B.A.PART-I **MATHEMATICS : PAPER-I**

SEMESTER-I **CALCULUS-I**

SECTION-A & B

PUNJABI UNIVERSITY PATIALA

Department of Distance Education

Punjabi University, Patiala

LESSON NO.

SECTION-A

SECTION-B

Note : Students can download the syllabus from department's website www.pbidde.org

LESSON NO. 1.1 **Author : Dr. Chanchal**

SUCCESSIVE DIFFERENTIATION

Structure :

I. Objectives

The prime goal of this unit is to enlighten the basic concepts of successive differentiation, multiple points and asymptotes, concavity and convexity etc. During the study in this particular lesson, our main objectives are

* To obtain n^{th} order derivatives of some standard functions by the method of mathematical induction.
* To discuss Leibnitz's theorem for finding the n^{th} order derivatives of the product of two functions.

II. Introduction

We are already familiar with the concept that derivative of a function of x is also a function of x. Thus the derivative of a function may have its derivative without any loss of genrality.

If y = f(x),

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} = f'(x)$$

is called the first differential coefficient or first derivative of f(x). If the process of differentiation be continued in succession, we obtain second, third and higher order derivatives, as follows :

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \lim_{\delta x \to 0} \frac{f'(x+\delta x) - f'(x)}{\delta x} = f''(x),$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \lim_{\delta x \to 0} \frac{f''(x+\delta x) - f''(x)}{\delta x} = f'''(x)$$

and so on. These are also denoted by

$$y_1 = \frac{dy}{dx} = Dy,\ y_2 = \frac{d^2y}{dx^2} = D^2y\ ,\ y_n = \frac{d^ny}{dx^n} = D^ny$$

III. Successive Differentiation of Some Standard Functions

Art 1.1 : Prove the following results :

(i) If $y = (ax + b)^m$, then $y_n = m\,(m-1)\,(m-2)\ldots.\,(m-n+1)\,(ax + b)^{m-n}\,a^n$.

Proof : Here $y = (ax + b)^m$

Differentiating both sides w.r.t. x, we get,

$y_1 = m\,(ax + b)^{m-1}.\ a = m\,(ax + b)^{m-1}.\ a^1$

∴ result is true for n = 1

Assume that the result is true for n = k, where k is positive integer.

∴ $y_k = m\,(m-1)\,(m-2)\ \ldots\ (m - k + 1)\,(ax + b)^{m-k}.\ a^k$

Differentiating both sides w.r.t.x, we get,

$y_{k+1} = m\,(m-1)\,(m-2)\ \ldots\ (m - k + 1)\,(m-k)\,(ax + b)^{m-k-1}.\ a\ .\ a^k$

or $y_{k+1} = m\,(m-1)\,(m-2)\ \ldots\ (m - k + 1)\,(m - k)\,(ax + b)^{m + (k+1)}.\ a^{k+1}$

∴ result is true for n = k + 1.

∴ if the result is true for any positive integer k, then it is also true for the next higher integer k + 1.

But the result is true for n = 1 also.

∴ By method of induction, the result is true for all positive integers n.

Cor. I. If m is a positive integer > n, then

$$y_n = \frac{m(m-1)\,(m-2)\ldots..(m-n+1)\,\underline{\lfloor m-n}}{\underline{\lfloor m-n}}\,(ax+b)^{m-n}.\ a^n$$

or
$$y_n = \frac{\underline{\lfloor m}}{\underline{\lfloor m-n}}\,(ax+b)^{m-n}\ .\ a^n$$

If m = n, then $y_n = n\,(n-1)\,(n-2)\ \ldots.\ 2.1\,(ax + b)^0.\ a^n$

or
$$y_n = \underline{\lfloor n}\ .\ a^n$$

$y_{n+1} = y_{n+2} = \ldots = 0$

$\therefore$ $y_n = 0 \ \forall \ n > m.$

(ii) If $y = \dfrac{1}{ax+b}$, then $y_n = \dfrac{(-1)^n \ \underline{|n} \, . \, a^n}{(ax+b)^{n+1}}, \ x \neq -\dfrac{b}{a}$.

