$$\frac{l}{2} = \frac{m}{3} = \frac{n}{6} = k \text{ (Say)}$$

From the last three members of (iii)

$$\frac{1-4k^2}{33} = \frac{1-9k^2}{13} \text{ or } k = 2/7$$

$$\therefore \qquad l = \frac{4}{7}, m = \frac{6}{7}, n = \frac{12}{7}$$

∴ the equation of the axis is

$$\frac{x}{4/7} = \frac{x}{6/7} = \frac{z}{12/7} \text{ or } \frac{x}{2} = \frac{y}{3} = \frac{z}{6}$$

The semivertical angle θ is given by

$$\cos^2\theta = \left(\frac{16}{49} + \frac{36}{49} + \frac{144}{49}\right)^{-1} = \frac{1}{4} \text{ or } \cos\theta = \frac{1}{2} \text{ or } \cos\theta = 60°$$

Example 4

Find the equation of the cone formed by rotating the line 2x + 3y = 6, z = 0, about y-axis.

Solution

The equation of the given line can be written as

$$\frac{x}{3}=\frac{y+2}{-2}=\frac{3}{0}$$

is meets the y-axis at (0, 2, 0)

The angle θ between this line and the y-axis is given by

$$\cos\theta=\frac{-2.1}{1\times\sqrt{9+4}}=-\frac{2}{\sqrt{13}}$$

Equation of the cone with vertex at the origin is

$$y\ 2-1\ .\ (x2+y2+z2)\ .\ \frac{4}{13}.$$

or $\quad 4x^2 - 9y^2 - 4z^2 = 0$

∴ Equation of the cone with vertex at (0, 2, 0) is

$$4x^2 - 9\ (y-2)^2 + 4z^2 = 0$$

or $\quad 4x^2 - 9y^2 + 4z^2 + 36 = 0$

2.3.3 Enveloping Cone

Definition :

The cone formed by the tangent lines to a surface drawn from a given point is called the enveloping cone of the surface and the given point called the vertex of the cone.

To find the equation of a cone whose vertex is at the point (α, β, γ) and whose generators touch the conicoid.

$$F(x, y, z) = ax^2 + by^2 + cz^2 + 2yfz + 2gzx + 2hxy + 2ux + 2vy +2wa +d = 0 \quad \ldots (9)$$

Let $f(x, y, z) = ax^2 + by^2 + cz^2 + 2yfz + 2gzx + 2hxy$

∴ (9) can be written as

$$f\ (x, y, z) = f(x, y, z) + 2ux + 2vy + 2wz + d = 0 \quad \ldots (10)$$

Equation of any line through (α, β, γ) and having direction ratios l, m, n is

$$\frac{x-a}{l}=\frac{y-a}{m}=\frac{z-\gamma}{n} \quad \ldots (11)$$

Any point on this line whose distance from (α, β, γ) is r has co-ordinates

$$(\alpha + lr, \beta + mr, \gamma + nr)$$

This point will lie on the conicoid F(x, y, z) = 0 if

$$F(\alpha + lr, \beta + mr, \gamma + nr) = 0$$

i.e. if $a(\alpha + lr)^2 +b\ (\beta + mr)^2 + c(\gamma +nr)^2+2f\ (\beta + mr)\ (\gamma+nr)+2g\ (\gamma+nr)\ (\alpha+lr)$

$$+2h\ (\alpha +lr)\ (\beta + mr) +2u\ (\alpha + lr) +2v\ (\beta+mr) +2w\ (\gamma+nr)+d = 0.$$

or if

$$F(\alpha, \beta, \gamma) + r\left(1\frac{\partial F}{\partial \beta} + m\frac{\partial F}{\partial \beta} + n\frac{\partial F}{\partial \gamma}\right) + r^2\ (1, m, n) = 0 \qquad \dots (12)$$

Hence $\frac{\partial F}{\partial \alpha} = \frac{\partial F(\alpha, \beta, \gamma)}{\partial \alpha}, \frac{\partial F}{\partial \beta} - \frac{\partial F\ (\alpha, \beta, \gamma)}{\partial \beta}, \frac{\partial F}{\partial \gamma} = \frac{\partial F(\alpha, \beta, \gamma)}{\partial \gamma}$

Equation (12) is quadratic in r and has therefore two values of r corresponding to each of which we get a point in which the line (11) meets the conicoid (9).

