Department of Distance Education

Punjabi University, Patiala

Class : B.A. I (Math) **Semester : 2**
Paper : 5 (Partial Differential Equations) **Unit : I**
Medium : English

Lesson No.

Department website : www.pbidde.org

LESSON NO. 1.1

PARTIAL DIFFERENTIAL EQUATIONS–I

1.1.1 Introduction and Basic Concepts

Partial differential equations occur in various physical and engineering problems when the functions involved depend on two or more independent variables. In this lesson, we shall be concerned with a discussion of some important partial differential equations and we shall restrict our study mostly to those equations involving two independent variables only.

An equation involving one or more partial derivatives of an unknown function of two or more independent variables is called a partial differential equation. The order of the highest derivative is called the order of the equation (as in ordinary differential equation).

Differential operator is $\frac{df}{dx}$ (Derivative of f. w.r.t. x)

Partial differential equations is $\frac{\alpha f}{a}$ (Partial derivative of f w.r.t. x)

We consider partial differential equation of order one and two only. The most general second order linear partial differential equation in two independent variables is

$$a\frac{\partial^2 z}{\partial x^2} + b\frac{\partial^2 z}{\partial x \partial y} + c\frac{\partial^2 z}{\partial y^2} + d\frac{\partial z}{\partial x} + e\frac{\partial^2}{\partial y} + fz = g \qquad (1)$$

where a, b, c, d, e, f and g are functions of x and y. If g = 0, then (1) is said to be homogeneous otherwise it is non-homogeneous. Some important linear partial differential equations of second order are

Homogeneous $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ (two dimensional Laplace equation) (2)

Non-homogeneous $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = f(x, y)$ (two dimensional Poisson equation) (3)

$\frac{\partial z}{\partial t} + c^2 \frac{\partial^2 z}{\partial x^2} = 0$ (one dimensional heat equation) (4)

$\frac{\partial^2 z}{\partial t^2} + c^2 \frac{\partial^2 z}{\partial x^2} = 0$ (one dimensional wave equation) (5)

where c is a constant, t is the time variable and (x, y) are cartesian coordinates.

1.1.2 Formation of Partial Differential Equations

Partial differential equations may be formed by the elimination of arbitrary constants from a given relation between the variables. Let z be a function of two variables x and y, let

$$f(x, y, z, a, b) = 0 \qquad (1)$$

where a and b are two arbitrary constants. Differentiating (1) partially w.r.t. 'x' and y and replacing $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ with p and q respectively, we have

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = 0 \qquad (2)$$

and $$\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial y} = 0 \qquad (3)$$

On eliminating 'a' and 'b' from (2) and (3) we get a partial differential equation of order one such as

$$g(x, y, z, p, q) = 0 \qquad (4)$$

where p and q are given by (2) and (3).

Now suppose the two arbitrary functions u and v are connected by the relation

$F(u, v) = 0$ (5)

where $u = f(x, y, z)$ (6)

$v = g(x, y, z)$

A partial differential equation may now be formed as mentioned below. Taking z as dependent variable, we get on differenting partially w.r.t. x and y, we get

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0 \quad (7)$$

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0 \quad (8)$$

On eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from (7) and (8), we get

$$\begin{vmatrix} \frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z} & \frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z} & \frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z} \end{vmatrix}=0$$

or $$\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y}+q\frac{\partial y}{\partial z}\right)-\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0$$

or $$\left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}\right)+p\left(\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}\right)+q\left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}\right)=0 \quad (9)$$

If we write

$$\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}=\lambda P$$

$$\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}=\lambda Q$$

$$\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}=\lambda R$$

where P, Q and R are functions of x, y and z then equation (9) becomes $Pp + Qq = R$

Example 1 : Derive a partial differential equation by eliminating a and b from $z = ax^2 y^3 + b$

Sol. : On differenting w.r.t.x and y and replacing $\frac{\partial z}{\partial x}$ by p and $\frac{\partial z}{\partial y}$ by q, we get

$$p = 2axy^3$$
$$q = 3ax^2y^2$$

$$\Rightarrow \quad \frac{p}{q} = \frac{2y}{3x} \text{ or } 3px - 2py = 0$$

is the desired partial differential equation.

Example 2 : Derive a partial differential equation by eliminating the arbitrary function f from the relation $f(x^2 + y^2, x^2 - z^2) = 0$.

Sol. : Define $u = x^2 + y^2$ (i)

and $v = x^2 - z^2$

We get the given relation as

$f(u, v) = 0$ (ii)

Differentiating (ii) partially w.r.t. 'x' and 'y', we get

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0, \text{ and}$$

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial u}(2x + p(o)) + \frac{\partial f}{\partial v}(2x + p(-2z)) = 0, \text{ and}$$

$$\frac{\partial f}{\partial u}(2y + q(o)) + \frac{\partial f}{\partial v}(o + q(-2z)) = 0$$

$$\Rightarrow \quad 2x\frac{\partial f}{\partial u} + (2x - 2pz)\frac{\partial f}{\partial v} = 0,$$

$$\text{and} \quad 2y\frac{\partial f}{\partial u} - 2qz\frac{\partial f}{\partial v} = 0$$

On eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$, we get $\begin{vmatrix} 2x & 2x - 2pz \\ 2y & -2qz \end{vmatrix} = 0$

$$\Rightarrow \quad -2\,qz \,.\, 2x - (2x - 2pz)\,(2y) = 0$$

or $4xqz + 4(xy - pyz) = 0$

or $xqz - pyz = -xy$

or $xq - py = \frac{-xy}{z}$

$\therefore$ $py - qx = \frac{xy}{z}$ is the required partial differential equation.

Example 3 : Eliminate, a, b, c from $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Sol. : On differentiating the given relation partially w.r.t. x and y, we have

$$\frac{2x}{a^2} + \frac{2zp}{c^2} = 0 \quad \text{(i)}$$

$$\frac{2y}{b^2} + \frac{2zq}{c^2} = 0 \quad \text{(ii)}$$

when $\frac{2}{c^2}[pq + 25] = 0$

$\Rightarrow$ $[pq + 25] = 0 \times \frac{c^2}{2}$

$\Rightarrow$ $pq + 25 = 0$

differentiate (i) partially w.r.t. y

$\frac{2}{c^2}[pq + zs] = 0$

$pq + z0 = 0$

where $\lambda = \frac{\partial^2 z}{\partial x \partial y}$

$pq + zx = 0$

If we differentiate (ii) partially OR

We get the same result as x, above $+\frac{2}{c^2}[pq + zs] = 0$

i.e. $pq + zx = 0$ is the required differential equation. In case, we differentiate (i) partially w.r.t. x again, we get

$$\frac{2}{a^2}+\frac{2}{c^2}\left[p^2+zr\right]=0$$

and $$\frac{2x}{a^2}+\frac{2zp}{c^2}=0$$

$$\frac{2}{a^2}\left[p^2x+zrx-zp\right]=0$$

$\Rightarrow$ $p^2x + xzr = pz$, which is another partial differential equation.

Similarly, we can get

$q^2y + yzt = qz$

Example 4 : Eliminate ϕ from $z = (x - y)\,\phi\,(x + y)$

Solution : On differentiating given relation partially w.r.t. x and y, we get

$p = \varphi\,(x + y) + (x - y)\,\varphi'\,(x + y)$

$q = -\,\varphi\,(x + y) + (x - y)\,\varphi'\,(x + y)$

which on subtracting

$p - q = 2\varphi(x + y)$

or $$p-q=\frac{2z}{(x-y)}$$

is the required partial differential equation.

1.1.3 Partial Differential Equations of Order One

The general linear partial differential equation of the first order in x, y, z is of the form

$$P\,(x,y,z)\,\frac{\partial z}{\partial x}+Q\,(x,y,z)\,\frac{\partial z}{\partial y}=R\,(x,y,z)$$

which may be written in the symbolic form as

$Pp + Qq = R$, where $p=\frac{\partial z}{\partial x},\ q=\frac{\partial z}{\partial y}$

For example, if we want to solve $p=\frac{\partial z}{\partial x}=x+y$

then its solution is, $z=\frac{x^2}{2}+xy+\varphi(y)$

where φ (y) is an arbitrary function of y. Please note that in the above differential equation p = x + y, P = 1, Q = 0 and R = x + y

Similarly, if we want to solve the equation :

$$y\frac{\partial p}{\partial y} + p = 2x \quad \text{or} \quad \frac{\partial p}{\partial y} + \frac{p}{y} = \frac{2x}{y}$$

Its solution (like ordinary differential equation) is given by :

py = 2xy + φ (x)

or $$\frac{\partial z}{\partial x} y = 2xy + \varphi(x)$$

or $$\frac{\partial z}{\partial x} = 2x + \frac{\varphi(x)}{y},$$

which on integration w.r.t. x, gives

$$z = x^2 + \frac{g(x)}{y} + \phi_1(y)$$, (where $g(x) = \int \phi(x)\,dx$ and $\phi_1(y)$ is an arbitrary function of y.)

∴ $$z = x^2 + \frac{g(x)}{y} + \phi_1(y)$$ is the required solution.

1.1.4 Lagrange's Method

An equation of the form Pp + Qq = R (i)

Can be solved with the help of Lagrange's method, which is described below :

Let u (x, y, z) = a

(be a solution of Pp + Qq = R)

Now differentiate (ii) w.r.t x, we get (ii)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x} = 0 \quad \text{or} \quad \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} = 0$$

Similarly, differentiating w.r.t 'y', we get : $\frac{\partial u}{\partial y} + q\,\frac{\partial u}{\partial z} = 0$ (iii)

Assuming $\frac{\partial u}{\partial z} \neq 0$, we obtain

$$\frac{\partial z}{\partial x} = p = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial z}} \quad \text{and} \quad \frac{\alpha z}{\alpha y} \; q = -\frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial z}}$$

Substituting the value of p and q from above in Pp + Qq = R, we get

$$-P\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial z}} - Q\frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial z}} = R$$

$$\Rightarrow \qquad P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0$$

Hence any solution of Pp + Qq = R is also a solution of

$$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0 \qquad \text{(iv)}$$

Now let u (x, y, z) = a be any solution of (iv). Then, we have

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial z} \quad \frac{\partial z}{\partial x}$$

and $$\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial z} \quad \frac{\partial z}{\partial y}$$

on using above values of $\frac{\alpha u}{\alpha x}$ and $\frac{\alpha u}{\alpha y}$ in

$$P\frac{\partial u}{\partial y} + Q\frac{\partial u}{\partial y} + R\frac{\partial z}{\partial z} = 0$$

$$\rightarrow \qquad -P\frac{\partial u}{\partial z}.\frac{\partial z}{\partial x} - Q\frac{\partial u}{\partial z}.\frac{\partial z}{\partial y} + R\frac{\partial u}{\partial z} = 0$$

or equivalent to $-P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} + R$, since $\frac{\partial u}{\partial z} \neq 0$

So from above, we conclude that any solution of

$$P\frac{\partial u}{\partial x}+Q\frac{\partial u}{\partial y}+R\frac{\partial u}{\partial z}=0$$

is also a solution of Pp + Qq = R

We shall now consider the subsidiary equations viz.

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \qquad \text{(vi)}$$

Taking $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\lambda$, we obtain

$$dx=\lambda P,\ dy=\lambda Q,\ dz=\lambda R \qquad \text{(vii)}$$

Multiplying (vii) by $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}$ respectively and adding, we get

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=\lambda P\frac{\partial u}{\partial x}+\lambda Q\frac{\partial u}{\partial y}+\lambda R\frac{\partial u}{\partial z}$$

$$\rightarrow \quad du=\lambda\left(P\frac{\partial u}{\partial x}+Q\frac{\partial u}{\partial y}+R\frac{\partial u}{\partial z}\right)\ \left(\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=du\right)$$

But du = 0 since u (x, y, z) = a

Hence, we have $P\frac{\partial u}{\partial x}+Q\frac{\partial u}{\partial y}+R\frac{\partial u}{\partial z}=0$

Hence any solution of $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ (A)

is also a solution of $P\frac{\partial u}{\partial x}+Q\frac{\partial u}{\partial y}+R\frac{\partial u}{\partial z}=0$. In other words any solution of (A) also satisfies Pp + Qq = R.

Now, we consider the relation $\phi(u_1, u_2)=0$ (B)

Where u_1 (x, y, z) = C_1, u_2 (x, y, z) = C_2 are the solutions of

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{Dz}{R}$$

From (B), we get

$$d\phi = \frac{\partial\phi}{\partial u_1} du_1 + \frac{\partial\phi}{\partial u_2} .du_2 = 0$$

where $du_1 = \frac{\partial u_1}{\partial x} dx + \frac{\partial u_1}{\partial y} dy + \frac{\partial u_1}{\partial z} dz$

and $du_2 = \frac{\partial u_2}{\partial x} dx + \frac{\partial u_2}{\partial y} dy + \frac{\partial u_2}{\partial z} dz$

Hence, we get

$$d\phi = \frac{\partial\phi}{\partial u_1}\left(\frac{\partial u_1}{\partial x} dx + \frac{\partial u_1}{\partial y} dy + \frac{\partial u_1}{\partial z} . dz\right) + \frac{\partial\phi}{\partial u_2}\left(\frac{\partial u_2}{\partial x} dx + \frac{\partial u_2}{\partial y} dy + \frac{\partial u_2}{\partial z} dz\right)$$

Using $dx = \lambda P$, $dy = \lambda Q$, $dz = \lambda R$, we get

$$d\phi = \frac{\partial\phi}{\partial u_1}\left(\frac{\partial u_1}{\partial x}\lambda P + \frac{\partial u_1}{\partial y}\lambda Q + \frac{\partial u_1}{\partial z}\lambda R\right) + \frac{\partial\phi}{\partial u_2}\left(\frac{\partial u_2}{\partial x}\lambda P + \frac{\partial u_2}{\partial y}\lambda Q + \frac{\partial u_2}{\partial z}\lambda R\right) = 0$$

or $d\phi = \frac{\partial\phi}{\partial u_1}\left(P\frac{\partial u_1}{\partial x} + Q\frac{\partial u_1}{\partial y} + R\frac{\partial u_1}{\partial z}\right) + \frac{\partial\phi}{\partial u_2}\left(P\frac{\partial u_2}{\partial x} + Q\frac{\partial u_2}{\partial y} + R\frac{\partial u_2}{\partial z}\right) = 0$

Since, u_1 and u_2 satisfy $P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0$

$\therefore$ $P\frac{au_1}{ax} + Q\frac{au_1}{ay} + R\frac{au_1}{az} = 0$

and $P\frac{au_2}{ax} + Q\frac{au_2}{ay} + R\frac{au_2}{az} = 0$

So we have shown that if $u_1 (x, y, z) = C_1$ and $u_2 (x, y, z) = C_2$ are two independent solutions of ordinary differential equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}, \text{ then } \phi(u_1, u_2) = 0$$

is the general solution of $Pp + Qq = R$

Any surface given by $\phi (u_1, u_2) = 0$

or $u_2 (x, y, z) = f [u_1 (x, y, z)]$

is called an integral surface of the given partial differential equation.

Example 5 : Solve $z\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x$

Solution : Subsidiary equations are $\frac{dx}{z} = \frac{dy}{y} = \frac{dz}{x}$

$$\frac{dx}{z} = \frac{dz}{x}$$

$\Rightarrow$ $xdx - zdz = 0$

or $x^2 - z^2 = C_1$

From $\frac{dx}{z} = \frac{dy}{y} = \frac{dz}{x} = \frac{dx + dy + dz}{x + y + z}$

$$\frac{dz}{y} = \frac{dx + dy + dz}{x + y + z}$$

We obtain ay = x + y + z

or x + z = (a – 1) y

$$\boxed{x + z = C_2 y}$$

Thus the general solution of the given differential equation is

$$f\left(x^2 - z^2, \frac{x + z}{y}\right) = 0$$

or $$x^2 - z^2 = \phi\left(\frac{x + z}{y}\right)$$

Example 6 : Solve (y + z) p + (x + z) q = x + y

Solution : The subsidiary equations are

$$\frac{dx}{y + z} = \frac{dy}{x + z} = \frac{dz}{x + y}$$

$$\frac{dx + dy + dz}{z(x + y + z)} = \frac{dy - dz}{z - y} = \frac{dx - dy}{y - x}$$

$$\frac{dx + dy + dz}{x + y + z} + 2\frac{(dy - dz)}{y - z} = 0$$

$$\frac{dx+dy+dz}{x+y+z}+2\left(\frac{dy-dz}{x-y}\right)=0$$

log $(x + y + z) + z \log (y - z) = \log C_1$

$(x + y + z)(y - z)^2 = C_1$

and $(x + y + z)(x - y)^2 = C_2$

or the general solution is $\phi\left((x-y)^2\ (x+y+z), \frac{x-y}{y-z}\right)=0$

Example 7 : Solve $yzp + xzq + 2xy = 0$

Solution : Its subsidiary equations are

$$\frac{dx}{yz}=\frac{dy}{xz}=\frac{dz}{2xy}$$

We get $xdx - ydy = 0$ and $2ydy - zdz = 0$

$$x^2-y^2=C_1 \text{ and } y^2-\frac{z^2}{2}=C_2$$

General solution is $\phi\left(x^2-y^2, y^2-\frac{z^2}{2}\right)=0$

Example 8 : Solve $px - qy = 2x - z$

Solution : Its subsidiary equations are

$$\frac{dx}{x}=\frac{dy}{-y}=\frac{dz}{2x-z}$$

$$\frac{dx}{x}+\frac{dy}{y}=0 \text{ or } xy=C_1$$

From the last pair $\frac{dx}{x}=\frac{dz}{2x-z}$

$2xdx - zdx = xdz$

$2xdx = zdx + xdz$

$x^2 = xz + C_2$

$x^2 - xz = C_2$

Hence general solution is $\phi(xy, x^2 - xz) = 0$

1.1.5 Integral Surface Passing Through a Given Curve

In the earlier section, we considered a method of finding the general solution of a first order partial differential equation. We shall now show that how such a general solution may be used to determine the integral surface which passes through a given curve.