Proof : Here $y = \dfrac{1}{ax+b} = (ax+b)^{-1}$

$$\therefore \quad y_1 = (-1)\,(ax+b)^{-2} \, . \, a = \frac{(-1)^{-1} \, . \, \underline{|1} \, . \, a^1}{(ax+b)^2}$$

$\therefore$ the result is true for n = 1.

Assume that the result is true for n = k, where k is a positive integer.

$$\therefore \quad y_k = \frac{(-1)^k \, . \, \underline{|k} \, . \, a^k}{(ax+b)^{k+1}} = (-1)^k \ \underline{|k} \ a^k \ (ax+b)^{-k-1}$$

Differentiating again w.r.t. x, we get,

$$y_{k+1} = (-1)^k \ \underline{|k} \ a^k \ (-k-1)\,(ax+b)^{-k-2} \, . \, a$$

$$= (-1)^{k+1} \ \underline{|k+1} \ a^{k+1} \ (ax+b)^{-(k+2)} = \frac{(-1)^{k+1} \ \underline{|k+1} \, . \, a^{k+1}}{(ax+b)^{k+2}}$$

$\therefore$ result is true for n = k + 1.

$\therefore$ if the result is true for n = k, then it is also true for n = k + 1.

But the result is true for n = 1.

$\therefore$ By method of induction, the result is true for all positive integers n.

(iii) If y = log (ax + b), then $y_n = \dfrac{(-1)^{n-1} \ \underline{|n-1} \, . \, a^n}{(ax+b)^n}, \ x > -\dfrac{b}{a}$.

Proof : Here y = log (ax + b)

Differentiating both sides w.r.t.x,

$$y_1 = \frac{1}{ax+b} \, . \, a = \frac{(-1)^{1-1} \, . \, \underline{|1-1} \, . \, a^1}{(ax+b)^1}$$

$\therefore$ the result is true for n = 1.

Assume that the result is true for n = k, where k is a positive integer.

$$\therefore \quad y_k = \frac{(-1)^{k-1} . \, \underline{|k-1} \, . \, a^k}{(ax+b)^k} = (-1)^{k-1} \ \underline{|k-1} \, . \, a^k \, . \, (ax+b)^{-k}$$

Differentiating both sides w.r.t.x, we get,

$$y_{k+1} = (-1)^{k-1} \underline{|k-1} \, a^k \, (-k)\,(ax+b)^{-k-1} \,.\, a$$

$$= (-1)^k \underline{|k} \, a^{k+1} \, (ax+b)^{-(k+1)} = \frac{(-1)^k \underline{|k} \,.\, a^{k+1}}{(ax+b)^{k+1}}$$

∴ result is true for n = k + 1

∴ if the result is true for n = k, then it is also true for n = k + 1

But the result is true for n = 1.

∴ By the method of induction, the result is true for all positive integers n.

Note : Same result will hold even if y = log |ax + b| where $x > -\frac{b}{a}$.

(iv) If $y = a^{mx}$, a > 0, then $y_n = a^{mx} . (\log a)^n . m^n$.

Proof : Here $y = a^{mx}$

Differentiating both sides w.r.t.x,

$y_1 = a^{mx}, \log a . m = a^{mx}. (\log a)^1 . m^1$

∴ the result is true for n = 1.

Assume that the result is true for n = k, where k is a positive integer.

∴ $y_k = a^{mx} . (\log a)^k . m^k$

Differentiating both sides w.r.t.x,

$y_{k+1} = [a^{mx} . (\log a) . (m)] . (\log a)^k . m^k = a^{mx} . (\log a)^{k+1} . m^{k+1}$

∴ result is true for n = k + 1.