The line will touch the conicoid if the two values of r are equal. This rquires

$$4F(\alpha, \beta, \gamma)\ f\ (l, m, n) = \left(1\frac{\partial F}{\partial \alpha} = m\frac{\partial F}{\partial \beta} = n\ \frac{\partial F}{\partial \gamma}\right)^3 \qquad \dots (13)$$

Therefore, the locus of the tangents drawn from the given point (α, β, γ) is obtained by eliminating l, m, n between (11) & (13).

Eliminating l, m, n from (11) & (13). The equation of the enveloping cone is

$$4F(\alpha, \beta, \gamma)\ F(x, y, z) = \left\{(x - \alpha)\ \frac{\partial F}{\partial \alpha} + (y - \beta)\ \frac{\partial F}{\partial \beta} + (z - \gamma)\ \frac{\partial F}{\partial \gamma}\right\}^2$$

But 4 (α + x – α, β + y – β, γ + z – γ)

$$= f(x–\alpha), y–\beta, z–\gamma) + (x–\alpha)\ \frac{\partial F}{\partial \alpha} + (y - \beta)\frac{\partial F}{\partial \beta} + (z - \gamma)\frac{\partial F}{\partial \gamma} + (\alpha, \beta, \gamma)$$

the above equation raduces to

$$4F\ (\alpha, \beta, \gamma)\ F(x, y, z) = \left[(x - \alpha)\ \frac{\partial F}{\partial \alpha} + (y - \beta)\ \frac{\partial F}{\partial \beta} + (z - \gamma)\frac{\partial F}{\partial \gamma} + 2F(\alpha, \beta, \gamma)\right]^2$$

$$= \left[x\frac{\partial F}{\partial \alpha} + y\frac{\partial F}{\partial \beta} + z\frac{\partial F}{\partial \gamma} + t\frac{\partial F}{\partial t}\right]^2$$

where t is an auriliary variable used to make the equation F(x, y, z) homogeneous in x, y, z and is replaced by unity after differentiation.

If we write

$S=ax^2 + by^2 + cz^2 +2yfz + 2gzx + 2hxy + 2ux + 2vy + 2wz+d$

$S_1 = a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta +2u\alpha + 2v\beta +2w\gamma +d$

$T=ax\ \alpha + by\beta + cz\gamma + f\ (y\gamma +z\beta) +g\ (z\alpha +x\gamma)+h\ (x\beta+y\alpha) + u(x+\alpha)+v\ (y+\beta)+w(z+\gamma)+d.$

Then the equation (14) of the cone can be written as

$SS_1 = T^2$ ... (15)

(15) is the required equation of the enveloping cone.

Example 5 :

Show that the plane z=0 cuts the enveloping cone of the sphere $x^2+y^2+z^2 = 11$ which has its vertex at (2, 4, 1) in a rectangular hypherbola.

Solution :

Here $S = x^2 + y^2 + z^2 - 11$, $S_1 = 4 + 16 + 1 - 11 = 10$

$T = 2x + 3y + z-11$

$\therefore$ the equation of the enveloping cone is

$10\,(x^2 + y^2 + z^2 - 11) - (2x + 4y + z - 11)^2$

It will meet the plane z = 0, whose

$10(x^2 + y^2 - 11) - (2x + 4y - 11)^2 = 0$

or $10x^2 + 10y^2 - 110 - 4x^2 - 16y^2 -$ `$6xy + 4yx + 88y - 121 = 0$

or $6x^2 - 6y^2 - 16xy + 44x + 88y - 231 = 0 \Rightarrow 6\,(x^2 - y^2) - 16xy + 44x + 88y - 231 = 0$

This curve is rectangular hipherbola because the sum of the coeffcients of x^2 and y^2 is zero.

Example 6

Find the locus of a luminous point if the ellipsoid

$$\frac{x^2}{\alpha^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

casts a circular shadow on the planze z = 0.