Let U (x, y, z) = C_1 and v (x, y, z) = C_2 be two solutions of the subsidiary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad (A)$$

of the linear partial differential euqation

Pp + Qq = R (A')

As we already that general solution of (A') is of the form

f (u, v) = 0 (B)

We now wish to determine the function f when the integral surface includes a specified curve in (xyz) space. Suppose, for instance, the curve is given by

ϕ_1 (x, yz) = 0 ϕ_2 (x, yz) = 0 (C)

Then, the determination of the function f in (B) is equivalent to finding a relationship between C_1 and C_2 in u (x, y, z) = C_1 and v (x, y, z) C_2 such that u (x, y, z) = C_1, v (x, y, z) = C_2 and C are compatible. If we now eliminate x, y, z from the last four equations, we obtain a relation connecting C_1 and C_2. Of this relation is

ψ (C_1, C_2) = 0 the required solution is thus given by

ψ (u, v) = 0

Example 9 : Find the solution of yp + xq + 1 = z representing a surface passing through the curve z = x^2 + y + 1 and y = 2x.

Solution : The given differential equation is yp + xq + = z – 1

comparing with Pp + Qq = R, we get

P = y, Q = x, R = z – 1, so that the subsidiary equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{z-1}$$

From the first two equations, we get

$-x^2 + y^2 = C_1$ (1)

Similarly from the pair $\frac{dx+dy}{y+x} = \frac{dz}{z-1}$

We have z – 1 = C_2 (x + y) (2)

We now wish to eliminate C_1 and C_2 from (1) and (2) with the help of given curves z = x^2 + y + 1, y = 2x

If we put $z = x^2 + y + 1$, $y = 2x$ in (2), we have $3C_2 = 2 + x$

From (1), we also get (if we replace $y = 2x$)

$$x = \sqrt{\frac{C_1}{3}}$$

So that $C_2 = \frac{1}{3}\left(2 + \sqrt{\frac{C_1}{3}}\right)$

Hence the required solution is $\frac{z-1}{x+y} = \frac{1}{3}\left(2 + \sqrt{\frac{y^2 - x^2}{3}}\right)$

Example 10 : Find an integral surface of $xp + yq = z$ passing through the curve $x + y = 1$, $x^2 + y^2 + z^2 = 25$.

Solution : Subsidiary equations are $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$

Easily we get $\frac{x}{y} = C_1, \frac{z}{y} = C_2$

Since $x + y = 1 \Rightarrow y\left(1 + \frac{x}{y}\right) = y\,(1 + C_1) = 1$

$$y = \frac{1}{(1 + C_1)}$$

$$x^2 + y^2 + z^2 = 25$$

$$y^2\left(\frac{x^2}{y^2} + \frac{z^2}{y} + 1\right) = 25$$

$$y^2\left(C_1^2 + C_2^2 + 1\right) = 25$$

$$\left(C_1^2 + C_2^2 + 1\right) = 25\left(1 + C_1\right)^2$$

which on substituting for C_1 and C_2 gives in the desired surface.

Solution of partial differential equations of first order of any degree (i.e. non-linear in p and q).

1.1.6 P.D.E is Independent of x, y and z (Type I)

When partial differential equation is independent of x, y and z i.e. it is of the form f (p, q) = 0.

In this case, its complete solution is given by

z = ax + by + c where f (a, b) = 0

because $\frac{\partial z}{\partial x} = a, \frac{\partial x}{\partial y} = b$

i.e. p = a, q = b, on putting these values in f (a, b) = 0 we get the equation f(b, q) = 0. Solving f (a, b) = 0, we can find b in terms of a and let b = g(a), then solution is z = ax + g(a) y + C (A)

To find singular solution (if any) which is envelope of the complete solution (A), to get singular solution we have to eliminate and c from

ϕ = z – ax - yg (a) – c = 0

$\frac{\partial b}{\partial a} = 0, \frac{\partial d}{\partial c} = 0$

i.e. x – g (a) = 0, – 1 = 0 which is not possible. Hence there is no singular solution. In fact such equations of type f (p, q) = 0 does not possess singular solution. To get the general solution, we take c = h (a) so that (A) becomes

z = ax + g(a) y + h (a) (B)

differentiate w.r.t. a

0 = x + g(a) y + n' (a) (C)

Now to find the general solution, we have to eliminate a from B and C.

Example 11 : Find the solution of $p^2 - q^2 = a^2$

Solution : So solution is z = Ax + By + C

where $A^2 - B^2 = a^2$

or $B^2 = A^2 - a^2$

or $z = Ax + \sqrt{A^2 - a^2}y + C \ \alpha > a$

To find general solution

$$z = Ax + \sqrt{A^2 - a^2}\ y + \lambda\,(a) \qquad (i)$$

$$v = x - \frac{a}{\sqrt{A^2 - a^2}}\,y + \lambda'(a) \qquad (ii)$$

So (i) and (ii) together represent the general solutionof the given p.d.e.

1.1.7 P.D.E is of the Form z = px + qy + f (p, q) (Type II)

When the partial differential equation (p.d.e.) is of the form

$z = px + qy + f(p, q)$

Then its complete solution is given by

$z = ax + by + f(a, b)$

where a and b are arbitrary constants.

To get singular solution, we eliminate a and b from

$F = z - ax - by - f(a, b) = 0$

$$\frac{\partial F}{\partial a} = 0, \frac{\partial F}{\partial b} = 0$$

To find general solution we take $b = \lambda(a)$ so that

$z = ax + by + f(a, b)$ becomes.

$z = ax + \lambda(a)\, y + f(a, \lambda(a))$

$z - ax - \lambda(a)\, y - f(a, \lambda(a) = 0$ (i)

$-x - \lambda'(a)\, y - f(a, \lambda(a) = 0$ (ii)

Now eliminant of a from (i) and (ii) gives us the general solution.

Example 12 : Solve $z = px + qy + \log(pq)$

Sol. : Its solution is $z = ax + by + \log(ab)$

Where a and b are arbitrary constants.

To find singular solution

Let $F = z - ax - by - \log ab$

$$\frac{\partial F}{\partial a} - x - \frac{1}{a}$$

$$\frac{\partial F}{\partial a} - y - \frac{1}{b}$$

Now we eliminate a and b from above three equations, which is

$$z + 2\log\frac{1}{xy} = 0$$

or $z + 2 + \log xy = 0$

which is the required singular solution. To get general solution, we take

$F = z - ax - g(a)\, y - \log(ag(a)) - 0$

$$\frac{\partial F}{\partial a} = -x - g'(a)\, y - \frac{1}{a} - \frac{g'(a)}{g(a)} = 0$$

The general solution is represented by the last two equations.

1.1.8 P.D.E f(z, p, q) = 0 (Type-III)

When the partial differential equation is of the form F (z, p, q) = 0 i.e. the partial differential equation has p, q and z only (x and y are absent). In such a case, we take u = x + ay as the trial solution so that

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{\partial z}{\partial u}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} = a \frac{\partial z}{\partial u}$$

equation becomes $F\left(z, \frac{\partial z}{\partial u}, a\frac{\partial z}{\partial u}\right) = 0$

which is an ordinary differential equation of first order and can be solved easily by integration. Hence, the complete solution is of the form

f (x, y, z, a, b) = 0

a, b being arbitrary constants.

Again to find general solution, take b = λ (a), then

f (x, y, z, a, λ(a)) = 0 and $\frac{\partial f}{\partial a} = 0$ is the general solution.

Example 13 : Solve $z^2 (p^2 + q^2 + 9) = 1$

Sol. : Here $z^2 (p^2 + q^2 + 9) - 1 = 0$

Take u = x + ay

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{\partial z}{\partial u}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} = a \frac{\partial z}{\partial u}$$

So that $z^2\left[\left(\frac{\partial z}{\partial u}\right)^2 \left(1 + a^2\right) + 9\right] - 1 = 0$

$$\left(\frac{\partial z}{\partial u}\right)^2 \left(1 + a^2\right) = \frac{1}{z^2} + 9$$

$$\frac{\partial z}{\partial u}\sqrt{1 + a^2} = \sqrt{\frac{1 - 9z^2}{z^2}} = \sqrt{\frac{1 - 9z^2}{z}}$$

or $$\frac{z}{\sqrt{1-9z^2}}\,dz = \frac{1}{\sqrt{1+a^2}}\,z$$

on integerating, we have

$$\frac{\left(1+a^2\right)}{81}\left(1-9z^2\right) = \left(x+ay+c\right)^2$$

$(1 + a^2)\ (1-9z^2) = 81\ (x + ay + c)^2$

$= (9x + 9ay + 9c)^2$

$(1 + a^2)\ (1 - 9z^2) = (9x + 9ay + b)^2$

where $b = 9c$

gives us the general solution of

$z^2 (p^2 + q^2 + 9) = 1$

1.1.9 P.D.E f(x, p) = g (y, z) (Type-IV)

When p.d.e. is of the form

$f(x, p) = g(y, q)$

The method to solve such type of equations will be explained on the following example.

Solve the equation $\sqrt{p} + \sqrt{q} = x + y$

We have $$\sqrt{p} - x = y - \sqrt{q} = k$$

So that $$\sqrt{p} = x + k, \text{ or } p = (x+k)^2$$

$\sqrt{q} = y - k$ and $q = (y - k)^2$

and $dz = pdx + qdy$

$dz = (x + k)^2\,dx + (y - k)^2\,dy$

or $$z = \frac{(x+k)^3}{3} + \frac{(y-k)^3}{3} + \frac{b}{3}$$

or $3z = (x + k)^3 + (y - k)^3 + b$ is the complete solution.

To get singular solution, we get

$F = 3z - (x + k)^3 - (y - k)^3 - b$

$$\frac{\partial F}{\partial k} = -3\,(x+k)^2 + 3\,(y-k)^2$$

$$\frac{\partial F}{\partial b} = -1$$

Singular solution is obtained by

$$F = 0, \frac{\partial F}{\partial k} = 0, \frac{\partial F}{\partial b} = 0$$

but $\frac{\partial F}{\partial b} = -1$ and $-1 \neq 0$

So singular solution (S.S.)

To get the general soluton (G.S.), we replace by g(k) and then both

$F = 3z - (x + k)^3 - (y - k)^3 - g(k)$ represent the G.S.

and $3(x + k)^2 - 3(y - k)^2 - g'(k) = 0$

1.1.10 SELF CHECK EXERCISE

1. (i) Show that $4xyz = 2px^2y + 2qy^2 + pq$ is the differential equation obtained when we eliminate C_1 and C_2 from $z = C_1x^2 + C_2y^2 + C_1C_2$

(ii) Find the differential equation when we eliminate f from $xyz = f(x + y + z)$

Solve the following :

2. $p^2 + q = q^2$

3. $p^2 = zq$

4. $pq = xy$

5. $z = px + qy - 2p - 4y$

6. $xzp + yzq = xy$

7. $(y + z)p + (z + x)q = x + y$

8. $p + q = q$

LESSON NO. 1.2

PARTIAL DIFFERENTIAL EQUATIONS–II

1.2.1 Homogeneous Linear Equation with Constant Coefficients

An equation of the form

$$\frac{\partial^2 z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \text{..........} + k_n \frac{\partial^n z}{\partial y^n} = F(x, y) \qquad \text{..........} (i)$$

In which k_i's are constants, is called a homogeneous linear partial differential equation of the nth order with constant coefficients. It is called homogeneous because all terms contain derivatives of the same order.

This can be written as, ϕ (D, D')z = F (x, y)

Its solution consists of two parts.

(i) the complementary function (C.F.): which is the complete solution of the equation ϕ (D, D') z = 0. It must contain n arbitrary functions where n is the order of the differential equation.

(ii) the particular integral (P.I.) which is a particular solution (free from arbitrary constants) of ϕ (D, D')z = F (x, y).

The complete solution of above differential equation is

z = C.F. + P.I.

1.2.2 Rules to Write Complementary Function

Consider the equation

$$\frac{\partial^2 z}{\partial x^2} + k_1 \frac{\partial^2 z}{\partial x \partial y} + k_2 \frac{\partial^n z}{\partial y^2} = 0 \qquad \text{..................} (i)$$

which is symbolic form is

$(D^2 + k_1DD' + k_2D'^2)\, z = 0$ .... (ii)

From the (A.E.) $m^2 + k_1 m + k_2 = 0$, by putting D = m and D' = 1 in (ii). Solve the (A.E.) and find its roots. If

(a) the roots of (A.E.) are different say m_1 and m_2 then
$z = \phi_2(y + m_1x) + \phi_2(y + m_2x)$ is the C.F.

(b) the roots of (A.E.) are equal each equal to say, m_1 then
$z = \phi_1(y + m_1x) + x\,\phi_2(y + m_1x)$ is the C.F.
In general if the (A.E.) has r roots equal, then
$z = \phi_1(y + mx) + \ldots\ldots\ldots\ldots\ldots + x^{r-1}\phi_r(y + m_1x)$.

1.2.3 Rules to Obtain Particular Integral

(i) When F (x, y) = $e^{\alpha x + by}$

$$\text{P.I.} = \frac{1}{\phi(D, D')}\, e^{\alpha x+by} = \frac{1}{\phi(a, b)}\, e^{\alpha x+by}$$

(i.e. put D = a and D' = b) provided $\phi(a, b) \neq 0$

If $\phi(a, b) = 0$: we have the case of failure in that case

$$\text{P.I.} = x \cdot \frac{1}{\frac{\partial \phi}{\partial D'}}\, e^{\alpha x+by} \text{ or } y \cdot \frac{1}{\frac{\partial \phi}{\partial D}}\, e^{\alpha x+by}$$

(ii) When F (x, y) = sin (ax + by)

$$\text{P.I.} = \frac{1}{\phi(D^2, DD', D^2)} \sin(ax + by)$$

$$= \frac{1}{\phi(-a^2, -ab, -b^2)} \sin(ax + by) \text{ i.e. put } D^2 = -a^2,\ DD' = -ab,\ D'^2 = -b^2$$

$$\text{provided } \phi\left(-a^2, -ab, -b^2\right) \neq 0$$

If $\phi(-a^2, -ab, -b^2) = 0$: then it is called a case of failure and we can repeat the process of (i). A similar rule holds when F(x, y) = cos (ax + by).

(iii) When F (x, y) = $x^p y^q$, where p and q are positive integers.

$$\text{P.I.} = \frac{1}{\phi(D, D')}\, x^p y^q = \left[\phi(D, D')\right]^{-1} x^p\ y^q$$

If $p < q$, expand $[\phi(D, D')]^{-1}$ in powers of $\frac{D}{D'}$

If $q < p$, expand $[\phi (D, D')]^{-1}$ in powers of $\frac{D'}{D}$

Also, we have $\frac{1}{D} F(x, y) = \int_{y\ constant} F(x, y)\, dx$ and

$$\frac{1}{D} F(x, y) = \int_{x\ constant} F(x, y)\, dy$$

(iv) If $f(x, y) = e^{ax+by} V(x, y)$ where V (x, y) is a function of x & y

$$P.I. = \frac{1}{\phi(D, D')} e^{ax+by} . V$$

$$= e^{ax+by} \frac{1}{\phi(D + a, D' + b)}$$

(v) A short method, when f(x, y) is a function of ax + by.

we may apply a shorter method to find the particular integral.

Working Rule : To get the particular integral of an equation $F(D, D') z = \phi(ax + by)$ where F (D, D') is a homogeneous function of D, D' of degree n.

Put ax + by = t, then integrate $\phi(t)$, n times with respect to t.

Put a for D and b for D' in F(D, D') to get F (a, b).

Then $P.I. = \frac{1}{F(a, b)} \times$ nth integral of $\phi(t)$ with respect to t, where t = ax + by.

In case of failure :

$$\frac{1}{(bD - aD')^n} \phi(ax + by) = \frac{x^n}{n!\, b^n} \phi(ax + by)$$

(vi) When F(x, y) = Any function

Then $P.I. = \frac{1}{\phi(D, D')} F(x, y)$

Resolve $\frac{1}{\phi(D, D')}$ into partial fractions.

Considering ϕ (D, D') as a function of D alone.

$$\text{P.I.} = \frac{1}{D - mD'} F(x, y) = \int F(x, c - mx)\, dx$$

where c is to be replaced by y + mx after integration.

1.2.4 Non Homogeneous Linear PDE

A linear PDE which is not homogeneous i.e. all the derivatives are not of the same order, is called a non-homogeneous linear partial differential equation.

F(D, D')z = f(x, y)

Where F(D, D') is non-homogenous in D and D'.

F (D, D') is not always resolvable into linear factors as in homogeneous linear equations.