∴ if the result is true for n = k, then it is also true for n = k + 1.

But the result is true for n = 1.

∴ By the method of induction, the result is true for all positive integers n.

Cor. 1. Put m = 1

∴ $y_n = a^x . (\log a)^n$

∴ $y = a^x \Rightarrow y_n = a^x . (\log a)^n$

Cor. 2. Put a = e

∴ $y_n = e^{mx} . (\log e)^n . m^n = e^{mx} . m^n$

∴ $y = e^{mx} \Rightarrow y_n = e^{mx} . m^n$

Cor. 3. Put a = e, m = 1

∴ $y_n = e^x . (\log e)^n . (1)^n = e^x$

∴ $y = e^x \Rightarrow y_n = e^x$.

(v) If y = sin (ax + b), then $y_n = a^n \sin\left(ax + b + \frac{n\pi}{2}\right) \forall\, x \in R$.

Proof : Here y = sin (ax + b)

Differentiating both sides w.r.t. x,

$$y_1 = \cos (ax + b)\ .\ a = a^1 \sin \left[ax + b + 1\,.\,\frac{\pi}{2}\right]$$

∴ the result is true for n = 1.

Assume that the result is true for n = k, where k is a positive integer.

$$\therefore \quad y_k = a^k \sin\left[ax + b + k\,\frac{\pi}{2}\right]$$

Differentiating again w.r.t.x,

$$y_{k+1} = a^k \cos\left[ax + b + k\,\frac{\pi}{2}\right].\,a = a^{k+1}\sin\left[\left(ax + b + k\,\frac{\pi}{2}\right) + \frac{\pi}{2}\right]$$

$$= a^{k+1}\sin\left[ax + b + (k+1)\,\frac{\pi}{2}\right]$$

∴ result is true for n = k + 1.

∴ if the result is true for n = k, then it is also true for n = k + 1.

But the result is true for n = 1.

∴ By the method of induction, the result is true for all positive integers n.

(vi) If y = cos (ax + b), then $y_n = a^n \cos\left(ax + b + n\,\frac{\pi}{2}\right)\ \forall\ x \in R$.

Proof : The proof is left as an exercise for the reader.

(vii) If y = e^{ax} sin (bx + c), then $y_n = (a^2 + b^2)^{\frac{\pi}{2}}\, e^{ax} \sin\left(bx + c + n\tan^{-1}\frac{b}{a}\right)$

Proof : Here y = e^{ax} sin (bx + c)

Differentiating both sides w.r.t.x,

$$y_1 = e^{ax}\,.\,\frac{d}{dx}[\sin(bx + c)] + \sin(bx + c)\,.\,\frac{d}{dx}(e^{ax})$$

= e^{ax} . cos (bx + c) . b + sin (bx + c) . e^{ax} . a

∴ $y_1 = e^{ax}$ [a sin (bx + c) + b cos (bx + c)]

Put a = r cos α and b = r sin α where r > 0.

Squaring and adding (2) and (3), we get,

$$a^2 + b^2 = r^2 \Rightarrow r = \sqrt{a^2 + b^2}$$

Dividing (3) by (2), $\tan\ \alpha = \frac{b}{a} \Rightarrow \iota = \tan^{-1}\frac{b}{a}$

$\therefore$ from (1), $y_1 = e^{ax}[r \cos\alpha \sin(bx + c) + r \sin\alpha \cos(bx + c)]$

$= e^{ax} . r [\sin(bx + c) \cos\alpha + \cos(bx + c) . \sin\alpha]$

$= r e^{ax} \sin(bx + c + \alpha)$

$\therefore$ $$y_1 = (a^2 + b^2)^{\frac{1}{2}} . e^{ax} \sin\left(bx + c + 1 . \tan^{-1}\frac{b}{a}\right)$$

$\therefore$ the result is true for n = 1.