Solution :

Let (α, β, γ) be the luminous point. The shadow cast by the ellipsoid on the z=0 plane will be a circle if the section of the enveloping cone of (α, β, γ) by the plane z=0 is a circle. P Equation of the enveloping cone is.

$$\left(\frac{x^2}{\alpha^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)\left(\frac{\alpha^2}{\alpha^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1\right) = \left(\frac{\alpha x}{\alpha^2} + \frac{y\beta}{b^2} + \frac{z\gamma}{c^2} - 1\right)^2$$

This meets the plane z = 0 where

$$\left(\frac{x^2}{\alpha^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{\alpha^2}{\alpha^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1\right) = \left(\frac{\alpha x}{\alpha^2} + \frac{y\beta}{b^2} - 1\right)^2$$

It will be a circle if

co-efficient of x^2 = co-efficient of y^2

co-efficient of xy = 0.

i.e. if $\frac{1}{\alpha^2}\left(\frac{\alpha^2}{\alpha^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1\right) - \frac{\alpha^2}{\alpha^4} = \frac{1}{b^2}\left[\frac{\alpha^2}{\alpha^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1\right] - \frac{\beta^2}{b^4}$

and $\alpha\beta = 0$

i.e. if $\frac{1}{\alpha^2}\left[\frac{\beta^2}{\alpha^2}+\frac{\gamma^2}{c^2}-1\right]=\frac{1}{b^2}\left[\frac{\alpha^2}{\alpha^2}+\frac{\gamma^2}{c^2}-1\right]$

and $\alpha\beta = 0$

when $\alpha = 0$, then $\frac{1}{\alpha^2}\left(\frac{\beta^2}{\alpha^2}+\frac{\gamma^2}{c^2}-1\right)=\frac{1}{b^2}\left(\frac{\gamma^2}{c^2}-1\right)$

when $\beta = 0$, then $\frac{1}{\alpha^2}\left[\frac{\gamma^2}{c^2}-1\right]=\frac{1}{b^2}\left(\frac{\alpha^2}{\alpha^2}+\frac{\gamma^2}{c^2}-1\right)$

Hence the locus of (α, β, γ) is

$$x-0, \frac{1}{\alpha^2}\left[\frac{y^2}{c^2}-\frac{z^2}{c^2}-1\right]=\frac{1}{b^2}\left(\frac{z^2}{c^2}-1\right)$$

or $$y=0, \frac{1}{\alpha^2}\left[\frac{z^2}{c^2}-1\right]=\frac{1}{b^2}\left(\frac{x^2}{a^2}+\frac{z^2}{c^2}-1\right)$$

i.e. $x = 0,\ \frac{z^2}{c^2}-\frac{y^2}{\alpha^2-b^2}-1, \Rightarrow\ z^2(a^2-b^2) - y^2c^2 = a^2c^2 - b^2c^2$

or $$y=0, \frac{x^2}{\alpha^2-b^2}+\frac{z^2}{c^2}=1$$

2.3.4 Tangent Line and Tangent Plane :

Let $P(\alpha, \beta, \gamma)$ be any point

Let $f(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

be the equation of the cone with vertex at the origin.

Any line through $P(\alpha, \beta, \gamma)$ is

$$\frac{x-\alpha}{1}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r\ (\text{say})$$

Any point on this line has Co-ordinates

$(\alpha + lr, \beta+mr, \gamma+nr)$

This will lie on the cone if

$a(\alpha + lr)^2, b(\beta+mr)^2, c(\gamma+nr)^2 + 2f(\beta+mr)(\gamma+nr)$
$+2g(\gamma+nr)(\alpha+lr) + 2h(\alpha+lr)(\beta+mr) = 0$

or $r^2 f(l, m, n) + 2r\,[1(a\alpha+h\beta+g\gamma)+m(h\gamma+b\beta+f\gamma)+n\,(g\alpha+f\beta+c\gamma)]+f(\alpha, \beta,\gamma)=0$ ... (16)

This is a quadratic in r and gives two values of corresponding to each of which we get a point of intersection of the cone and the line. If the point P (α, β, γ) lies on the cone then $f(\alpha, \beta, \gamma) = 0$

Which from (16) implies that one root of the equation (16) is zero.

If both points of intersection are coincident at P (α, β, γ) then the other not of (16) is also zero.

For this

$l(a\alpha+h\beta+g\gamma) + m\,(h\alpha+b\beta+f\gamma)+n(g\alpha + f\beta+c\gamma) = 0$

and in this case the line

$\dfrac{x-\alpha}{1} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$ is a tengent to the cone.

at P (α, β, γ).