Therefore, we classify linear differential operators F (D, D') into two following types.

(i) F (D, D') cannot be resolved into linear factors for example $D^2 - D'$.

(ii) F (D, D') can be expressed as product of linear factors of the form $(\alpha D + \beta D + \gamma)$ where α, β and γ are constants.

Method of finding C.F. of non-homogenous linear PDEs :

F(D, D')z = f(x, y)

Case I. When F (D, D') cannot be factorized into linear factors.

In such cases we apply trial method consider the equation

$$(D - D'^2)z = 0 \qquad \text{............... (i)}$$

Let a trial solution of (i) be

$$z = Ae^{hx + ky} \qquad \text{............. (ii)}$$

where, A, h and k are constants

from (ii), $Dz = \frac{\partial z}{\partial x} = hAe^{hx+ky}$

and $D'^2 z = \frac{\partial^2 z}{\partial y^2} = k^2 Ae^{hx+ky}$

Putting these in (i), we get

$(h - k^2)\, Ae^{hx + ky} = 0$

or $h = k^2$ .... (iii)

Putting the value of h in (ii) a solution (which is also C.F.) is taken as

$$z = Ae^{k^2 x + ky} \qquad \text{............. (iv)}$$

Since all values of k satisfy the given equation (i), a more general solution (which is also C.F.) is taken as

$$z = \sum Ae^{k^2x+ky} \qquad \text{........... (v)}$$

Where A and k are arbitrary constants.

Example 1 : Solve $(2D^4 - 3D^2 D' + D'^2)\, z = 0$

Solution : The given equation can be written as

$(2D^2 - D')\,(D^2 - D')\, z = 0$ .... (1)

consider $(2D^2 - D')\, z = 0$ .... (2)

Let $z = Ae^{hx + my}$ be a trial solution of (1). Then we have

$D^2z = Ah^2 e^{hx + ky}$ and $D'z = Ah^2e^{hx + ky}$. Putting these values in (2), we get

$A(2h^2 - k)\ e^{hx + ky} = 0$ so that $2h^2 - k = 0$ or $k = 2h^2$ hence the most general solution of (2) is $z = \sum Ae^{hx+2h^2y}$ .... (3)

Next, consider $(D^2 - D')\, z = 0$........... (4)

Let $z = A'e^{h'x+k'y}$ be a trial solution of (1). Then we have

$D^2z = A'h'^2 e^{h'x+k'y}$ and $D'z = A'k'e^{h'x+k'y}$. Putting these values in (4), we get

$A'(h' - k')\,e^{h'x+k'y} = 0$ so that $h' - k' = 0$ or $h'^2 = k'$

Hence the most general solution of (3) is

$$z = \sum A'e^{h'x+2h'^2 y} \qquad \text{................ (5)}$$

From (3) and (5) the most general solution of (1) is

$\sum Ae^{hx+2h^2y} + \sum Ae^{h'x+2h'^2 y}$ A, A', h, h', k, k' being arbitrary constants.

Example 2 : Solve $\left(\frac{\partial^2 z}{\partial x^2}\right) + \left(\frac{\partial^2 z}{\partial y^2}\right) = n^2 z$

Solution : The given equation can be written as

$(D^2 + D'^2 - n^2)\, z = 0$ .... (1)

Let a trial solution of (1) be

$z = Ae^{hx + ky}$ .... (2)

$\therefore$ $D^2z = Ah^2e^{hx + ky}$ and $D'^2 = Ak^2\, e^{h'x + k'y}$

Hence (1) gives

$A\,(h^2 + k^2 - n^2)\ e^{hx + ky} = 0$

or $h^2 + k^2 = n^2$ .... (3)

Taking a as parameter, we see that (2) is satisfied if h = n cos q and k = n sin a Putting these values in (2) the required general equation is

$$z = \sum A\, e^{n\,(x \cos \alpha + \gamma \sin \alpha)}$$

A and a being arbitrary constants.

Case II : When F (D, D') can be linear factor of F(D, D'). To find C.F. corresponding to this factor we consider the most simple non-homogeneous equation.

$$(\alpha D + \beta D' + \gamma)\, z = 0$$

or $\alpha p + \beta q = -\gamma z$ .... (i)

Which is of Lagrange's form

$$\frac{dx}{\alpha} = \frac{dy}{\beta} = \frac{dz}{-\gamma z} \qquad \text{........... (ii)}$$

From first and second ratio of (ii)

$\alpha dy - \beta dx = 0$ i.e. $\alpha y - \beta x = C$ .... (iii)

Again from first and third ratios of (ii), we get

$$\frac{dz}{z} = \frac{-\gamma}{\alpha}\, dx$$

Integrating, $\log z = \frac{-\gamma}{\alpha} x + \log C_1$

$$z = C_1\, e^{\frac{-\gamma}{\alpha}x},\; z = e^{\frac{-\gamma}{\alpha}x}\, \phi(C),\; z = e^{\frac{-\gamma}{\alpha}x}\, \phi(\alpha y - \beta x)$$

From (iii)

Thus the part of C.F. corresponding to linear factor

$\alpha D + \beta D' + \gamma$ is $e^{\frac{-x}{\alpha}}\, \phi\, \alpha y - \beta x$ .... (iv)

where ϕ is an arbitrary function. Similarly it can be shown that if F(D, D') has non-repeated linear factors of the type.

$F(D, D') = (\alpha_1 D + \beta_1 D' + \gamma_1)(\alpha_2 D + \beta_2 D' + \gamma_2)$ .... $(\alpha_n D + \beta_n D' + \gamma_n)$

then C.F. of equation F (D, D') z = F (x, y)

$$= e^{\frac{\gamma_1}{\alpha_1}}\phi(\alpha_1 y + \beta_1 x) + e^{\frac{\gamma_2}{\alpha_2}}\phi(\alpha_2 y + \beta_2 x) + \ldots\ldots\; e^{\frac{\gamma_n}{\alpha_n}}\phi(\alpha_n y + \beta_n x)$$

Also corresponding to a repeated factor

$(\alpha D + \beta D' + \gamma)^k$, the part of C.F. is

$$e^{\frac{-\gamma}{\alpha}x}\phi_1(\alpha y-\beta x)+x\phi_2(\alpha y-\beta x)+........+x^{k-1}\phi_k(\alpha y-\beta x)$$

Remarks :

1. Corresponding to each non repeated factor $(D - mD' - \gamma)$ the part of C.F. is $e^{\gamma x}\phi(y+mx)$
2. If the factor $(D-mD'-\gamma)$ repeats k times then the part of C.F. corresponding to it is $e^{\gamma x}(\phi_1(y+mx)+x\phi_2(y+mx)+.............x^{k-1}\phi_k(y+mx))$.
3. If a factor $(\beta D'+\gamma)$ occurs only once then C.F. corresponding to it is

   $e^{\frac{-\gamma}{\beta}}\phi(\beta x)$. In case $\beta D'+\gamma$ repeats k times the part of C.F. is

   $$e^{\frac{-\gamma_1}{\beta}y}[\phi_1(\beta x)+x\phi_2(\beta x)+......................x^{k-1}\phi_{k-1}(\beta x)]$$

1.2.5 Method of Finding P.I of Non-Nomogeneous Linear PDE

$F(D, D')z = f(x, y)$

$$P.I.=\frac{1}{F(D,D')}f(x,y)$$

Case 1 : When $F(x, y) = e^{\alpha x+by}$ and $F(a, b) \neq 0$

$$P.I.=\frac{1}{F(D,D')}e^{ax+by}=\frac{1}{F(a,b)}e^{ax+by}$$

Example 3 : Solve $(D^2 - D'^2 + D - D') = z = e^{2x+3y}$

Solution : The given equation can be re-written as

$[(D - D')(D + D')]z = e^{2x+3y}$

or $(D + D')(D + D'+1)z = e^{2x+3y}$

$\therefore$ C.F. $= \phi_1(y + x) + e^{-x}\phi_2(y - x) + \phi_1\phi_2$

being arbitrary functions and P.I. is

$$\frac{1}{(D-D')(D+D'+1)}e^{2x+3y}$$

$$=\frac{1}{(2-3)(2+2+1)}e^{2x+3y}$$

$$=-\frac{1}{6}e^{2x+3y}$$ Hence the required general solution is

z = C.F. + P.I. i.e.

$$\phi_1(y+x)+e^{-x}\phi_2(y-x)-\left(\frac{1}{6}\right)e^{2x+3y}$$

Case II : When f(x, y) = sin (ax + by) or cos (ax + by)

$$\text{P.I.}=\frac{1}{F(D,D^1)}\sin(ax+by)$$

$$=\frac{1}{F\left(D^2,DD^1,D^2,D,D^1\right)}$$

which can be evaluated further.

Example 4 : Solve $(D^2 + DD' + D' - 1)\,z = \sin(x + 2y)$

Solution : The given equation can be written as

$$(D + 1)(D + D' - 1)\,z = \sin(x + 2y)$$

Hence he required general Solution is z = C.F. + P.I. Do Yourself

Case III : When $f(x, y) = x^m y^n$, m and n being positive integers.

$$\text{P.I.}=\frac{1}{F(D,D^1)}x^m y^n=\left[F(D,D')^{-1}x^m y^n\right]$$

Example 5 : Solve $r - s + 2q - z = x^2y^2$

Solution : $\left(\frac{\partial^2 z}{\partial x^2}\right)-\left(\frac{\partial^2 z}{\partial x\partial y}\right)+2\left(\frac{\partial z}{\partial y}\right)-z=x^2y^2$

$(D^2 - DD' + 2D' - 1)\,z = x^2y^2$ .... (1)

Since $(D^2 - DD' + 2D' - 1)$ cannot be resolved into linear factors in D and D', hence C.F. of (1) is obtained by considering the equation.

$(D^2 - DD' + 2D' - 1)\,z = 0$ .... (2)

Let a trial solution of (2) be

$z = Ae^{hx + ky}$ .... (3)

∴ $D^2z = Ah^2e^{hx + ky}$, $DD'z = Ahke^{hx + ky}$, $D'z = Ake^{hx + ky}$

then (2) gives

$A(h^2 - hk + 2k - 1)\,e^{hx + ky} = 0$ or $h2 - hk + 2k - 1 = 0$

so that $k=\frac{\left(1-h^2\right)}{(2-h)}$ .... (4)

∴ From (3),C.F. $\Sigma\, Ae^{hx + ky}$ where A, h, k are arbitrary constants and h and k are related by (4). Now P.I. =

$$\frac{1}{D^2 - DD' + 2D' + 1} x^2y^2 = \frac{1}{1 - \left(D^2 - DD' + 2D'\right)} x^2y^2$$

$= [1 - (D^2 - DD' + 2D')]^{-1} x^2y^2$

$= [-[1 + (D^2 - DD' + 2D')] + (D^2 - DD' + 2D')^2 + [D^2 + D' (2 - D)]^3 + \ldots\ldots] x^2y^2$

$= - [1 + (D^2 - DD' + 2D') + (D^2D^2 + 4D^2 D' - 4DD'^2 + \ldots\ldots\ldots)$

$+ 3D^2 D'^2 (2 - D)^2 + \ldots\ldots\ldots] x^2y^2$

$= -[1 + (D^2 - DD' + 2D') + D^2D'^2 + 4D^2D' - 4DD'^2 + 12D^2 D'^2 + \ldots\ldots] x^2y^2$

$= -x^2 y^2 - 2y^2 + 4xy - 2x^2 y - x^2 - 16x - 16y - 52$

Hence the required general solution is

z = C.F. + P.I.

$z = \Sigma\, Ae^{hx + ky} - x^2y^2 - 2y^2 + 4xy - 2x^2 y - x^2 - 16x - 16y - 52$

Case IV : When f(x, y) $Ve^{ax + by}$ where v is a function of x and y, then

$$\text{P.I.} = \frac{1}{F(D, D')}\left(Ve^{ax+by}\right) = e^{ax+by}$$

$$= \frac{1}{F\left(D + a, D'b\right)}$$

which can be evaluated further.

Example 6 : Solve $(D - 3D' - 2)^2 z = 2e^{2x} \tan(y + 3x)$

Solution : Hence

C.F. $= e^{2x} [\phi, (y + 3x) + x\phi_2 (y + 3x)]$

and P.I. $= \dfrac{1}{(D - 3D' - 2)^2} 2e^{2x+0.y} \tan(y + 3x)$

$$= 2e^{2x+0.y} \frac{1}{\left\{(D + 2) - 3(D' + 0) - 2\right\}^2} \tan(y + 3x)$$

$$= 2e^{2x} \frac{1}{(D - 3D')^2} . \tan(y + 3x) = 2e^{2x} \frac{x^2}{1^2.2!} \tan(y + 3x$$

$= x^2 e^{2x} \tan(y + 3x)$

$$z = e^{2x}\left[\phi, (y + 3x) + x\phi_2 (y + 3x)\right] + x^2e^{2x} \tan(y + 3x)$$

Remark : if $f(x, y) = e^{ax + by}$ and $F(a, b) = 0$ then we have

$$P.I. = \frac{1}{F(D.D')} e^{ax+by} = e^{ax+by}$$

$$= \frac{1}{F(D + a\ D' + b)} . 1$$

which can be evaluated further.

LESSON NO. 1.3

PARTIAL DIFFERENTIAL EQUATIONS–III

1.3.1 Classification of Linear Partial Differential Equation of Second Order in Two Independent Variables

Let us consider the equation of second order in two independent variables x and y

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}\right) = 0 \qquad \text{.......... (1)}$$

where A is positive.

Here $\phi = A\delta_1^2 + B\delta_1 + C\delta_2^2$

The equation (1) is

(i) elliptic if $B^2 - 4AC < 0$.

(ii) Hyperbolic if $B^2 - 4AC > 0$, and

(iii) Parabolic if $B^2 - 4AC = 0$

Note 1. If A, B, C are constants then the nature of the equation (1) will be the same in the whole region i.e. for all values of x and y. The nature will depend on $B^2 - 4AC$.

The equation (1) will be elliptic if $B^2 - 4AC < 0$

The equation (1) will be hyperbolic if $B^2 - 4AC > 0$,

The equation (1) will be Parabolic if $B^2 - 4AC = 0$

Note 2. If A, B, C are functions of x and y then the nature of equation (1) will not be same in the whole region i.e. for all values of x and y.

The equation (1) will be elliptic in the region where $B^2 - 4AC < 0$

The equation (1) will be hyperbolic in the region where $B^2 - 4AC > 0$,

The equation (1) will be Parabolic in the region where $B^2 - 4AC = 0$

Example 1 : Classify the following :

(i) $$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial x^2}$$

(ii) $$\frac{\partial^2 u}{\partial t^2}-4\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial x^2}$$

(iii) $$\frac{\partial^2 u}{\partial t^2}+4\frac{\partial^2 u}{\partial x\partial t}+4\frac{\partial^2 u}{\partial x^2}$$

Solution :

(i) Here A = 1, B = 1, C = 1 and so
$B^2 - 4AC = 1 - 4 = -3 < 0$
Therefore, the given operator is elliptic.

(ii) Here A = 1, B = –4, C = 1 and so
$B^2 - 4AC = 16 - 4 = 12 > 0$
Therefore, the given operator is hyperbolic.

(iii) Here A = 1, B = 4, C = 4 and so
$B^2 - 4AC = 16 - 16 = 0$
Therefore, the given operator is parabolic.

Example 2 : Classify the following equations :

(i) $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \text{ (Laplace equation)}$$

(ii) $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \text{ (Wave equation)}$$

(iii) $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{c^2}\frac{\partial u}{\partial t} \text{ (Heat equation)}$$

Solution :

(i) Here the operator

$$\phi=\delta_1^2+\delta_2^2+\delta_3^2,\ a_{11}=a_{22}=a_{33}=1$$

$$a_{13} = a_{23} = a_{31} = 1$$

ϕ is +ve for all real values of $\delta_1 + \delta_2 + \delta_3$ and it reduces to zero only when
$\delta_1 + \delta_2 + \delta_3 = 0$
Hence, the given Laplace's equation is elliptic.

(ii) Here the operator

$$\phi = \delta_1^2 + \delta_2^2 + \delta_3^2 - \frac{1}{c^4}\delta_4^2$$

This can be both positive or negative. Hence the equation is hyperbolic.

(iii) Here $a_{11} = a_{22} = a_{33} = a_{44} = 0$

and $a_{12} = a_{13} = a_{14} = a_{21} = a_{23} = a_{24} = a_{31} = a_{34} = a_{41} = a_{42} = a_{43} = 0$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

Hence the equation is parabolic.

Example 3 : Classify the equation.

$$(1-x)\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} + (1-y)\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + 3x^2 y\frac{\partial z}{\partial y} - 2z = 0$$

Solution : Consider the operator

$$\phi = A\delta_1^2 + B\,\delta_1\delta_2 + C\delta_2^2 \text{where } \delta_1 \equiv \frac{\partial}{\partial x}\ \delta_2 \equiv \frac{\partial}{\partial y}$$

Here $A = 1 - x^2$, $B = -2xy$, $C = 1 - y$ and so

$$B^2 - 4AC = 4x^2y^2 - 4\,(1 - x^2)\,(1 - y^2)$$
$$= 4\,(-1 + x^2 + y^2)$$

Since A, B, C are functions of x and y, the given differential equation is hyperbolic in the region where $B^2 - 4AC > 0$ i.e. $x^2 + y^2 > 1$, parabolic in the region where $B^2 - 4AC = 0$ i.e. at points on the circle $x^2 + y^2 = 1$, and elliptic in the region where $B^2 - 4AC < 0$ i.e. $x^2 + y^2 < 1$.