Assume that the result is true for n = k, where k is a positive integer

$\therefore$ $$y_k = (a^2 + b^2)^{\frac{k}{2}} . e^{ax} \sin\left(bx + c + k \tan^{-1}\frac{b}{a}\right)$$

or $y_k = r^k e^{ax} \sin(bx + c + k\alpha)$

Differentiating again w.r.t. x, we get,

$y_{k+1} = r^k . [e^{ax} . \cos(bx + c + k\alpha) . b + \sin(bx + c + k\alpha) . ae^{ax}]$

$= r^k . e^{ax}[a \sin(bx + c + k\alpha) + b \cos(bx + c + k\alpha)]$

$= r^k . e^{ax}[r \cos\alpha \sin(bx + c + k\alpha) + r \sin\alpha \cos(bx + c + k\alpha)]$

$= r^{k+1} . e^{ax} \sin[(bx + c + k\alpha) + \alpha] = r^{k+1} . e^{ax} \sin[bx + c + (k + 1)\alpha]$

$\therefore$ $$y_{k+1} = (a^2 + b^2)^{\frac{k+1}{2}} . e^{ax} \sin\left(bx + c + (k + 1) \tan^{-1}\frac{b}{a}\right)$$

$\therefore$ the result is true for n = k + 1

$\therefore$ if the result is true for n = k, then it is also true for n = k + 1.

But the result is true for n = 1.

$\therefore$ By the method of induction, the result is true for all positive integers.

(viii) If $y = e^{ax} \cos(bx + c)$, then

$$y_n = (a^2 + b^2)^{\frac{n}{2}} . e^{ax} \cos\left(bx + c + n \tan^{-1}\frac{b}{a}\right)$$

Proof : The proof is left as an exercise for the reader.

IV. Some Important Examples

Example 1 : If y = cosh (log x) + sinh (log x), prove that $y_n = 0$ for n > 1.

Sol. y = cosh (log x) + sinh (log x)

Differentiating both sides w.r.t.x, we get

$$y_1 = \sinh(\log x) \cdot \frac{1}{x} + \cosh(\log x) \cdot \frac{1}{x}$$

$\therefore$ $xy_1 = \sinh(\log x) + \cosh(\log x)$

or $xy_1 = y$

Again differentiating w.r.t.x, we get

$xy_2 + y_1 = y_1$ or $xy_2 = 0$

$\therefore$ $y_2 = 0$

$\therefore$ $y_n = 0$ for $x > 1$.

Example 2 : If $p^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta$, prove that $p + \frac{d^2p}{d\theta^2} = \frac{a^2b^2}{p^3}$

Sol. Here $p^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$

$$\Rightarrow \quad p^2 = a^2 \left(\frac{1 + \cos 2\theta}{2}\right) + b^2 \left(\frac{1 - \cos 2\theta}{2}\right)$$

$\Rightarrow$ $2p^2 = a^2 (1 + \cos 2\theta) + b^2 (1 - \cos 2\theta)$

$\Rightarrow$ $2p^2 - (a^2 + b^2) = (a^2 - b^2) \cos 2\theta$ ... (1)

Differentiating w.r.t. θ, we get,

$$4p \frac{dp}{d\theta} = -2(a^2 - b^2) \sin 2\theta$$

or $$-2p \frac{dp}{d\theta} = (a^2 - b^2) \sin 2\theta \quad \text{... (2)}$$

Squaring (1), (2) and adding, we get,

$$4p^4 + (a^2 + b^2)^2 - 4p^2 (a^2 + b^2) + 4p^2 \left(\frac{dp}{d\theta}\right)^2 = (a^2 - b^2)^2$$

or $$4p^4 - 4p^2(a^2 + b^2) + 4p^2 \left(\frac{dp}{d\theta}\right)^2 + (a^2 + b^2)^2 - (a^2 - b^2)^2 = 0$$

or $$4p^4 - 4p^2(a^2 + b^2) + 4p^2 \left(\frac{dp}{d\theta}\right)^2 + 4a^2b^2 = 0$$

Dividing both sides by $4p^2$, we get,

$$p^2 - (a^2 + b^2) + \left(\frac{dp}{d\theta}\right)^2 + \frac{a^2b^2}{p^2} = 0$$

Dividing by $2\frac{dp}{d\theta}$, we get,

$$p + \frac{d^2p}{d\theta^2} = \frac{a^2b^2}{p^3}$$

Example 3 : Find the nth derivative of $\sqrt{ax+b}$.