The locus of all such lines is a plane at P known as the tangent plane at P to the cone and its equation is obtained by eliminating, l, m, n between (17) and the equation of the line.

Thus we get the equation of the tangent plane at P to the cone as

$(x-\alpha)\,(a\alpha+h\beta+g\gamma) + (y-\beta)\,(h\alpha + b\beta + f\gamma) + (z-\gamma)\,(g\alpha + f\beta + c\gamma) = 0$

or $x(a\alpha + h\beta + g\gamma) + y\,(h\alpha + b\beta + f\gamma) + z\,(g\alpha + f\beta + c\gamma) + f(\alpha, \beta, \gamma) = 0$

or $x(a\alpha + h\beta + g\gamma)\; y\,(h\alpha + b\beta + f\gamma) + z\,(g\alpha + f\beta + c\gamma) = 0$

or $$\alpha\frac{\partial f}{\partial x} + \beta\frac{\partial f}{\partial y} + \gamma\frac{\partial f}{\partial z} = 0 \qquad \text{... (18)}$$

$\{\therefore f(\alpha, \beta, \gamma) = 0\}$

(18) is the required equation of the tangent plane.

It may be observed that plane (18) passes through the vertex of the cone which is of the origin O. Therefore, the generator O P lies in the plane (18). Infact (18) is a tangent plane at every point of O.P. For this reason O P is called the generator of contact.

There is a Unique tangent plane at every point of the surface of the cone except the vertex, through which all tangent planes pass. This point is said to be a singular point of the surface.

2.3.5 Condition of Tangency

To find the condition that the plane

$lx + my + nz = 0$ ... (19)

my touch the cone

$ax^2 + by^2 + cz^2 + 2\,f\,yz + 2gzx + 2hxy = 0$ ... (20)

If the plane touches the cone (20) say at the point (α, β, γ) then plane (19) must be at Tangent plane to (20) at (α, β, γ). Now equation of the tangent plane to (20) at (α, β, γ) is

$$x(a\alpha+h\beta+g\gamma)+y(h\alpha+b\beta+f\gamma)+z(g\alpha+f\beta+c\gamma)=0 \quad \dots (21)$$

This equation must be identical to (19)

$$\therefore \frac{a\alpha+h\beta+g\gamma}{l}=\frac{h\alpha+b\beta+f\gamma}{m}=\frac{g\alpha+f\beta+c\gamma}{n}=k \text{ (say)}$$

$$\therefore a\alpha+h\beta+g\gamma-kl=0 \quad \dots (22)$$

$$h\alpha+b\alpha+f\gamma-km=0 \quad \dots (23)$$

$$g\alpha+f\beta+c\gamma-kn=0 \quad \dots (24)$$

(α, β, γ) lies on (19)

$$\therefore l\alpha+m\beta+n\gamma=0. \quad \dots (25)$$

Eliminating α, β, γ, k between (22), (23), (24) & (25) we get

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & o \end{vmatrix}=0$$

or $\quad Al^2+Bm^2+cn+2fmn+2Gnl+2Hlm=0 \quad \dots (26)$

where A, B, C, F, G, H are the co-factors of a, b, c, f, g, h respectively in the diterminant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=abc-af^2-h^2c+hgf+ghf=g^2b$$

$$abc-af^2-h^2c-g^2b+2fgh$$

Example 8 :

Find the locus of chords of the cone

$$f(x, y, z)=ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad \dots (i)$$

which are bisected at (α, β, γ).

Solution

Equation of any line through (α, β, γ) is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

any point on it has co-ordinates $(\alpha+lr, \beta+mr, \gamma+mr)$ $\quad \dots$ (ii)

This point will be on the cone (i) if

$r^2 f(l, m, n) + 2r[l (a\alpha + h\beta + g\gamma) + m(h\alpha + b\beta + f\gamma] + n(g\alpha + f\beta + c\gamma\_]$

$+f (\alpha, \beta, \gamma) = 0$ ... (iii)

Since (α, β, γ) is mid point $\therefore$ the two values of r given by (iii) must equal in magnitude and opposite sign. Therefore, their sum must be zero.