Example 4 : Find where the following operator is hyperbolic, parabolic and elliptic.

(i) $$\frac{\partial^2 u}{\partial t^2} + t\frac{\partial^2 u}{\partial x \partial t} + x\frac{\partial^2 u}{\partial x^2}$$

(ii) $$x^2\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u$$

(iii) $$t\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial x \partial t} + x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}$$

Solution : (i) Here A = 1, B = t, C = x

$\therefore$ $B^2 - 4AC = t^2 - 4x$

Thus the operator is hyperbolic if $t^2 - 4x > 0$ i.e. if $t^2 - 4x > 0$, parabolic if $t^2 = 4x$ and elliptic if $t^2 < 4x$.

(ii) Here $A = x^2$, B = 0, C = –1

$\therefore$ $B^2 - 4AC = 4x^2$

Thus the operator is hyperbolic if $4x^2 > 0$ i.e. if $x^2 > 0$, i.e. if $x < 0$ parabolic if $4x^2 = 0$ i.e. if x = 0.

Since $4x^2$ cannot be negative so the operator cannot be elliptic.

(iii) Here A = t, B = 2, C = x

$\therefore$ $B^2 - 4AC = 4 - 4tx$

Thus the operator is hyperbolic if $4 - 4tx > 0$ i.e. $tx < 1$, parabolic if $tx = 1$ and elliptic if $tx > 1$.

Example 5 : Shot that the equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial t^2}$ is hyperbolic.

Solution : Here $A = c^2$, B = 0, C = -

$\therefore$ $B^2 - 4AC = 4c^2 > 0$

Hence the given equation is hyperbolic.

Example 6 : Classify the following as elliptic, parabolic or hyperbolic.

(i) $\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y}$ (ii) $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$

(iii) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

Solution : Do yourself.

Answer : (i) Parabolic (ii) Hyperbolic (iii) Elliptic

1.3.2 Canonical Forms (Method of Transformation)

Now we shall consider the equation of the type

$$Rr + Ss + Tt + F(x, y, z, p, q) = 0 \qquad \ldots\ldots\ldots\ldots (1)$$

Where R, S, T are continuous functions of x and y possessing continuous partial derivatives of as high an order as necessary. We shall show that any equation of the type (1) can be reduced to one of the three canonical forms by a suitable change of the independent variables. Suppose, we change the independent variables from x, y to u, v were

$$u = u(x, y),\ v = v(x, y) \qquad \ldots\ldots\ldots\ldots (2)$$

Then, we have

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial x}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial y}$$

$$\therefore \quad \frac{\partial}{\partial x} \equiv \frac{\partial u}{\partial x}.\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}.\frac{\partial}{\partial v}$$

$$\frac{\partial}{\partial y} \equiv \frac{\partial u}{\partial y}.\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}.\frac{\partial}{\partial v}$$

Now, $r = \dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right) = \left(\dfrac{\partial u}{\partial x}.\dfrac{\partial}{\partial u} + \dfrac{\partial u}{\partial x}.\dfrac{\partial}{\partial v}\right)$

$$\left(\frac{\partial u}{\partial x}.\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}.\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial u \partial v}.\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2}$$

$$\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

$$S = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y}\right) = \left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}.\frac{\partial}{\partial v}\right)$$

$$\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial u \partial v}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}.\frac{\partial v}{\partial x}\right)$$

$$\frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y \partial x}$$

and $t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial u} + \frac{\partial v}{\partial y}.\frac{\partial}{\partial v}\right)$

$$\left(\frac{\partial u}{\partial y}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial y}.\frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial^2 z}{\partial u \partial v}\frac{\partial u}{\partial y}$$

$$\frac{\partial v}{\partial y} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2}$$

Substituting these values of p, q, r, s and t, in (1), it takes the form

$$A = \frac{\partial^2 z}{\partial u^2} + 2B\frac{\partial^2 z}{\partial u \partial v} + C\frac{\partial^2 z}{\partial y^2} + \left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) = 0 \qquad \text{............. (3)}$$

$$\text{Where } A = R\left(\frac{\partial u}{\partial x}\right)^2 + S\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} + T\left(\frac{\partial u}{\partial y}\right)^2 \qquad \text{............. (4)}$$

$$B = R\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{1}{2}S\left(\frac{\partial u}{\partial x}.\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}.\frac{\partial v}{\partial x}\right) + T\frac{\partial u}{\partial y}.\frac{\partial v}{\partial y} \qquad \text{............. (5)}$$

$$C = R\left(\frac{\partial v}{\partial x}\right)^2 + S\frac{\partial v}{\partial x}.\frac{\partial v}{\partial y} + T\left(\frac{\partial v}{\partial y}\right)^2 \qquad \text{................ (6)}$$

and the function F is the transformed form of the function f.

Now the problem is to determine u and v so that the equation (3) takes the simplest possible form. The procedure is simple when the discriminant $S^2 - 4RT$ of the quadratic equation.

$R\lambda^2 + S\lambda + T = 0$ .... (7)

is everywhere either positive, negative or zero, and we shall discuss these three cases separately.

1.3.3 Case–I : $S^2 - 4RT > 0$: If this condition is satisfied then the roots λ_1, λ_2 of the equation (7) are real and distinct. The coefficient of $\frac{\partial^2 z}{\partial u^2}$ and $\frac{\partial^2 z}{\partial v^2}$ in the eqution (3) will vanish if we choose u and v such that

$$\frac{\partial u}{\partial x} = \lambda_1 \frac{\partial u}{\partial y} \qquad \text{...................... (8)}$$

and $$\frac{\partial u}{\partial x} = \lambda_2 \frac{\partial u}{\partial y} \qquad \text{.....................} (9)$$

The differential equation (8) and (9) will determine the form of u and v as functions of x and y. For this, from (8), lagranges auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-\lambda_1} = \frac{du}{0}$$

The last member gives du = 0, i.e.

u = constant

The first two members given

$$\frac{\partial y}{\partial x} = \lambda_1 = 0 \qquad \text{....................} (10)$$

Let f_1 (x, y) = constant be the solution of the equation (10).

Then the solution of the equation (8) can be taken as

$$u = f_1 (x, y) \qquad \text{..................} (11)$$

Similarly, if f_2 (x, y) = constant is a solution of

$$\frac{\partial y}{\partial x} = \lambda_2 = 0$$

Then the solution of the equation (9) can be taken as

$$u = f_2 (x, y) \qquad \text{.....................} (12)$$

Also it can be easily seen that, in general

$$AC - B^2 = (4RT - S^2)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2$$

So that when A and C are zero

$$B^2 = (S^2 - 4RT)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2 \qquad \text{....................} (13)$$

It follows that $B^2 > 0$ since $S^2 - 4RT > 0$ and hence we can divide both sides of the equation by it.

Thus making the substitution defined by the equations (11) and (12) the equation (1) transforms to the form.

$$\frac{\partial^2 z}{\partial u\, \partial v} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) \qquad \text{.....................} (14)$$

which is the canonical form in this case.

1.3.4 Case–II : $S^2 - 4RT = 0$: In this case the roots of the equation (7) are equal. We define the function u as in Case I and take v to be any function of x and y, which is independent of u. Then we have as before, A = 0.

Since $S^2 - 4RT = 0$, hence from (13), $B^2 = 0$ i.e. B = 0 and dividing by C, we see that in this case the canonical form of the equation (1) is

$$\frac{\partial^2 z}{\partial v^2} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) \qquad \text{.................. (15)}$$

1.3.5 Case–III : $S^2 - 4RT < 0$: Formally it is the same as Case I expect that now the roots of the equation (7) are complex.

Proceeding as in Case I, we find that the equation (1) reduces to the form (14) but that the variables u, v are not real but are in fact complex conjugates.

To find a real canonical form let

$u = \alpha + i\beta, \qquad v = \alpha - i\beta$

So that $\alpha = \frac{1}{2}(u+v), \beta = \frac{1}{2}(u-v)$.

Now $\quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial u} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial u} = \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} - i\frac{\partial z}{\partial \beta}\right)$

Similarly $\frac{\partial z}{\partial v} = \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} - i\frac{\partial z}{\partial \beta}\right)$

$\therefore \qquad \frac{\partial^2 z}{\partial u\, \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right)$

$$= \frac{1}{4}\left(\frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \beta}\right)\left(\frac{\partial z}{\partial \alpha} - i\frac{\partial z}{\partial \beta}\right)$$

$$= \frac{1}{4}\left(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2}\right)$$

Thus, transforming the independent variables u, v, and α, β the desired canonical form is

$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \psi\left(\alpha, \beta, z, \frac{\partial z}{\partial \alpha}, \frac{\partial z}{\partial \beta}\right) \qquad \text{.......... (16)}$$

Second order partial differential equations of the type (1) are classified by their canonical forms; we say that an equation of this type is :

(i) Hyperbolic, if $S^2 - 4RT > 0$

(ii) Parabolic, if $S^2 - 4RT = 0$

(iii) Elliptic, if $S^2 - 4RT < 0$

Solved Examples :

Example 7 : Reduce the equation

$$(y-1)^2 r - (y^2-1)s + y(y-1)t + p - q = 2ye^{2x}(1-y)^3 \quad \ldots\ldots\ldots\ldots (1)$$

to canonical form and hence solve it.

Solution : Comparing the equation (1) with

$Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we have

$R = (y-1)$, $S = -(y^2-1)$, $T = y(y-1)$

The quadratic equation $R\lambda^2 + S\lambda + T = 0$

therefore, becomes

$(y-1)\lambda^2 - (y^2-1)\lambda + y(y-1) = 0$

or $\lambda^2 - (y+1)\lambda + y = 0$

or $(\lambda - 1)(\lambda - y) = 0$

$\Rightarrow$ $\lambda = 1$ (real and distinct roots)

The equation $\dfrac{dy}{dx} + 1 = 0$

and $\dfrac{dy}{dx} + y = 0$

These on integration give

$x + y =$ constant and $y^{ex} =$ constant,

so that to change the independent variables from x, y, to u, v, we take

$u = x + y$ and $v = ye^x$.

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u} + ye^x\frac{\partial z}{\partial v} = \frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u} + e^x\frac{\partial z}{\partial v}$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2} + 2v\frac{\partial^2 z}{\partial u\,\partial v} + v^2\frac{\partial^2 z}{\partial v^2} + v\frac{\partial z}{\partial v}$$

$$s = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + e^x\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + e^x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) + e^x\frac{\partial z}{\partial v}$$

$$= \left(\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}\right) + e^x\left(\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial v}\right) + e^x\frac{\partial z}{\partial v}$$

$$= \frac{\partial^2 z}{\partial u^2} + \left(e^x + v\right)\frac{\partial^2 v}{\partial u\partial v} + ve^x\frac{\partial^2 z}{\partial v^2} + e^x\frac{\partial z}{\partial v}$$

and $$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} + e^x\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + e^x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y} + e^x$$

$$\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right]$$

$$= \frac{\partial^2 z}{\partial u^2} + 2e^x\frac{\partial^2 z}{\partial u\,\partial v} + e^{2x}\frac{\partial^2 z}{\partial v^2}$$

Substituting these values in (1) it reduces to

$$\left(1 - y\right)^3 e^x\frac{\partial^2 z}{\partial u\,\partial v} = 2y\,e^{2x}(1 - y)^3$$

which is the canonical form of the equation (1).

Integrating (2) w.r.t. v, we get

$$\frac{\partial z}{\partial u} - v^2 + \phi_1 (u) \qquad \text{.......... (3)}$$

where $\phi_1 (u)$ is an arbitrary function of u.

Again integrating (3) w.r.t. u, we get

$z = uv^2 + \Psi_1 (u) + \Psi_2 (v)$

where Ψ_1 is an integral ϕ_1 and Ψ_2 is an arbitrary function.

or $z = (x + y)\, y^2 e^{2x} + \Psi_1 (x + y) + \Psi_2 (ye^x)$

Example 8 : Reduce the equation $\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$ to canonical form.

Solution : The given equation can be written as

$$r - x^2 t = 0$$

Comparing the equation (1) with

$Rr + Ss + Tt + (x, y, z, p, q) = 0$, we have

$R = 1,\ S = 0,\ T = -x^2$.

The quadratic equation $R\lambda^2 + S\lambda + T = 0$

therefore becomes

$\lambda^2 - x^2 = 0 \Rightarrow \lambda = x, -x$ (real and distinct roots).

The equations $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$ becomes

$$\frac{dy}{dx} + x = 0 \text{ and } \frac{dy}{dx} - x = 0$$

These on integration give

$$y + \frac{1}{2}x^2 = \text{constant and } y\text{-}\frac{1}{2}x^2 - \text{cons tan t}$$

So that to change the independent variables from x, y, to u, v, we take

$$u = v + \frac{1}{2}x^2 = v = y - \frac{1}{2}x^2$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= x\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial z}{\partial x}\left\{x\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right\}$$

$$= x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + 1\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right] + \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= x^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}\right) + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \text{ and}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}$$

Substituting these values in (1), it reduces to

$$\frac{\partial^2 z}{\partial u\,\partial v} = \frac{1}{4x^2}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$\frac{\partial^2 z}{\partial u\,\partial v} = \frac{1}{4\,(u - v)}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

which is the required canonical form of the given equation.

Department of Distance Education

Punjabi University, Patiala

Class : B.A. I (Math) **Semester : 2**
Paper : 5 (Partial Differential Equations) **Unit : 2**
Medium : English

Lesson No.

Department website : www.pbidde.org

PARTIAL DIFFERENTIAL EQUATIONS -IV

2.1.0 LINEAR PARTIAL DIFFERENTIAL EQUATION

A partial differential equation in which dependent variable and its derivatives are of degree one and coefficients are constants is called a linear partial differential equation with constant coefficients. It can be written as

$$\frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + A_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \ldots + A_n \frac{\partial^n z}{\partial y^n} + B_0 \frac{\partial^{n-1} z}{\partial x^{n-1}} + B_1 \frac{\partial^{n-1} z}{\partial_x^{n-2}\partial y}$$

$$+ \ldots\ldots + M\frac{\partial z}{\partial x} + N\frac{\partial z}{\partial y} + Pz = \phi(x, y) \qquad \ldots.(1)$$

If we denote D for $\frac{\partial}{\partial x}$ and D' for $\frac{\partial}{\partial y}$, this equation can be written as

$$\left[D^n + A_1 D^{n-1}D' + \ldots + A_n D'^n + B_0 D^{n-1} + B_1 D^{n-2}D' + \ldots MD + ND' + P\right]z = \phi(x, y)$$

or in short $f(D, D')z = \phi(x, y)$

The equation of such type is solved in two following steps.

(i) Finding complementary function by putting $f(D, D')z = 0$,

(ii) Particular Integral

$$z = \frac{1}{f(D, D')}\phi(x, y).$$

The complete solution of the equatin will be z = C.F. + P.I.

2.1.2 HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

The equation of the type

$$(D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \ldots a_n D'^n)z = f(x, y) \qquad \ldots..(1)$$

where $a_1, a_2, \ldots\ldots\ldots\ldots a_n$ are constants, is called homogeneous linear partial differential equation of the nth order with the constant coefficients.

Note: It is called homogeneous because all terms contain derivatives of same order. The simplest case is $(D - mD')z = 0$ ....(2)

i.e., $p - mq = 0$

On solving with Lagrange's method

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{0} \Rightarrow dz = 0 \text{ and } dy = -mdx, \text{ integrating}$$

we get $z = C_1$ and $y = -mx + C_2$ or $z = C_1$ and $y + mx = C_2$

Hence solution of (2) is $z = \phi(y + mx)$

2.1.3 METHOD OF FINDING COMPLEMENTARY FUNCTION OF HOMOGENEOUS EQUATION

Let $z = \phi(y + mx)$ be the solution of the equation

$$(a_0 D^n + a_1 D^{n-1} D' + \ldots\ldots a_n D'^n)z = 0 \qquad \ldots(1)$$

$$D^n z = D^n \phi(y + mx) = m^n \phi^{(n)}(y + mx)$$

$$D'^n z = D'^n \phi(y + mx) = \phi^{(n)}(y + mx)$$

$$\therefore (a_0 D^n + a_1 D^{n-1} D' + \ldots a_n D'^n)z = (a_0 m^n + a_1 m^{n-1} + \ldots a_n)\phi^{(n)}(y + mx) = 0$$

or $a_0 m^n + a_1 m^{n-1} + \ldots. + a_n = 0$ ...(2)

The equation (2) is called the auxilary equation.

Auxilary equation can be written by replacing D by m and D' by 1 in the given equation (1)

Case 1. If solution of equation (2) $m_1, m_2, \; ..... m_n$ are all distinct, then the solution of the differential equation is,

$$z = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + ... + \phi_n(y + m_n x) \qquad ....(3)$$

Case II. When auxilary equation has equal roots. Let *m, m* be two equal roots of equation (1), then $(D - m D')^2$ is a factor of the equation.