Sol. Let $y = \sqrt{ax+b} = (ax+b)^{\frac{1}{2}}$

$\therefore \quad y_1 = \frac{1}{2}(ax+b)^{\frac{1}{2}-1} . a = \frac{1}{2}(ax+b)^{-\frac{1}{2}} a \quad \frac{(-1)^1 .1}{2^2}$

$\therefore \quad y_2 = \frac{1}{2}\left(-\frac{1}{2}\right)(ax+b)^{-\frac{3}{2}} a^2 = y_3 = \frac{(-1)^2\ 1.3}{2}(ax+b)^{\frac{1}{2}-3} a^3$

$\therefore \quad y_3 = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) . (ax+b)^{-\frac{5}{2}} a^3$

$\therefore \quad y_3 = \frac{(-1)^2\ 1.3}{2}(ax+b)^{\frac{1}{2}-3} a^3$

2

....

$\therefore \quad y_n = \frac{(-1)^{n-1}\ 1.3.5\ldots(2n-1)\ (ax+b)^{\frac{1}{2}-n} a^n}{2^n}$

$\therefore \quad y_n = \frac{(-1)^{n-1} 1.3.5\ldots(2n-1)}{2^n (ax+b)^{\frac{2n-1}{2}}} . a^n$ where $x \neq -\frac{b}{a}$

Example 4 : Find y_n if $y = \frac{2x+1}{(x-2)(x-1)^3}$.

Sol. $y = \frac{2x+1}{(x-2)(x-1)^3}$

Put $\quad \frac{2x+1}{(x-2)(x-1)^3} \equiv \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$

Multiplying both sides by $(x-2)(x-1)^3$, we get

$2x + 1 \equiv A(x-1)^3 + B(x-2)(x-1)^2 + C(x-2)(x-1) + D(x-2)$ ... (1)

Putting $x - 2 = 0$ i.e. $x = 2$ in (1), we get

$5 = A \Rightarrow A = 5$

Putting $x - 1 = 0$ i.e. $x = 1$ in (1), we get

$3 = -D \Rightarrow D = -3$

(1) can be writing as

$2x + 1 = A(x^3 - 3x^2 + 3x - 1) + B(x^3 - 4x^2 + 5x - 2) + C(x^2 - 3x + 2) + D(x - 2)$ ... (2)

Equating coefficients in (2) of

x^3) $A + B = 0 \Rightarrow 5 + B = 0 \Rightarrow B = -5$

x^2) $-3A - 4B + C = 0 \Rightarrow -15 + 20 + C = 0 \Rightarrow C = -5$

$$\therefore \quad \frac{2x+1}{(x-2)(x-1)^2} \equiv \frac{5}{x-2} - \frac{5}{x-1} - \frac{5}{(x-1)^2} - \frac{3}{(x-1)^3}$$

$$\therefore \quad y = \frac{5}{x-2} - \frac{5}{x-1} - \frac{5}{(x-1)^2} - \frac{3}{(x-1)^3}$$

$$\therefore \quad y_n = 5\frac{(-1)^n \underline{|n}}{(x-2)^{n+1}} - 5\frac{(-1)^n \underline{|n}}{(x-1)^{n+1}} - 5\frac{(-1)^n \underline{|n+1}}{(x-2)^{n+2}} - 3\frac{(-1)^n \underline{|n+2}}{(x-1)^{n+3}}$$

$$1$$

$$\therefore \quad y_n = (-1)^n \underline{|n}\left[\frac{5}{(x-2)^{n+1}} - \frac{5}{(x-1)^{n+1}} - \frac{5(n+1)}{(x-1)^{n+2}} - \frac{3(n+2)(n+1)}{(x-1)^{n+3}}\right]$$

Example 5 : Find the nth derivative of $y = e^{3x}\sin^2 2x$.