Hence $l(a\alpha+h\beta+g\gamma) + m (h\alpha + b\beta +g\gamma) + n (g\alpha + f\beta +c\gamma) = 0$ ... (iv)

Eliminating (l, m, n) between (ii) & (iv) we get the locus of all such chords as

$(x-\alpha) (a\alpha + h\beta +g\gamma) + (y-\beta) (h\alpha + b\beta +f\gamma) + (z-\gamma) (g\alpha+f\beta+c\gamma) = 0$

Example 9

Find the condition that the plane

$$lx + my + nz = 0 \quad \text{... (i)}$$

may touch the cone

$$fyz + gzx + hxy = 0 \quad \text{... (ii)}$$

Solution : Let the plane (i) touch the cone (ii) at (α, β, γ) Equation of the tangent plane at (α, β, γ) to the cone (ii) is

$$f(y\gamma + z\beta) + g(z\alpha +x\gamma) +h (x\beta +y\alpha) = 0$$

or $(g\gamma+h\theta) x + (h\alpha + f\gamma) y+ (f\beta+g\alpha) z = 0$ ... (iii)

Now equation (i) & (iii) represent the same plane

$$\therefore \frac{g\gamma - h\beta}{l} = \frac{h\alpha - f\gamma}{m} = \frac{f\beta - g\alpha}{n} = k \text{ (say)}$$

$\therefore$ $g\gamma+h\beta-lk = 0$ ... (iv)

$ha+f\gamma-mk = 0$ ... (v)

$f\beta=g\alpha-nk = 0$ ... (vi)

Solving these equations for α, β, γ we get.

$$\alpha = \frac{k(gm + hn - fl)}{2gh}, \beta = \frac{k(hn + fl - gm)}{2fh}, \gamma = \frac{k(fl + gm - hn)}{2fg}$$

But (α, β, γ) lies on (i)

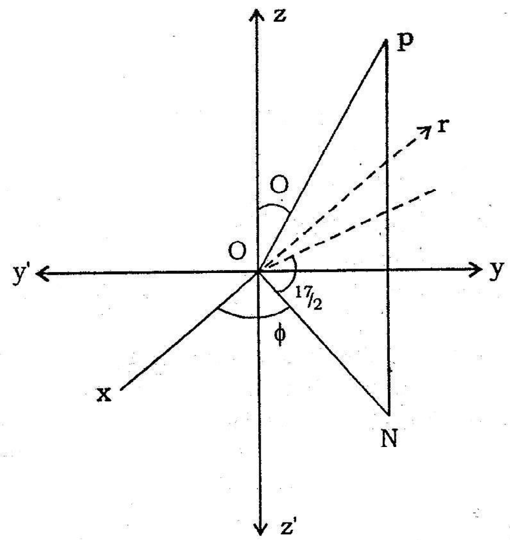
or $f^2l^2 + g^2+m^2+h^2n^2-ghmn-2hfnl-2fglm=0$

which is the required condition.

Art : Equation of the Surface of Revolution obtained by rotating the Curve f(y, z)=0 about the z-axis.

Cylindrical Co-ordinates.

Proof : Let XOX', YOY', ZOZ' be the rectangular axis. Suppose P is any point. From P draw PN perpendicular to the XOY plane. The position of the point P can lie determined if the quantities ON, i.e. distance of N from the Z-axis, angle XON i.e. the angle which ON makes with the X-axis and PN the length of the perpendicular from P on the XOY plane are known.



These three quantities are denoted by

ON = u, $\angle$ XON=ϕ, PN = z

(u, ϕ, z) are called the cylinderical co-ordinates of P.

If the certesian co-ordinates of P are (x, y, z). Then the cardesian co-ordinates of N are (x, y, o). In the plane XOY cardesian co-ordinates if N are (x, y) and polar coordinates of N are (u, ϕ).