$(D - m D')^2 z = 0$ can be written as

$$(D - m D')(D - m D')z = 0 \qquad .... (4)$$

Let $(D - m D') z = u$ .... (5)

The equation (4) can be written as

$$(D - m D') u = 0 \quad \Rightarrow \quad u = \phi(y + mx)$$

Putting the value of u in (5), we have

$$(D - m D') z = \phi(y + mx)$$

or $\quad p - mq = \phi(y + mx)$

Solving by Lagranges method, we get

$$\frac{dx}{1} = \frac{dx}{-m} = \frac{dz}{\phi(y + mx)} \Rightarrow dy = -mdx \;\; and \;\; dz = \phi(y + mx)dx$$

On integration, we have $y + mx = a$

and $\quad dz = \phi(y + mx)dx = \phi(a)dx \qquad ....(6)$

or $\quad z = x\phi(a) + b = x\phi(y + mx) + \phi_1(y + mx) \qquad ....(7)$

Combining (6) and (7) we have

$$z = x\,\phi(y + mx) + \phi_1(y + mx).$$

If the root m is repeated n times then

$$z = x^{n-1}\phi_1(y + mx) + x^{n-2}\phi_2(y + mx) + ... + \phi_n(y + mx)$$

is the required complementary function.

Example 1. *Find the general solution of the partial differential equation:*

$$\frac{\partial^3 z}{\partial x^3} - \frac{2\partial^3 z}{\partial x\,\partial x\,\partial y} = 0$$

Solution. The given equation is $\frac{\partial^3 z}{\partial x^3} - \frac{2\partial^3 z}{\partial x\,\partial x\,\partial y} = 0$

It can be written as

$$(D^3 - 2D^2 D')z = 0$$

The auxilary equation is

$$m^3 - 2m^2 = 0$$

$m^2(m-2) = 0 \Rightarrow m = 0, 0, 2$ are its roots. Hence 0 occurs twice.

The general solution of the given equation is

$$z = \phi_1(y + 0x) + x\phi_2(y + 0x) + \phi_3(y + 2x)$$

or $$z = \phi_1(y) + x\phi_2(y) + \phi_3(y + 2x).$$

2.1.4 PARTICULAR INTEGRAL

Particular integral of the equation F(D, D') $z = f(x, y)$ is written as

$z = \frac{1}{F(D, D')} f(x, y)$. Now we shall discuss the methods of finding P.I. for different functions.

Case I. When f(x,y) is a function of ax + by.

Let F(D, D') $\equiv a_0 D^n + a_1 D^{n-1} D' + ... + a_n D'^n$

be a homogeneous function of D and D' of degree n

$$Df(ax+by) = af'(ax+by) \qquad D'f(ax+by) = bf'(ax+by)$$

$$D^2 f(ax+by) = a^2 f''(ax+by) \qquad D'^2 f(ax+by) = b^2 f''(ax+by)$$

::: ::

$$D^n f(ax+by) = a^n f^{(n)}(ax+by) \qquad D'^{(n)} f(ax+by) = b^n f^{(n)}(ax+by)$$

F (D, D') $f(ax+by) = \{a_0 D^n + a_1 D^{n-1} D' + ... + a_n D'^n\} f(ax+by)$

$$\{a_0 a^n + a_1 a^{n=1} b + ... + a_n b^n\} f^{(n)}(ax+by)$$

F (D, D) $f(ax+by) = F(a,b) f^{(n)}(ax+by)$

$$\therefore \qquad \frac{1}{F(D, D')} f^{(n)}(ax+by) = \frac{1}{F(a,b)} f(ax+by)$$

or

$$\frac{1}{F(D,D')}f(ax+by)=\frac{1}{F(a,b)}\underbrace{\iint\cdots\cdots\int}_{n\,times} f(t)\,dt\,dt\,dt......where\ t=ax+by \qquad ...(3)$$

provided F $(a,b)\neq 0$

Exceptional case : F *(a, b) = 0*

If F (*a, b) = 0* then (*b*D - *a*D') is a factor of F(D, D')

or F(D, D') = (*b*D -*a*D') G (D, D') ...(1)

P.I. is $$z=\frac{1}{(bD-aD')G(D,D')}f(ax+by) \qquad ...(2)$$

Consider $(bD-aD')z=f(ax+by)$ ...(3

i.e. $bp-aq=f(ax+by)$

or $$\frac{dx}{b}=\frac{dy}{-a}=\frac{dz}{f(ax+by)} \qquad ...(4)$$

From first and second members, we have a $dx+b\ dy=0 \Rightarrow ax+by=c$

(By Integration)

From first and third members, $\frac{dx}{b}=\frac{dz}{f(c)}$ $\Rightarrow z\frac{x}{b}f(c)=\frac{x}{b}f(ax+by)$

By integration)

Thus solution of (3) is $z=\frac{x}{b}f(ax+by)$

$\therefore$ Solution of (2) is $z=\frac{x}{b}\frac{\psi(ax+by)}{G(a,b)}$ ...(5)

where $\psi(ax+by)$ is integration $f(ax+by)$ as many times as is the degree of G (D, D')

Now $F(D,D')=(bD-aD')G(D,D')$

$F'(D,D')=bG\ (D,D')+(bD-aD')\ G'(D,D')$

$\therefore$ $F'(a,b)=bG\ (a,b)$ ...(6)

Putting in (5), we get

$$z=\frac{1}{F(D,D')}f(ax+by)=\frac{x}{F(D,D')}\psi(ax+by)$$

Working Rule - To evalute $\frac{1}{F(D,D')}f(ax+by)$ when F (a, b) = 0

(i) Differentiate F (D, D') w.r.t D partially and multiply the expression by x and replace D and D' by a and b.

(ii) If F' (a, b) is also zero, differentiate again and multiply again by x and replace a, b for D and D' respectively.

(iii) When $F^{(r)}(a,b) \neq 0$ obtain $\psi(ax+by)$ as in the previous article.

Case II. Particular Integral when f (x,y) is a polynomial in x, y

$$P.I = \frac{1}{F(D,D')}f(x,y) = \frac{1}{D^n\left[1+\phi\left(\frac{D}{D'}\right)\right]}f(x,y)$$

$$= \frac{1}{D^n}\left[1+\phi\left(\frac{D'}{D}\right)\right]^{-1} f(x,y)$$

Expanding by Binomial theorem and taking $\frac{1}{D}$ as integral w.r.t. x and D' as derivative w.r.t. y we can find the P.I.

Case III. P.I. when f(x, y) is either sin(mx+ny) or cos(mx + ny)

$$D'^2 \sin(mx+ny) = -n^2 \sin(mx+ny)$$

$$D'^2 \sin(mx+ny) = -n^2 \sin(mx+ny)$$

$$DD' \sin(mx+ny) = -mn \sin(mx+ny)$$

then $F(D^2, DD', D'^2)\sin(mx+ny) = f(-m^2, -mn, -n^2)\sin(mx+ny)$

$$\therefore P.I. = \frac{1}{F(D^2,DD',D'^2)}\sin(mx+ny) = \frac{1}{f(-m^2,-mn,-n^2)}\sin(mx+ny),$$

provided $f(-m^2,-mn,-n^2) \neq 0$

Similarly $\frac{1}{f(D^2,DD',D'^2)}\cos(mx+ny) = \frac{1}{f(-m^2,-mn,-n^2)}\cos(mx+ny)$

Method. *Replace* D^2 *by* $-m^2$, D'^2 *by* $-n^2$ *and* DD' *by* $-nm$ *to get P.I.*

Case IV. P.I. when F(x, y) is of the form e^{ax+by}

Let F(D, D') $\equiv a_0 D^n + a_1 D^{n-1} D' + \ldots + a_n D'^n$

$D^n e^{ax+by} = a^n e^{ax+by}$

$D^{n-1} D' e^{ax+by} = a^{n-1} b\, e^{ax+by}$

$D^{n'} e^{ax+by} = b^n e^{ax+by}$

$$\therefore\ F(D,D')e^{ax+by} = \{a_0 d^n + a_1 d^{n-1} b + \ldots + a_n b^n\} e^{ax+by} = F(a,b) e^{ax+by}$$

$$\frac{1}{F(D,D')} e^{ax+by} = \frac{1}{F(a,b)} e^{ax+by} \quad \textit{provided } F(a,b) \neq 0 .$$

2.1.5 GENERAL METHOD OF FINDING THE PARTICULAR INTEGRAL

Let $$z = \frac{1}{D - mD'} f(x,y) \quad \ldots(1)$$

or $$(D - mD')z = f(x,y)$$

$$p - mq = f(x,y)$$

Its subsidiary equations are:

$$\frac{dx}{1} = \frac{dx}{-m} = \frac{dz}{f(x,y)} \quad \ldots(2)$$

From first and second relations of (2), we have $dy + m\, dx = 0$

$$\therefore \quad y + mx = c \quad \ldots(3)$$

From first and third relations of (2), we have

$$dz = f(x,y)dx = f(x, c - mx)dx \quad \text{[Using (3)]}$$

$$\therefore \quad z = \int f(x, c - mx)dx$$

Thus $$z = \frac{1}{D - mD'} f(x,y) = \int f(x, c - mx)dx$$

where c is to be replaced by $y + mx$ after integration

Example 2. Solve $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y} = \cos x \cos 2y.$

Solution. The symbolic form of the given equation is

$$(D^2 - DD')z = \cos x \cos 2y$$

It auxiliary equation is $m^2 - m = 0 \Rightarrow m = 0, 1$

$\therefore C.F.\ is\ \phi_1(y+0x)+\phi_2(y+x)\ i.e.\ \phi_1(y)+\phi_2(y+x)$

Now P.I. $= \dfrac{1}{D^2 - DD'}\cos x \cos 2y$

$= \dfrac{1}{2}\dfrac{1}{D^2 - DD'}\left[\cos(x+2y)+\cos(x-2y)\right]$

$= \dfrac{1}{2}\left[\dfrac{1}{D^2 - DD'}\cos(x+2y)+\dfrac{1}{D^2 - DD'}\cos(x-2y)\right]$

$\dfrac{1}{2}\left[\dfrac{1}{-1^2-1(-1.2)}\cos(x+2y)+\dfrac{1}{-1^2-(-1.(-2))}\cos(x-2y)\right]$

[obtained by replacing D² by -1² and DD' by -1.2] [obtained by replacing D² by -1² and DD' by -1(-2)]

$= \dfrac{1}{2}\left[\dfrac{1}{-1+2}\cos(x+2y)+\dfrac{1}{-1-2}\cos(x-2y)\right]$

$= \dfrac{1}{2}\left[\cos(x+2y)-\dfrac{1}{3}\cos(x-2y)\right]$

Hence complete solution is

$$z = C.F. + P.I$$

$$= \phi_1(y)+\phi\,(y+x)+\frac{1}{2}\cos(x+2y)-\frac{1}{6}\cos(x-2y)$$

Example 3. $Solve: (D^3 - 7DD'^2 - 6D'^3)z = x^2 + xy^2 + y^3 + \cos(x-y) + e^{x+2y}$

Solution. $A.E. is\ m^3 - 7m - 6 = 0$

or $(m+1)(m^2 - m - 6) = 0$

$m = -1, -2, 3$

$\therefore\ C.F. = \phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x)$ ...(1)

P.I. for $x^2 + xy^2 + y^3$ is $\dfrac{1}{D^3 - y\,DD'^2 - 6D'^3}(x^2 + xy^2 + y^3)$

$$= \frac{1}{D^3\left[1-\frac{7D'^2}{D^2}-\frac{6D'^3}{D^3}\right]}(x^2+xy+y^3)$$

$$= \frac{1}{D^3}\left[1-\frac{7D'^2}{D^2}-\frac{6D'^3}{D^3}\right]^{-1}(x^2+xy+y^3)$$

$$= \frac{1}{D^3}\left[1+\frac{7D'^2}{D^2}+\frac{6D'3}{D^3}\right](x^2+xy+y^3)$$

$$= \frac{1}{D^3}\left[(x^2+xy+y^2)+\frac{7}{D^2}(6y)+\frac{6}{D^3}(6)\right]$$

$$= \frac{1}{D^3}(x^2+xy+y^2)+\frac{42}{D^5}(y)+\frac{36}{D^6} \qquad \left[\because \frac{1}{D} \text{ stands for Integration w.r.t. x}\right]$$

$$= \frac{x^5}{60}+\frac{x^4y}{24}+\frac{x^3y^3}{6}+42y\frac{x^5}{120}+36.\frac{x^6}{720}=\frac{x^5}{60}+\frac{x^4y}{24}+\frac{x^3y^3}{6}+\frac{7}{20}x^5y+\frac{x^6}{20} \qquad \ldots(2)$$

P.I. for $\cos(x-y)$ *is* $\frac{1}{D^3-7DD'^2-6D'^3}\cos(x-y)$

$$= \frac{1}{(D+D')(D+2D')(D-3D')}\cos(x-y)$$

$$= \frac{1}{(D+D')(D+2D')(1+3)}\int \cos t\, dt, \text{ where } t=x-y$$

[By putting D=1, D' =-1 in D-3D']

$$= \frac{1}{4(D+D')(D+2D')}\sin t$$

$$= \frac{1}{4(D+D')(D+2D')}\sin(x-y)$$

$$= \frac{1}{4(D+D')(1-2)}\int \sin t\, dt$$ [By putting D=1, D' = -1 in D+2D')

$$= \frac{-1}{4(D+D')}(-\cos t)$$

$$= \frac{1}{4(D+D')}\cos(x-y)$$ [Here D+D' = 1-1=0 $\therefore$ rule fails]

[$\therefore$ multiply numerator by x and differentiate the denominator]

$$= \frac{x\cos(x-y)}{4}$$

...(3)

P.I. for $e^{x+2y} = \dfrac{1}{D^3 - 7DD'^2 - 6D'^3} e^{x+2y} = \dfrac{1}{1-28-48} = -\dfrac{e^{x+2y}}{75}$ ...(4)

[obtained by putting D=1, D'=2 in the denominator]

$\therefore$ Complete solution is $= z = C.F. + P.I.$

i.e. $z = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x) + \dfrac{x^5}{60} + \dfrac{x^4 y}{24} + \dfrac{x^3 y^3}{6} + \dfrac{7x^5 y}{20} + \dfrac{x^6}{20}$

$$+ \frac{x\cos(x-y)}{4} - \frac{e^{x+2y}}{75}$$

Art 1: Prove that :

$$\frac{1}{f(D,D')} e^{ax+by}.V = e^{ax+by} \frac{1}{f(D+a, D'+b)}.V$$

Proof. $D(e^{ax+by}V) = e^{ax+by}DV + ae^{ax+by}.V = e^{ax+by}(D+a)V$

$$D'(e^{ax+by}V) = e^{ax+by}D'V + be^{ax+by}V = e^{ax+by}(D'+b)V$$

$$\therefore \quad f(D,D')e^{ax+by}.V = e^{ax+by} f(D+a, D'+b)$$

$$\therefore \quad e^{ax+by}V = \frac{1}{f(D,D')} e^{ax+by} f(D+a,\ D'+b)V$$

Let $f(D+a, D'+b)V = Q$ then $V = \dfrac{1}{f(D+a,\ D'+b)} Q$

Then from (1), we have

$$e^{ax+by} \frac{1}{f(D+a,\ D'+b)} Q = \frac{1}{f(D,D')} e^{ax+by}.Q$$

Replacing Q by V and interchanging the sides we get,

$$\frac{1}{f(D,D')} e^{ax+by} V = e^{ax+by} \frac{1}{f(D+a,\ D'+b)} V$$

2.1.6 NON- HOMOGENEOU'S LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Definition: If the partial derivatives occuring in the equation are not of same order, then it is called **Non-homogenous linear partial differential equation with constant coefficients.**

e.g. (i) $3\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial^2 z}{\partial y^2}-5\frac{\partial z}{\partial y}=e^{x+4y}$ (ii) $(D_x^2-3D_y+5)(D_x-D_y)=(x+3y)e^{-3x+y}$

Complementary function:

Case I. When $f(D_x,D_y)$ can be factorized in terms of linear factors in D_x *and* D_y

We know, the solutions of $f(D_x,D_y)z=0$ ....(i)

are, C.F.'s of $f(D_x,D_y)z=F(x,y)$ ....(ii)

(subcase i)

Now if $f(D_x,D_y)$ can be factorized into distinct linear factors

Let $f(D_x,D_y)z=0$

becomes $(a_1D_x+\beta_1 D_y+\gamma_1)(a_2 D_x+\beta_2 D_y+\gamma_2)$.....

$$(a_i D_x+\beta_i D_y+\gamma_i)....(a_n D_x+\beta_n D_y+\gamma_n)z=0 \qquad ...(iii)$$

Clearly any solution of $(a_i D_x+\beta_i D_y+\gamma_i)z=0$, $1\le i\le n$ ... (iv)

is the solution of $f(D_x,D_y)z=0$

Equation (iv) $\Rightarrow a_i(D_xz)+\beta_i(D_yz)=-\gamma_i z$ or $a_i p+\beta_i q=-\gamma_i z$

so the lagrange's A.E.'s are $\frac{dx}{a_i}=\frac{dy}{\beta_i}=\frac{dz}{-\gamma_i z}$ ...(v)

Taking first two members of (*v*)

We get $\beta_i\,dx=a_i\,dy$

Integrating $\beta_i x-a_i y+c$

where c is constant of integration

$\Rightarrow \beta_i x-a_i y=c$ ...(vi)