Sol. $y = e^{3x}\sin^2 2x = e^{3x}\,\frac{1-\cos 4x}{2} = \frac{1}{2}e^{3x} - \frac{1}{2}e^{3x}\cos 4x$

$$\therefore \quad y_n = \frac{1}{2}e^{3x} \,.\, 3^n - \frac{1}{2}(9+16)^{\frac{n}{2}}\, e^{3x} \cos 4x + n\tan^{-1}\frac{4}{3}$$

$$\therefore \quad y_n = \frac{1}{2}e^{3x}\; 3^n - 5^n \cos 4x + n\tan^{-1}\frac{4}{3}.$$

V. Leibnitz's Theorem

Statement : If u and v are functions of x possessing nth order derivatives, then

$(uv)_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \dots + {}^nC_r u_{n-r} v_r + \dots + {}^nC_n u v_n$

where u_r denotes the rth order derivative of u and ${}^nC_r$ denotes the number of combinations out of n different things taken r at a time.

Proof : We have

$(uv)_1 = u_1 v + uv_1 = {}^1C_0\, u_1 v + {}^1C_1\, uv_1$

$\therefore$ theorem is true for n = 1.

Assume that the theorem is true for n = m, where m is a positive integer.

$\therefore$ $(uv)_m = {}^mC_0\, u_m v + {}^mC_1\, u_{m-1} v_1 + {}^mC_2 u_{m-2} v_2 + \ldots$
$+ {}^mC_{r-1}\, u_{m-r+1} v_{r-1} + {}^mC_r\, u_{m-r} v_r + \ldots + {}^mC_m\, uv_m$

Differentiating both sides w.r.t.x, we get,

$(uv)_{m+1} = {}^mC_0\, u_{m+1} v + {}^mC_0\, u_m v_1$
$+ {}^mC_1\, u_m v_1 + {}^mC_1\, u_{m-1} v_2$
$+ \ldots \ldots \ldots \ldots \ldots$
$+ {}^mC_{r-1}\, u_{m-r+2} v_{r-1} + {}^mC_{r-1}\, u_{m-r+1} v_r$
$+ {}^mC_r\, u_{m-r+1} v_r + {}^mC_r\, u_{m-r} v_{r+1}$
$+ \ldots \ldots \ldots \ldots \ldots$
$+ {}^mC_m\, u_1 v_m + {}^mC_m\, uv_{m+1}$

$\therefore$ $(uv)_{m+1} = {}^mC_0\, u_{m+1} v + ({}^mC_0 + {}^mC_1)\, u_m v_{1+} ({}^mC_1 + {}^mC_2)\, u_{m-1} v_2$
$+\ldots+ ({}^mC_{r-1} + {}^mC_r)\, u_{m-r+1} v_r + \ldots + {}^mC_m\, uv_{m+1}$

But ${}^mC_0 = 1 = {}^{m+1}C_0$

${}^mC_0 + {}^mC_1 = {}^{m+1}C_1$

${}^mC_1 + {}^mC_2 = {}^{m+1}C_2$

$\ldots \ldots \ldots \ldots \ldots$

${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r$

${}^mC_m = 1 = {}^{m+1}C_{m+1}$

$\therefore$ we have

$(uv)_{m+1}\ {}^{m+1}C_0\, u_{m+1} v + {}^{m+1}C_1 u_m v_1 + {}^{m+1}C_2\, u_{m-1} v_2 + \ldots$
$+ {}^{m+1}C_r\, u_{m-r+1} v_r + \ldots + {}^{m+1}C_{m+1}\, uv_{m+1}$

$\therefore$ theorem is true for n = m + 1.