$\therefore$ x=u cos ϕ, y=u sin ϕ or $x^2+y^2=u^2$,

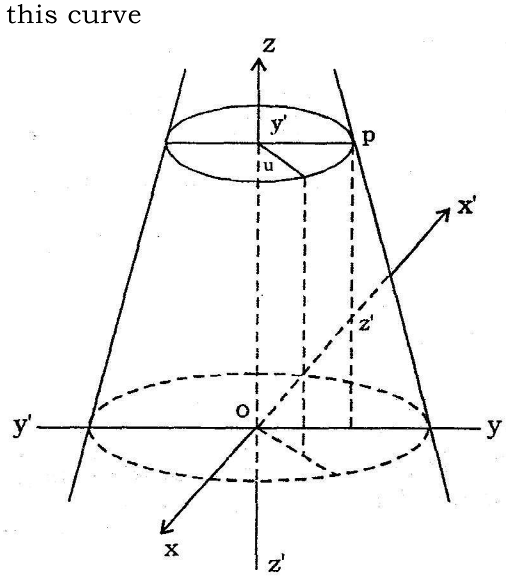
and tanϕ = y/x ... (27)

2.3.6 Surface of Revolution

Let the equation of any curve in Y O Z plane be f(y, z) = 0.

Let P (0, y_1, z_1) be any point on this curve

$\therefore$ f(y, 1, z) = 0 ... (28)



If we rotate the curve f(y, z) = 0 about the z-axis we will get a surface of revolution. As P moves round the surface, its Z-co-ordinates i.e. Z_1 remains unaltered and u its distance from the z-axis is always y_1.

Therefore the cylindrical co-ordinates of P satisfy the equation f(y, z) = because of (28) therefore, f(u, z) = 0. But P is a point on the surface, and therefore, the cylindrical equation to the surface is

f(u, z) = 0

Hence the certerian equation of the surface of revolution is

$$f(\sqrt{x^2+y^2}, z) = 0 \quad [\therefore \text{ of } (27)]$$

Since the distance of the point (x, y, z) from the y-axis is $\sqrt{z^2+x^2}$, Therefore, it similarly follows that the surface formed by rotating the curve f(y, z) = 0, x=0, about the y-axis will be $f(y, \sqrt{z^2+x^2}) = 0$ and with axis as x-axis it will be $f(\sqrt{y^2+z^2}, x) = 0$.

2.3.7 Self Check Exercise

1. Find the equation of the right circular cone with vertex at the origin and having the lines.

$$\frac{x}{-1} = \frac{y}{2} = \frac{z}{2}, \frac{x}{2} = -\frac{y}{1} = \frac{z}{2}, \frac{-x}{2} = -\frac{y}{3} = \frac{z}{6}$$

as generators.

2. If θ be the semivertical angle of the right circular cone with which passes through the lines OY, OZ and x = y = z show that

$$\theta = \cos^{-1} (9 - 4\sqrt{3})^{\frac{-1}{2}}$$

3. If ax2+by2+cz2+2fyz+2gzx+2hxy=0 represents a right circular cone with semivertical angle θ show that

$$\frac{gh}{f} - a = \frac{hf}{g} - b = \frac{fg}{h} - c = \left(\frac{gh}{f} + \frac{hf}{g} + \frac{fg}{h}\right)\cos^2\theta$$

4. Find the equation of the cone with vertex at (α, β, γ) and whose generators touch the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

5. Prove that the plane z = a meets any enveloping cone of the sphere $x^2+y^2+z^2=a^2$, in a conic which has a focus at the point (0, 0, a).
6. Find the plane which touches the cone $x^2+2y^2+4z^2+-6yz+4zx-5xy=0$ along the generators having direction ratios, (1, 1, 1).
7. Find the locus of the mid-points of chords of the cone $ax^2+by^2+2fyz+2gzx+2hxy$ which one parallel to the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

8. Find the condition that the plane lx + my + nz = 0 may touch the cone $ax^2+by^2+cz^2 = 0$
9. Find the equation of surface of the revolution obtained by rotating the parabola $y^2=4ax$, z=0 about its axis.
10. Show that the surfaces generated by rotating the ellipse

$\frac{x^2}{\alpha^2}=\frac{y^2}{b^2}=1,$ z = 0 about its axis are given by

$$\frac{x^2}{\alpha^2}+\frac{y^2+z^2}{b^2}=1 \text{ and } \frac{x^2+z^2}{\alpha^2}+\frac{y^2}{b^2}=1$$

11. Find the equation of the surface generated by revolving the circle $x^2+y^2+2ax+b^2=0$, z=0 about the y-axis.