Taking first and last member of (*v*)

we get $\frac{dz}{z}=\frac{\gamma_i}{a_i}dx$ $\Rightarrow \log z=-\frac{\gamma_i}{a_i}x+\log d$,

where d>0 is constant of integration

$$\Rightarrow \log z-\log d=-\frac{\gamma_i}{a_i}x \quad \Rightarrow \quad \log\frac{z}{d}=-\frac{\gamma_i}{a_i}x$$

$$\Rightarrow \frac{z}{d}=e^{-\frac{\gamma_i}{(a_i)}x} \quad \Rightarrow \quad z=d\,e^{-\frac{\gamma_i}{(a_i)}x}$$

$\Rightarrow \ z = e^{-\frac{\gamma_i}{(a_i)}x} \ \phi_i(c)$ (Put $d = \phi_i(c)$)

$\Rightarrow \ z = e^{-\frac{\gamma_i}{(a_i)}x} \ \phi_i(\beta_i x - a_i y)$ (using (vi)
for $a_i \neq 0$

If $a_i = 0$ and $\beta_i \neq 0,$ then taking last two members of *(v)*

We have $\dfrac{dy}{\beta_i} = \dfrac{dz}{-\gamma_i z}$ $\Rightarrow$ $\dfrac{dz}{z} = -\dfrac{\gamma_i}{\beta_i} dy$

Integrating log $z = -\dfrac{\gamma_i}{\beta_i} y + \log \lambda$ $\Rightarrow$ $\log z - \log \lambda = -\dfrac{\gamma_i}{\beta_i} y$

$\Rightarrow \ \log \dfrac{z}{\lambda} = -\dfrac{\gamma_i}{\beta_i} y$

$\therefore \ \dfrac{z}{\lambda} = e^{-\frac{\gamma_i}{\beta_i}y}$ $\Rightarrow z = \lambda \ e^{-\frac{\gamma_i}{\beta_i}y}$

$\Rightarrow \ z = \phi_i(c) \ e^{-\frac{\gamma_i}{\beta_i}y}$ (Putting $\lambda = \psi_i(c)$)

$$z = e^{-\frac{\gamma_i}{\beta_i}y} \ \psi(\beta_i x - a_i y)$$

which is solution of $f(D_x, D_y)z = 0$

so that for each factor $(1 \leq i \leq n)$, we got a solution and general solution is

$$z = e^{-\frac{\gamma_1}{a_1}x} \ \phi_1(\beta_1 x - a_1 y) + e^{-\frac{\gamma_2}{a_2}x} \ \phi_2(\beta_2 x - a_2 y) + ... + e^{-\frac{\gamma_n}{a_n}} \phi(\beta_n x - a_n y)$$

assuming $a_1, ... a_n$ are all non zero

...(vii)

or $z = e^{-\frac{\gamma_1}{\beta_1}x} \ \psi_1(\beta_1 x - a_1 y) + e^{-\frac{\gamma_2}{\beta_2}} \psi_2(\beta_2 x - a_2 y) + ... + e^{-\frac{\gamma_n}{\beta_n}x} \ \psi_n(\beta_n x - a_n y)$ ...(viii)

assuming $\beta_1, \beta_2 .... \beta_n$ are non zero

WHEN FACTORS ARE REPEATED *(subcase ii)*

If some factors of $f(D_x, D_y)$ are repeated then let us suppose that first two factors are same *i.e.,* $a_2 D_x + \beta_2 D_y + \gamma_2 = a_1 D_x + \beta_1 D_y + \gamma_1$

Then, general solution is given by

$$z = e^{-\frac{\gamma_1}{a_1}x}(\phi_1+\phi_2)(\beta_1 x - a_1 y) + e^{-\frac{\gamma_3}{a_3}x}\phi_3(\beta_3 x - a_3 y) + \ldots + e^{-\frac{\gamma_n}{a_n}x}\phi_n(\beta_n x - a_n y)$$

But it contains $n-1$ arbitrary functions so it cannot be general solution.

To find general solution

Now relation *(iii)* becomes

$(a_1 D_x + \beta_1 D_y + \gamma_1)^2 (a_3 D_x + \beta_3 D_y + \gamma_3).......(a_n D_x + \beta_n D_y + \gamma_n)z = 0$ ....(ix)

Clearly For $3 \le i \le n$, the solution of $(a_i\ D_x + \beta_i D_y + \gamma_i)z = 0$ is also solution of *(ix)* and hence of *(iii)*

$$\therefore \quad z = e^{-\frac{\gamma_i}{a_i}x}\phi_i(\beta_i x - a_i y)$$

where $3 \le i \le n$ *if* $a_i \ne 0$ *is a solution of* (ix) *and hence of* (iii) *or*

$z = e^{-\frac{\gamma_i}{\beta_i}y}\psi(\beta_i\ x - a_i y)$ where $3 \le i \le n$ if $\beta_i \ne 0$ is a solution of *(ix)* and hence of *(iii)*

Hence sum of these solutions is a solution of *(iii)* ...*(x)*

Now solution of $(a_1 D_x + \beta_1 D_y + \gamma_1)^2 z = 0$ is a solution of *(ix)* and hence of *(iii)*

$$\Rightarrow (a_1\ D_x + \beta_1 D_y + \gamma_1)(a_1 D_x + \beta_1 D_y + \gamma_1)z = 0$$

$$\Rightarrow (a_1 D_x + \beta_1 D_y + \gamma_1)u = 0$$

where $u = (a_1 D_x + \beta_1 D_y + \gamma_1)z$

$\Rightarrow u = e^{-\frac{\gamma_1}{a_1}x}\phi(\beta_1 x - a_1 y)$ if $a_1 \ne 0$ or $u = e^{-\frac{\gamma_1}{\beta_1}y}\psi(\beta_1 x - a_1 y)$ if $\beta_1 \ne 0$.

Let $a_1 \ne 0$

$\therefore \quad (a_1 D_x + \beta_1 D_y + \gamma_1)z = u \qquad \Rightarrow (a_1 D_x + \beta_1 D_y + \gamma_1)z = e^{-\frac{\gamma_1}{a_1}x}\phi(\beta_1 x - a_1 y)z$

$$\Rightarrow a_1 p + \beta_1 q = e^{-\frac{\gamma_1}{a_1}x}\phi(\beta_1 x - a_1 y) - \gamma_1 z$$

which is lagrange's linear equation

$\therefore$ Lagrange's A. Equations are

$$\frac{dx}{a_1}=\frac{dy}{\beta_1}=\frac{dz}{e^{-\frac{\gamma_1}{a_1}x}\phi(\beta_1 x-a_1 y)-\gamma_1 z}$$

Taking first two members

We get $\beta_1\, dx = a_1\, dy$ after Integrating, we get $\beta_1 x - a_1 y = c$

Now taking first and third members

We get $\dfrac{dx}{a_1}=\dfrac{dz}{e^{-\frac{\gamma_1}{a_1}x}\phi(c)-\gamma_1 z} \Rightarrow \dfrac{dz}{dx}=\dfrac{1}{a_1}\left(e^{-\frac{\gamma_1}{a_1}x}\phi(c)-\gamma_1 z\right)$

$$\Rightarrow \frac{dz}{dx}+\frac{\gamma_1}{a_1}z=\frac{1}{a_1}e^{-\frac{\gamma_1}{a_1}x}\phi(c)$$

which is linear differential equation

$\therefore$ I.F. (Integrating factor) = $e^{\int\frac{\gamma_1}{a_1}dx}=e^{\frac{\gamma_1}{a_1}x}$

so sol of this equation is

$$z\, e^{\frac{\gamma_1}{a_1}x}=\int e^{\frac{\gamma_1}{a_1}x}\left(\frac{1}{a}e^{-\frac{\gamma_1}{a_1}x}\phi(c)\right)dx+d=\frac{1}{a}\phi(c)\int 1.dx+d=\frac{\phi(c)}{a}x+d$$

Let $d=\phi_1(c)$ and $\dfrac{1}{a}\phi(c)=\phi_1(c)$

$$\therefore \quad z\, e^{\frac{\gamma_1}{a_1}x}=\phi_1(c)x+\phi_2(c)=x\phi_1(\beta_1 x-a_1 y)+\phi_2(\beta_1 x-a_1 y)$$

$$\Rightarrow \quad z=e^{-\frac{\gamma_1}{a_1}x}(x\phi_1(\beta_1 x-a_1 y)+\phi_2(\beta_1 x-a_1 y)) \qquad \text{.....(xi)}$$

Combining (x) and (xi)

$$z=e^{-\frac{\gamma_1}{a_1}x}(x\phi_1(\beta x-a_1 y)+\phi_2(\beta_1 x-a_1 y)+\phi_3(\beta_1 x-a_1 y)+.....+\phi_n(\beta_n x-a_1 y)$$

is general sol of *(ix)* and hence of *(iii)* ($\because$ it contains n arbitrary functions) correct

Similarly if $\beta_1 \neq 0$

Let $(a_1 D_x + \beta_1 D_y + \gamma_1)z = u = e^{-\frac{\gamma_1}{\beta_1}y}\ \psi(\beta_1 x - a_1 y)$

$$\Rightarrow \quad a_1 p + \beta_1 q = e^{-\frac{\gamma_1}{\beta_1}y}\ \psi\ (\beta_1 x - a_1 y) - \gamma_1 z$$

which is Lagrange's linear equation

its A.E.'s are

$$\frac{dx}{a_1} = \frac{dy}{\beta_1} = \frac{dz}{e^{-\frac{\gamma_1}{\beta_1}y}\psi\ (\beta_1 x - a_1 y) - \gamma_1 z}$$ taking first two members

we get $\beta_1\, dx = a_1\ dy \ \Rightarrow\ \beta_1 x - a_1\ y = c'$

Now taking IInd and IIIrd members

we get $\dfrac{dy}{\beta_1} = \dfrac{dz}{e^{-\frac{\gamma_1}{\beta_1}y}\ \psi(c') - \gamma_1 z} \quad \Rightarrow \quad \dfrac{dz}{dy} = \dfrac{1}{\beta_1}\left(e^{-\frac{\gamma_1}{\beta_1}y}\ \psi(c') - \gamma_1 z\right)$

$$\Rightarrow \quad \frac{dz}{dy} = \frac{\gamma_1}{\beta_1} z = \frac{1}{\beta_1} e^{-\frac{\gamma_1}{\beta_1}y}\ \psi(c')$$

which is linear equation.

Its I.F. = $e^{\int \frac{\gamma_1}{\beta_1}dy} = e^{\frac{\gamma_1}{\beta_1}y}$

$\therefore$ sol is $z\, e^{\frac{\gamma_1}{\beta_1}y} = \int \dfrac{1}{\beta} e^{-\frac{\gamma_1}{\beta_1}y}\ \psi(c')\ e^{\frac{\gamma_1}{\beta_1}y} dy + d' = \dfrac{1}{\beta}\psi(c')\int dy + d' = \dfrac{\psi(c')}{\beta} y + d'$

Let $d' = \psi_2(c')$ and $\dfrac{\psi(c')}{\beta} = \psi_1(c')$

$$\therefore \quad z\, e^{\frac{\gamma_1}{\beta_1}y} = \psi_1(c')y + \psi_2(c') = y\psi_1(\beta_1 x - a_1 y) + \psi_2(\beta_1 x - a_1 y)$$

$$\Rightarrow \ z = e^{-\frac{\gamma_1}{\beta_1}}(y\psi_1(\beta_1 x - a_1 y) + \psi_2(\beta_1 x - a_1 y) \qquad \ldots(xii)$$

Combining *(x)* and *(xii)*

$$z = e^{-\frac{\gamma_1}{\beta_1}y}(y\psi_1(\beta_1 x - a_1 y) + \psi_2(\beta_1 x - a_1 y) + \psi_3(\beta_3 x - a_3 y) + ... + \psi_n(\beta_n x - a_n y))$$

is general sol of *(ix)* and hence of *(iii)* ($\because$ it contains n arbitrary functions)

Note: If $a_1 D_x + \beta_1 D_y + \gamma_1$ is repeaed K times, Then general solution is

$$z = e^{-\frac{\gamma_1}{a_1}x}\{(x^{K-1}\psi_1(\beta_1 x - a_1 y) + x^{K-2}\psi_2(\beta_1 x - a_1 y) + ... + x\psi_K(\beta_1 x - a_1 y)\}$$

$$+\psi_{K+1}(\beta_{K+1}\ x - a_{K+1}y) + ... + \psi_n(\beta_n x - a_n y))$$

OR

$$z = e^{-\frac{\gamma_1}{\beta_1}y}(y^{K-1}\psi_1(\beta_1 x - a_1 y) + y^{K-2}\psi_2(\beta_1 x - a_1 y) + ... + y\psi_K(\beta_1 x - a_1 y)$$

$$+\psi_{K+1}(\beta_{K+1}x - a_{K+1}y) + ... + \psi_n(\beta_n x - a_n y))$$

Case *(ii)*. When $f(D_x, D_y)$ can not be factorized into linear factors

Let $z = Ae^{ax+\beta y}$, where a, β, A are constants ...(i)

be C.F. of $f(D_x, D_y)z = 0$ ...(ii)

Here $D_x z = A a e^{ax+\beta y}$	and $D_y z = A\ \beta\ e^{ax+\beta y}$
$D_x^2 z = A a^2 e^{ax+\beta y}$	$D_y^2 z = A\beta^2 e^{ax+\beta y}$
....	....
$D_x^l z = A a^l\ e^{ax+\beta y}$	$D_y^m z = A\beta^m\ e^{ax+\beta y}$

$\therefore\ D_x^l D_y^m x\ = D_x^l\ D_y^m\ (A\ e^{ax+\beta y)}$

= $D_x^l(A\ \beta^m\ e^{ax+\beta y}) = A\beta^m\ D_x^l(e^{ax+\beta y}) = A\beta^m(a^l\ e^{ax+\beta y}) = A\beta^m\ a^l\ e^{ax+\beta y}$

$\Rightarrow\ f(D_x, D_y)z = f(a,\beta)(Ae^{ax+\beta y})$

so that $z = A\ e^{ax+\beta y}$ is sol of *(ii)* if $f(a,\beta) = 0$

where A is arbitrary constant. For any value of a, we can find β such that $f(a,\beta) = 0$ or for any value of β, we can find a such that $f(a,\beta) = 0$

$\therefore$ there are infinite pairs (a_i, β_i) such that $f(a_i, \beta_i) = 0$.

Therefore $z = \Sigma A_i e^{a_i x + \beta_i y}$

where $f(a_i, \beta_i) = 0$ is general sol.

Example 4. Find the general solution of $r-s-2t+2p+2q=0$

Sol. We have $r-s-2t+2p+2q=0$

i.e. $\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial x\,\partial y}-2\frac{\partial^2 z}{\partial y^2}+2\frac{\partial z}{\partial x}+2\frac{\partial z}{\partial y}=0$

In symbol form, $(D_x^2-D_x D_y-2D_y^2+2D_x+2D_y)z=0$

$\Rightarrow (D_x^2-2D_x D_y+D_x D_y-2D_y^2+2(D_x+D_y))z=0$

$\Rightarrow (D_x(D_x-2D_y)+D_y(D_x-2D_y)+2(D_x+D_y))z=0$

$\Rightarrow ((D_x+D_y)(D_x-2D_y)+2(D_x+D_y))z=0 \;\; \Rightarrow (D_x+D_y)(D_x-2D_y+2)z=0$

$\Rightarrow (1\cdot D_x+1\cdot D_y+0)(1\cdot D_x-2\cdot D_y+2)z=0$

$\therefore$ The general solution is given by

(Here $a_1=1,\ \beta_1=1, \gamma_1=0\ a_2=1, \beta_2=-2, \gamma_2=2$)

$z = e^{-\frac{0}{1}x}\phi_1(1\cdot y-1\cdot x)+e^{-\frac{2}{1}x}\phi_2(-2\cdot y-1\cdot x)$

$z=\phi_1(y-x)+e^{-2x}\phi_2(-2y-x)$ where ϕ_1, ϕ_2 are arbitrary functions.

2.1.7 Exercise

Solve the following differential equations:

(i) $(D_x^2+2D_xD_y+D_y^2)z=e^{2x+3y}$

(ii) $3\frac{\partial^2 z}{\partial x^2}-10\frac{\partial^2 z}{\partial x\,\partial y}+3\frac{\partial^2 z}{\partial y^2}=10\sin(2x+y)$

(iii) $r+t=\cos a x \cos \beta y$

(iv) $r+3s+2t=2x+3y$

$r+s-6t=x^2\sin(x+y)$

(v) $(D_x^2 + 4D_xD_y - 2D_y^2)z = \sqrt{x+2y}$

(vi) $(D_x^3 - 4D_x^2\ D_y + 4D_xD_y^2)z = 4\sin(2x+y)$

(vi) $(D_x^2 - 4D_xD_y + 4D_y^2)z = \tan\ (y+2x)$

(vii) Solve the following partial diff. equation

(viii) $r + s + p + q + z = 0$

LESSON NO. 2.2 AUTHOR: DR.RAJ KUMAR

PARTIAL DIFFERENTIAL EQUATIONS -V

2.2.1 METHODS TO FIND P.I. OF NON-HOMOGENOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT CEFFICIENTS

Here we are discussing some rules (methods) to find P.I. of

$$f(D_x, D_y)z = F(x, y)$$

TYPE-I When $F(x, y) = e^{ax+\beta y}$ form

Then P.I. = $\dfrac{1}{f(D_x, D_y)} e^{ax+\beta y} = \dfrac{1}{f(a, \beta)} e^{ax+\beta y}$

if $f(a, \beta) \neq 0$ *i.e.,* change D_x by a and D_y by β.

TYPE-II When $F(x, y) = \sin(ax+\beta y)$ or $\cos(ax+\beta y)$

Then P.I. = $\dfrac{1}{f(D_x, D_y)} \sin(ax+\beta y)$

or $\cos(ax+\beta y) = \dfrac{1}{change\ D_z^2\ by -a^2, D_y^2\ by -\beta^2\ and\ D_x D_y\ by -a\beta}$

$\sin(ax+\beta y$ or $\cos(ax+\beta y)$ if denominator $\neq 0$

TYPE-III When $F(x,y) = x^l y^m$

Then P.I. = $\dfrac{1}{f(D_x,D_y)} x^l y^m (f(D_x,D_y))^{-1}(x^l y^m)$

Here expand $(f(D_x,D_y))^{-1}$ in as cending powers of

$\dfrac{D_x}{D_y}$ or $\dfrac{D_y}{D_x}$.

TYPE-IV When $F(x,y) = e^{ax+\beta y} U(x,y)$

Then P.I. = $\dfrac{1}{f(D_x,D_y)} e^{ax+\beta y} U(x,y) = e^{ax+\beta y} \dfrac{1}{f(D_x+a, D_y+\beta} U(x,y)$

Note: If $f(a,\beta)=0$ when $F(x,y) = e^{ax+\beta y}$

Then $\dfrac{1}{f(D_x,D_y)} e^{ax+\beta y} = \dfrac{1}{f(D_x,D_y)} (e^{ax+\beta y}\ 1)$ Here U(x, y)=1

= $e^{ax+\beta y} \dfrac{1}{f(D_x+a,\ D_y+\beta)}(1)$.

Example 1. Find the general solution of $r - t - 3\ p + 3\ q = x\ y + e^{x+2y}$.

Sol. We have $r - t - 3p + 3q = xy + e^{x+2y}$

$\Rightarrow (D_x^2 - D_y^2 - 3D_x + 3D_y)z = xy + e^{x+2y}$

$\Rightarrow (D_x - D_y)(D_x + D_y - 3)z = xy + e^{x+2y}$

$\therefore$ C.F. is given by

$$z = e^{-\frac{0}{1}x}\phi_1(-1\cdot x - 1\cdot y) + e^{-\frac{3}{1}x}\phi_2(-1.x - 1.y)$$

$$= \phi_1(-x-y) + e^{3x}\phi_2(-x-y)$$

$$= \psi_1(x+y) + e^{3x}\psi_2(x+y)$$

where ψ_1, ψ_2 are arbitrary functions

And P.I. = $\dfrac{1}{(D_x - D_y)\,(D_x + D_y - 3)}(x\,y + e^{x+2y})$

$$= \frac{1}{(D_x - D_y)(D_x + D_y - 3}x\,y + \frac{1}{(D_x - D_y)(D_x + D_y - 3)}e^{x+2y} \qquad \ldots(1)$$

Here $\dfrac{1}{(D_x - D_y)(D_x + D_y - 3)}x\,y$

$$= \frac{1}{D_x\left(1 - \frac{D_y}{D_x}\right)(-3)\left(1 - \frac{D_x + D_y}{3}\right)}(x\,y)$$

$$= \frac{1}{-3\,D_x}\left(1 - \frac{D_y}{D_x}\right)^{-1}\left(1 - \frac{D_x + D_y}{3}\right)^{-1}(x\,y)$$

$$= -\frac{1}{3}\frac{1}{D_x}\left(1 + \frac{D_y}{D_x}\right)\left(1 + \frac{1}{3}(D_x + D_y) + \frac{1}{9}(D_x + D_y)^2\right)(x\ y)$$

$$= -\frac{1}{3}\frac{1}{D_x}\left(1 + \frac{D_y}{D_x}\right)\left(x\ y + \frac{1}{3}(D_x + D_y)(x\ y) + \frac{2}{9}D_x\,D_y(x\ y)\right) + 0$$

$$= -\frac{1}{3}\frac{1}{D_x}\left(1 + \frac{D_y}{D_x}\right)\left(x\ y + \frac{y + x}{3} + \frac{2}{9}\right)$$

$$= -\frac{1}{3}\frac{1}{D_x}\left(\left(x\,y+\frac{y+x}{3}\right)+\frac{2}{9}+\frac{1}{D_x}D_y\left(x\,y+\frac{y+x}{3}\right)\right)$$

$$= -\frac{1}{3}\frac{1}{D_x}\left(x\,y+\frac{y+x}{3}+\frac{2}{9}+\frac{1}{D_x}\left(x+\frac{1}{3}\right)\right)$$

$$= -\frac{1}{3}\frac{1}{D_x}\left(x\,y+\frac{x+y}{3}+\frac{2}{9}+\frac{x^2}{2}+\frac{x}{3}\right)$$

$$= -\frac{1}{3}\left(\frac{x^2}{2}y+\frac{x^2}{6}+\frac{2}{9}x+\frac{x\,y}{3}+\frac{x^3}{6}+\frac{x^2}{6}\right)$$

$$= -\frac{x^2 y}{6}-\frac{x^2}{18}-\frac{2}{27}x-\frac{x\,y}{9}-\frac{x^3}{18}-\frac{x^2}{18}$$

$$= -\frac{x^2 y}{6}-\frac{x^2}{9}-\frac{x\,y}{9}-\frac{x^3}{18}-\frac{2}{27}x\,.$$

And $\dfrac{1}{(D_x+D_y-3)(D_x-D_y)}e^{x+2y}$

$$= \frac{1}{(D_x+D_y-3)(D_x-D_y)}e^{x+2y}$$

$$= \frac{1}{D_x+D_y-3}\left(\frac{1}{-1}\right)e^{x+2y}$$ (It is case of failure)

$$= -e^{x+2y}\frac{1}{D_x+1+D_y+2-3}(1)$$

$$= -e^{x+2y}(D_x+D_y)^{-1}(1)$$

$$= -e^{x+2y}\frac{1}{D_x}\left(1+\frac{D_y}{D_x}\right)^{-1}(1)$$

$$= -e^{x+2y}\frac{1}{D_x}\left(1-\frac{D_y}{D_x}\right)(1)$$

$$= -e^{x+2y}\left(\frac{1}{D_x}(1-0)\right)$$

$$= -e^{x+2y}(x)$$

Put in *(i)*

$$\text{P.I.} = -\frac{x^2y}{6}-\frac{x^2}{9}-\frac{xy}{9}-\frac{x^3}{18}-\frac{2}{27}x-xe^{x+2y}$$

Hence the general solution is

$$z=\psi_1(x+y)+e^{3x}\psi_2(x+y)-\frac{x^2y}{6}-\frac{x^2}{9}-\frac{x^3}{18}-\frac{2x}{27}-x\,e^{x+2y}$$. correct

2.2.2 HEAT, WAVE AND LAPLACE'S EQUATION

Physical applications of partial differential equations involve the setting up and solution of differential equations which involve physical problems like waves on string, heat diffusion in metal bar etc. The differential equation together with these boundary conditions, constitute a boundary value problem.

A number of problems in engineerng give rise to the following well known partial differential equations:

(i) Wave equation : $\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}$.

(ii) One dimensional heat flow equation : $\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}$.

(iii) The dimensional heat flow equation which in steady becomes the two dimensional Laplace's equation :

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

(iv) Transmission line equations.

(v) Vibrating membrane. Two dimensional wave equations.

(vi) Laplace's equation in three dimensions.

Beside these, the partial differential equations frequently occur in the theory of elasticity.

2.2.3 Method of Separation of Variables

The method is used in solving second order linear partial differential equations *i.e.* of the form $p\,P + q\,Q + r\,R + s\,S + t\,T = w$ ...(1)

where P, Q, R, S, T, W are functions of *x, y* only

Put z = X*(x)* Y *(y)* ...(2)

Then equation (1) becomes

$$\frac{1}{X} f(D)X = \frac{1}{Y} g(D')Y \qquad ...(3)$$

It can be solved by method of separation of variables. In this case, equation (1) is known as separable in *x, y.*

Now equation (3) is possible only when each side is constant λ (say) because L.H.S. is a function of *x* only and R.H.S. is a function of *y* only

Then X and Y are given by $f(D)X = \lambda X$ ...(4)

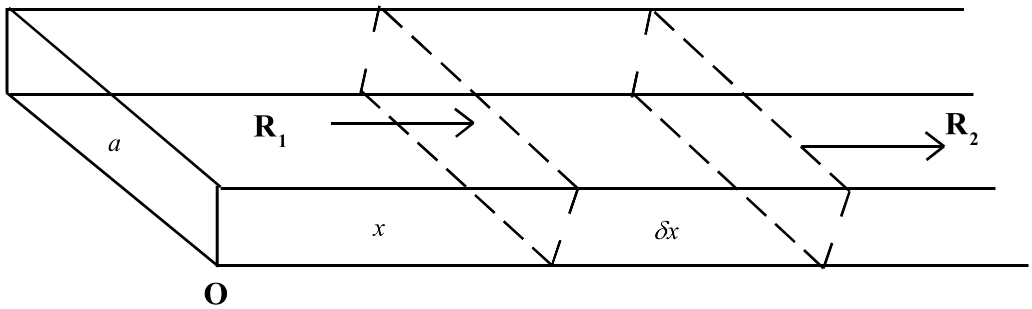
and $g(D')Y = \lambda Y$ ...(5)

Equations (4) and (5) are ordinary linear second order differential equations and can be solved.

Knowing X, Y the solution of (1) is given by (2)

2.2.4 Heat Diffusion Equation

Consider a homogenous bar of uniform cross-section *a*. Suppose that the sides are covered with a material imperivious to heat so that the stream lines of heat-flow are all parallel and perpendicular to the area *a*. Take one end of the bar as the origin and the direction of flow as the positive *x*-axis. Let *p* be the density, *s* the specific heat and *k* the thermal conductivity.



Let $u(x,t)$ be the temperature at a distance x from 0. If δu be the temperature change in a slab of thickness δx of the bar then the quantity of heat in this slab.

$$= s\,p\,a\,\delta x\,\delta u$$

Hence the rate of increase of heat in this slab is

$$s\,p\,a\,\delta x\frac{\partial u}{\partial t} = R_1 - R_2,$$

where R_1 and R_2 are respectively the rate of inflow and out flow of heat. These rates are given by

$$R_1 = -k\,a\left(\frac{\partial u}{\partial x}\right)_x \; and \; R_2 = -ka\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{k}{sp}\left\{\frac{(\partial u/\partial x)_{x+\delta x} - (\partial u/\partial x)_x}{\delta x}\right\}$$

Writing $k/sp = c^2$, called the diffusivity of the substance and taking the limit as $\delta x \to 0$, we get

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$$

This is one dimensional heat-flow (diffusion) equation.

Example 2. Find the solution of $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ satisfying boundary conditins $u(0,t) = 0 = u(l,t)$ and $u(x,0) = (l-x)x, 0 \le x \le l.$

Sol. by using the method of variable separation we get,

$$u(x,t)=\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} e^{-\frac{c^2n^2\pi^2}{l^2}t} \quad ....(1) \qquad \text{satisfying } u(0,t)=0=u(l,t)$$

Using intitial condition $u = (l - x)x$ *when* $t = 0$

we get $(l-x)x=\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$

$$\Rightarrow \quad a_n=\frac{2}{l}\int_0^l (l-x)x \sin\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\left(\left(lx-x^2\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\Bigg\}_0^l-\int_0^l (l-2x)\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right)$$

$$=\frac{2}{l}\left((0-0)+\frac{l}{n\pi}\int_0^l (l-2x)\cos\frac{n\pi x}{l}dx\right)$$

$$=\frac{2}{l}\times\frac{l}{n\pi}\left((l-2x)\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\Bigg\}_0^l-\int_0^l(-2)\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}dx\right)$$

$$=\frac{2}{n\pi}\left(\frac{l}{n\pi}(-l)\sin n\pi-\frac{1}{nx}(l)\sin 0+\frac{21}{n\pi}\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^l\right)$$

$$=\frac{2}{n\pi}\left(0-0+\frac{2l^2}{n^2\pi^2}(-\cos n\pi+\cos 0)\right)$$

$$=\frac{4l^2}{n^3\pi^3}\left(-(-1)^n+1\right)=\frac{4l^2(1-(-1)^n)}{n^3\pi^3}$$

Putting in (1), we, get

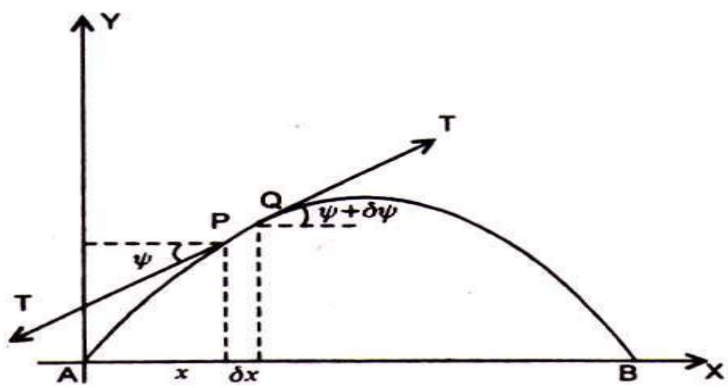
$$u(x,t)=\frac{4l^2}{\pi^3}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^3}\right)\sin\frac{n\pi x}{l}e^{-\frac{c^2n^2\pi^3}{l^2}t}$$ is the required solution.

2.2.5 Vibrations of Stretched String-Wave Equation

Consider a tightly stretched elastic string of length l and fixed ends A and B subjected to constant tension T. The tension T will be considered to be large as compared to the weight of the string so that the effects of gravity are negligible.

Let the string be released from rest and and allowed to vibrate. We shall study the subsequent motion of string, with no external forces acting on it, assuming that each point of the string makes small vibrations at right angles to the equilibrium position AB of the string entirely in one plane.

Take the end A as the origin, AB as the x-axis and AY perpendicular to it as the y-axis; so that the motion takes place entirely in the xy-plane. The above fig shows the string in the position APB at time t. Consider the motion of the element



PQ of the string between its points P (x, y) and Q $(x+\delta x, y+\delta y)$, where the tangents make angles ψ and $\psi+\delta\psi$ with x-axis. Clearly the element is moving upwards with the acceleration $\partial^2 y/\partial t^2$. Also the vertical component of the force acting on this element.

$$= T\sin(\psi+\delta\psi) - T\sin\psi$$
$$= T\left[\sin(\psi+\delta\psi) - \sin\psi\right]$$
$$= T\left[\tan(\psi+\delta\psi) - \tan\psi\right] \text{since } \psi \text{ is small}$$
$$= T\left[\left\{\frac{\partial y}{\partial x}\right\}_{x\delta+x} - \left\{\frac{\partial y}{\partial x}\right\}_{x}\right]$$

If m be the mass per unit length of the string, then by Newton's second law of motion, we have

mass X acceleration = Net force

$$(m\delta x)\cdot\frac{\partial^2 y}{\partial t^2} = T\left[\left\{\frac{\partial y}{\partial x}\right\}_{x+\delta x} - \left\{\frac{\partial y}{\partial x}\right\}_{x}\right]$$

i.e. $$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\left[\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_{x}}{\delta x}\right]$$

Taking limits as $Q \to P, i.e.\ \delta x \to 0,$ we have

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}, \qquad .....(1) \qquad \text{where } c^2 = \frac{T}{m}$$

This is the partial differential equation giving the transverse vibrations of the string. It is also called the one dimension wave equation.

(II) Solution of the wave equation :

Assume that a solution of (1) is of the form

$$y = X(x)T(t) \qquad ...(2)$$

Where X is a function of x a nd T is a function of t only.

Then $\frac{\partial^2 y}{\partial t^2} = X \cdot T''$ and $\frac{\partial^2 y}{\partial t^2} = X''T$

Substituting these in equation (1), we get

$$X\,T'' = c^2\ X''T \qquad \text{i.e.} \qquad \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} \qquad .....(3)$$

Clearly the left hand side of (3) is a function x only and the right side is a function of t only. Since x and t are independent variables, (2) can hold good if each side is equal to a constant k (say). Then (3) leads to the ordinary differential equations:

$$\frac{d^2 X}{dx^2} - kX = 0 \qquad ...(4)$$

and $$\frac{d^2 T}{dt^2} - kc^2 T = 0 \qquad ...(5)$$

Solving (4) and (5), we get

(i) When k is positive and $k = p^2$, say

$X = c_1 e^{px} + c_2 e^{-px}$ and $T = c_3 e^{cpt} + c_4 e^{-cpt}$

(ii) When k is negative and $k = -p^2$, say

$X = c_5 \cos px + c_6 \sin px$ and $T = c_7 \cos cpt + c_8 \sin cpt.$

(iii) When k is zero

$X = c_9 x + c_{10}$ and $T = c_{11} t + c_{12}$

Thus the various possible solutions of wave-equation (1) are

$$y = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt}) \quad \text{...(6)}$$

$$y = (c_5 \cos px + c_6 \sin px)\,(c_7 \cos cpt + c_8 \sin cpt) \quad \text{...(7)}$$

$$y = (c_9 x + c_{10})(c_{11} t + c_{12}) \quad \text{...(8)}$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we will be dealing with problems on vibrations, y must be a periodic function of x and t. Hence their solution must involve trigonometric terms. Accordingly the solution given by (7) *i.e.* of the form.

$$y = (c_1 \cos p + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt) \quad \text{...(9)}$$

is the only suitable solution of the wave equation.

Example 3. A string fixed at ends distant π is initially in the shape of the string $y(x,0) = \dfrac{x(\pi - x)}{\pi}$ and is released from rest. Find $y(x,t)$.

Sol. The displacement $y(x,t)$ is solution of wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{...(1)}$$

with boundary conditions $y(0,t) = 0 = y(\pi,t)$ for all $t \geq 0$ and initial displacement is given by

$$y(x,0) = f(x) = \frac{x(\pi - x)}{\pi} \text{ for } 0 \leq x \leq \pi$$

and initial velocity = 0

by using method of separation variable we get,

Here $l=\pi$

We have

$$y(x,t)=\sum_{n=1}^{\infty}E_n\cos\left(\frac{n\pi ct}{\pi}\right)\sin\left(\frac{n\pi x}{\pi}\right)$$

$$=\sum_{n=1}^{\infty}E_n\cos(nct)\sin(nx)$$

where $E_n=\dfrac{2}{\pi}\displaystyle\int_0^{\pi}f(x)\sin\left(\frac{n\pi x}{\pi}\right)dx$

$$=\frac{2}{\pi}\int_0^{\pi}\frac{x(\pi-x)}{\pi}\sin nx\,dx$$

$$=\frac{2}{\pi^2}\left(\left.x(\pi-x)\left(\frac{-\cos nx}{n}\right)\right\}_0^{\pi}-\int_0^{\pi}(\pi-2x)\left(-\frac{\cos nx}{n}\right)dx\right)$$

$$=\frac{2}{\pi^2}\left((0-0)+\frac{1}{n}\int_0^{\pi}(\pi-2x)\cos nx\,dx\right)$$

$$=\frac{2}{n\pi^2}\left(\left.(\pi-2x)\frac{\sin nx}{n}\right\}_0^{\pi}-\int_0^{\pi}-2\frac{\sin nx}{n}dx\right)$$

$$=\frac{2}{n\pi^2}\left((0-0)+\frac{2}{n}\left(\frac{-\cos nx}{n}\right)_0^{\pi}\right)$$

$$=\frac{4}{n^2\pi^2}\left(-\frac{1}{n}\right)(\cos nx-\cos 0)$$

$$=\frac{4}{n^3\pi^2}((-1)^n-1)=\begin{cases}0 & \text{if } n \text{ even}\\ \dfrac{8}{n^2(2p-1)^3} & \text{if } n\ odd \text{ i.e. } n=2p-1\end{cases}$$

$$\therefore\quad y(x,t)=\sum_{p=1}^{\infty}\frac{8}{\pi^2(2p-1)^3}\cos((2p-1)ct)\sin((2p-1)x).$$

$$\therefore \ \frac{\partial y}{\partial x}=\frac{\partial y}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial y}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial y}{\partial u}+\frac{\partial y}{\partial v}$$

2.2.6 D' Almebert's Solution of the Wave Equation

The wave equation is

$$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

Let us introduce the new independent variables $u=x+ct,\ v=x-ct$ so that y becomes a function of u and v.

And $$\frac{\partial^2 y}{\partial x^2}=\frac{\partial}{\partial x}\left[\frac{\partial y}{\partial u}+\frac{\partial y}{\partial v}\right]=\left[\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right]\left[\frac{\partial y}{\partial u}+\frac{\partial y}{\partial v}\right]$$

$$=\frac{\partial}{\partial u}\left[\frac{\partial y}{\partial u}+\frac{\partial y}{\partial v}\right]\cdot 1+\frac{\partial}{\partial v}\left[\frac{\partial y}{\partial u}+\frac{\partial y}{\partial v}\right]\cdot 1$$

$$=\frac{\partial^2 y}{\partial u^2}+\frac{\partial^2 y}{\partial u\,\partial u}+\frac{\partial^2 y}{\partial v^2}$$

Similarly $$\frac{\partial^2 y}{\partial t^2}=c^2\left[\frac{\partial^2 y}{\partial u^2}-2\frac{\partial^2 y}{\partial u\,\partial v}+\frac{\partial^2 y}{\partial v^2}\right].$$

Substituting this in equation (1), we get

$$\frac{\partial^2 y}{\partial u\,\partial v}=0 \qquad ...(2)$$

Integrating (2) wrt v, we get $\frac{\partial y}{\partial u}=f(u)$ ...(3)

Where $f(u)$ is an arbitrary function of u. Now integrating (3) w.r.t. u we obtain.

$$y=\int f(u)\,du+\psi(v).$$

Where $\psi(v)$ is an arbitrary function of v. Since the integral is a function of u alone, we may denote it by $\phi(u)$.. Thus

$$y = \phi(u) + \psi(v)$$

i.e. $y(x,t) = \phi(x+ct) + \psi(x-ct)$ ...(4)

This is the geneal solution of the wave equation (1)

Now to determine ϕ and ψ, suppose initially

$$u(x,0) = f(x) \text{ and } \frac{\partial y(x,0)}{\partial t} = 0.$$

Differentiating (4) w.r.t. t, we get $\dfrac{\partial y}{\partial t} = c\phi'(x+ct) - c\psi'(x-ct)$

At $t = 0$, $\phi'(x) = \psi'(x)$ ...(5)

and $y(x,0) = \phi(x) + \psi(x) = f(x)$ ...(6)

$\therefore$ (5) gives $\phi(x) = \psi(x) + k.$

$\therefore$ (5) becomes $2\psi(x) + k = f(x)$

or $\psi(x) = \dfrac{1}{2}\left[f(x) - k\right]$ and $\phi(x) = \dfrac{1}{2}\left[f(x) + k\right]$

Hence the solution of (4) takes the form

$$y(x,t) = \frac{1}{2}\left[f(x+ct) + k\right] + \frac{1}{2}\left[f(x-ct) - k\right]$$

$$\Rightarrow y(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct)) \quad ...(7)$$

which is the d' Alembert's solution of the wave equation (1).

Example 4. Using D' Alembert's method, find the deflection of a vibrating string of unit length having fixed ends with initial velocity zero and initial deflection

$$f(x) = a(x - x^3)$$

Sol. The vibrations of an elastic string are governed by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

under conditions $u(0,t) = 0 = u(1,t)$

Here $u(x,0) = f(x) = a(x - x^3)$

and $u_t(x,0) = 0$

By D' Alembert's method, the solution is

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct))$$

$$= \frac{1}{2}(a(x+ct-(x+ct)^3) + a(x-ct-(x-ct)^3)$$

$$= \frac{a}{2}(2x-(2x^3+6xc^2t^2)$$

$$= ax(1-x^2-3c^2t^2)$$

$\Rightarrow$ $u(x,0) = ax(1-x^2) = f(x)$

and $\dfrac{\partial u(x,t)}{\partial t} = ax(-3c^2(2t))$

$\Rightarrow$ $\left.\dfrac{\partial u(x,t)}{\partial t}\right)_{t=0} = 0,$ i.e. boundary conditions are satisfied.

Hence solution is $u(x,t) = ax(1-x^2-3c^2t^2)$

2.2.7 SOLUTION OF LAPLACE'S EQUATION IN TWO DIMENSIONAL

The Laplace's Equation is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ ...(1)

Let $u = X(x)Y(y)$ be a solution of (1)

Putting in (1), we get $Y\dfrac{d^2X}{dx^2} + X\dfrac{d^2Y}{dy^2} = 0$

$$\Rightarrow \quad \frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} \qquad ...(2)$$

Since x and y are independent variables so (2) can hold good only if each side of (2) is equal to constant k (say). Then (2) leads to the ordinary differential equations.

$$\frac{d^2X}{dx^2} - kX = 0 \quad \text{and} \quad \frac{d^2Y}{dx^2} - kY = 0 \qquad ...(3)$$

Solving these equations, we have to discuss three cases

Case I. When k is +ve *i.e.* $k=\lambda^2$

Equation (3) becomes $\frac{d^2X}{dx^2}-\lambda^2 x=0$ and $\frac{d^2Y}{dy^2}+\lambda^2 y=0$

A.E.'s are $D^2-\lambda^2=0$ and $D'^2+\lambda^2=0$

$\Rightarrow$ $D=\pm\lambda$ and $D'=\pm i\lambda$

so that $X=a_1 e^{\lambda x}+a_2 e^{-\lambda x}$

and $Y=b_1 \cos\lambda y+b_2 \sin\lambda y$

Or

$= a_1 \cosh\lambda x+a_2 \sinh\lambda x$

$\because e^{\lambda x}=\cosh\lambda x+\sinh\lambda x$ and $e^{-\lambda x}=\cosh\lambda x-\sinh\lambda x)$

Thus solution of (1) is

$$\left.\begin{aligned} u(x,y)&=(a_1 e^{\lambda x}+a_2 e^{-\lambda x})(b_1\cos\lambda y+b_2\sin\lambda y)\\ &\quad or\\ &=(a_1\cosh\lambda x+a_2\sinh\lambda x)(b_1\cos\lambda y+b_2\sin\lambda y)\end{aligned}\right\} \quad ...(4)$$

Case II. When $k=0$

Equation (3) becomes $\frac{d^2X}{dx^2}=0$ and $\frac{d^2Y}{dy^2}=0$

Solving $X=a_3 x+a_4$ and $Y=b_3 y+b_4$

Thus solution of (1) is

$$u(x,y)=(a_3 x+a_4)(b_3 y+b_4) \quad ...(5)$$

Case III. When k is -ve *i.e.* $k=-\lambda^2$

Equation (3) becomes $\frac{d^2X}{dx^2}+\lambda^2 x=0$ and $\frac{d^2Y}{dy^2}+\lambda^2 y=0$

A.E.'s are $D^2+\lambda^2=0$ and $D'^2-\lambda^2=0$

$\Rightarrow$ $D \quad \pm, \lambda$ and $D' = \pm \lambda$

so that

$$X = a_5 \cos \lambda x + a_6 \sin \lambda x \quad \text{and} \quad Y = b_5 e^{\lambda y} + b_6 e^{-\lambda y}$$

OR

$$b_5 \cosh \lambda y + b_6 \sinh \lambda y$$

Thus solution of (1) is

$$\left.\begin{aligned} u(x,y) &= (a_5 \cos \lambda x + a_6 \sin \lambda x)(b_5 e^{\lambda y} + b_6 e^{-\lambda y}) \\ &\quad Or \\ &= (a_5 \cos \lambda x + a_6 \sin \lambda x)(b_5 \cosh \lambda y + b_6 \sinh \lambda y) \end{aligned}\right\} \quad \ldots(6)$$

Example 5. Solve $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

subject to $u(0,y)=u(\pi,y)=0; 0 \le y \le \pi$

and $u(x,\pi)=0, u(x,0)=\sin^2 x; 0 \le x \le \pi$

Sol. Do as in Example 2

Here $u(x,0)=F(x)=\sin^2 x$

Here $l=m=\pi$

$$\therefore \quad A_n = \frac{2}{\pi}\text{cosech} \int_0^{\pi} \sin^2 x \sin\left(\frac{n\pi}{\pi}x\right)dx$$

$$= \frac{2}{\pi}\text{cosech } n\pi \int_0^{\pi}\left(\frac{1-\cos 2x}{2}\right)\sin(nx)\,dx$$

$$= \frac{2}{2\pi}\text{cosech } n\pi\left(\int_0^{\pi}\sin nx\,dx - \int_0^{\pi}\sin nx \cos 2x \; dx\right)$$

$$=\frac{1}{\pi}\operatorname{cosech} n\pi\left(\left(\frac{-\cos nx}{n}\right)_0^{\pi}-\frac{1}{2}\int_0^{\pi}(\sin(n+2)x+\sin(n-2)x)\,dx\right)$$

$$=\frac{1}{\pi}\operatorname{cosech} n\pi\left(\frac{-1}{n}\left((-1)^n-1\right)-\frac{1}{2}\left(-\frac{\cos(n+2)x}{n+2}-\frac{\cos(n-2)x}{n-2}\right)_0^{\pi}\right)$$

$$=\frac{1}{\pi}\operatorname{cosech} n\pi\left(\frac{-1}{n}\left((-1)^n-1\right)+\frac{1}{2}\left\{\left(\frac{\cos(n+2)\pi}{n+2}+\frac{\cos(n-2)\pi}{n-2}\right)\right\}\right)$$

where $n \neq 2$

$$=\left\{\frac{1}{\pi}\operatorname{cosech} n\pi\left(\frac{-1}{n}\left((-1)^n-1\right)+\frac{1}{2}\left\{\left(\frac{(-1)^n}{n+2}+\frac{(-1)^n}{n-2}\right)-\left(\frac{1}{n+2}+\frac{1}{n-2}\right)\right\}\right.$$

$$(\because \cos(n+2\pi=\cos(n\pi+2\pi)=(-1)^n\cos 2\pi=(-1)^n$$

and cos $(n-2)\pi=\cos(n\pi-2\pi)=\left((-1)^n\cos 2\pi=(-1)^n\right)$

$$=\frac{1}{\pi}\operatorname{cosech} n\pi\left(\frac{1}{n}(1-(-1)^n+\frac{(-1)^n}{2}\left(\frac{2n}{n^2-4}\right)-\frac{1}{2}\left(\frac{2n}{n^2-4}\right)\right)$$

$$=\frac{1}{\pi}\operatorname{cosech} n\pi\left(\frac{2}{n}-\frac{1}{2}\left(\frac{2n}{n^2-4}\right)-\frac{1}{2}\left(\frac{2n}{n^2-4}\right)\right)\text{if } n \text{ is odd}$$

$$=\frac{1}{\pi}\operatorname{cosech} n\pi\left(\frac{2}{n}-\frac{2n}{n^2-4}\right)\text{if } n \text{ is odd}$$

$$=\frac{-8\operatorname{cosech} n\pi}{\pi\, n\,(n^2-4)}\text{ if } n \text{ is odd.}$$

And $A_n=0$ if n is even

And when $n=2$, then $\int_0^{\pi}\sin^2 x\sin nx\,dx=\int_0^{\pi}\sin^2 x\sin 2x\,dx$

$$=\int_0^{\pi}\sin^2 x\,(2\sin x\cos x)dx$$

$$=2\int_0^{\pi} \sin^3 x \cos x\, dx = 2\left(\frac{\sin^4 x}{4}\right)_0^{\pi} = 0$$

Hence sol is given by

$$u(x,y) = \sum_{n=1}^{\infty} \frac{-8\operatorname{cosech} n\pi}{\pi n\,(n^2-4)} \sin\,(n(\pi - y))\sin(nx)$$

where n is odd

$$= \sum_{p=1}^{\infty} \frac{-8\operatorname{cosech}\left((2p-1)\pi\right)}{\pi(2p-1)\left((2p-1)^2-4\right)} \sinh\left((2p-1)(x-y)\sin(2p-1)x\right).$$

2.2.8 EXERCISE

1. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to $u(0,y) = u(l,y) = 0; 0 \le y \le m$

and $u(x,m) = 0,\ u(x,0) = lx - x^2, 0 \le x \le l$

2. A rectangular plate with insulated surface is 8 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by

$$v(x,0) = 100\sin\left(\frac{\pi x}{8}\right), 0 < x < 8$$

3. A long rectangular plate of width a cms withinsulated surface has its temperature v equal to zero on both the long sides and one of the shortsides so that

$$v(0,y) = 0, v(a,y) = 0, v(x,\infty) = 0, v(x,0) = kx$$

show that the steady state temperature within the plate is

$$v(x,y) = \frac{2ak}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{\frac{-n\pi y}{a}} \sin\frac{n\pi x}{a}$$

(4) Solve $\frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial u}{\partial t}$, given that

(i) $u = 0$ when $x = 0$ and l for $t \ge 0$ *(ii)* $u = \frac{100\,x}{l}; 0 < x < l.$

(5) A homogeneous rod of conducting material of length 100 cms has its ends kept at zero temperature and temperature initially is

$$u(x,0)=\begin{cases} x & 0 \le x \le 50 \\ 100-x, & 50 \le x \le 100 \end{cases}$$

(6) Find the deflection $u(x,t)$ of the vibrating string (length $=\pi$ and $c^2=1$) corresponding to zero initial velocity and initial deflection.

(7) A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = a \sin \frac{\pi x}{l}$ from which it is released at time *t=0*. Show that the displacement of any point at a distance x from one end at time t is given by

$$y(x,t) = a \sin\left(\frac{\pi x}{l}\right) \cos\left(\frac{\pi ct}{l}\right).$$

(8) A tight string of length l has its ends $x=0, x=l$ fixed. The point where $x=\frac{1}{3}$ is drawn aside a small distance h and released at time $t=0$. At any subsequent time $t>0$ the displacement $Y(x,t)$ of the string satisfies the one dimensional wave equation

$$\frac{\partial^2 Y}{\partial t^2} = c^2 \frac{\partial^2 Y}{\partial x^2}$$

Determine $Y(x,t)$ at any time $t>0$.