$\therefore$ if the theorem is true for n = m, then it is also true for n = m + 1

But the theorem is true for n = 1.

$\therefore$ By the method of induction, theorem is true for all positive integers n.

V.(a) Some Important Examples

Example 6 : Prove that $\frac{d^n}{dx^n}\left[\frac{\log x}{x}\right] = \frac{(-1)^n \lfloor n}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{n}\right]$

Given that x > 0.

Sol. Here $y = \frac{\log x}{x} = \log x \,.\, \frac{1}{x}$

Let $V = \log x$ $\qquad U = \frac{1}{x}$

$V_1 = \frac{1}{x} = x^{-1}$ $\qquad U_1 = (-1)\ x^{-2}$

$V_2 = (-1)\ x^{-2}$ $\qquad U_2 = (-1)\ (-2)\ x^{-3}$

$V_3 = (-1)\ (-2)\ x^{-3}$ $\qquad U_3 = \frac{(-1)^3 \underline{|3}}{x^4}$

and so on $\qquad$ and so on

$V_n = \frac{(-1)^{n-1} \underline{|n-1}}{x^n}$ $\qquad U_n = \frac{(-1)^n \underline{|n}}{x^{n-1}}$

By Leibnitz's rule

$$\frac{d^n y}{dx^n} = \frac{d^n}{dx^n}(U\,.\,V) = \frac{d^n}{dx^n}\left(\frac{\log x}{x}\right)$$

$$= {}^nC_0\ \frac{(-1)^n \underline{|n}}{x^{n+1}}\,.\,\log x + {}^nC_1\ \frac{(-1)^{n-1}\underline{|n-1}}{x^n}\,.\,\frac{1}{x}$$

$$= {}^nC_2\ \frac{(-1)^{n-2}\ \underline{|n-2}}{x^{n+1}}\,.\,\frac{(-1)}{x^2} + ... + {}^nC_n\,.\,\frac{1}{x}\ \frac{(-1)^{n-1}\underline{|n-1}}{x^n}$$

$$= \frac{(-1)^n\underline{|n}}{x^{n-1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - ... - \frac{1}{n}\right].$$

Example 7 : If $y = (\sin^{-1} x)^2$, find $y_n(0)$.

Sol. $y = (\sin^{-1} x)^2$

Differentiating w.r.t. x,

$$y_1 = 2\,(\sin^{-1} x)\,.\,\frac{1}{\sqrt{1-x^2}}$$

Squaring and cross-multiplying

$(1 - x^2)\ y_1^{\,2} = 4\ (\sin^{-1} x)^2 \qquad \Rightarrow (1 - x^2)\ y_1^{\,2} = 4y$

$\therefore \qquad (1 - x^2)\ y_1^{\,2} - 4y = 0$

Differentiating w.r.t.x, again we get,

$(1 - x^2)\ 2y_1 y_2 - 2x\ y_1^{\,2} - 4y_1 = 0$

Dividing by $2y_1$, we get

$(1 - x^2)\, y_2 - xy_1 - 2 = 0$ ... (3)

Differentiating n times (3) by Leibnitz's rule,

$$1.\ y_{n+2}(1-x^2) + \frac{n}{1} y_{n+1}(-2x) + \frac{n(n-1)}{2.1} y_n(-2) - 1 \,.\, y_{n+1}\, x - \frac{n}{1} y_n \,.\, 1 - 0 = 0$$

$(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} - n^2 y_n = 0$ ... (4)

Putting x = 0 in (1), (2), (3) and (4) we get,

$y(0) = 0$ ... (5)

$y_1(0) = 0$ ... (6)

$y_2(0) = 2$ ... (7)

$y_{n+2} = n^2 y_n(0)$ ... (8)

Putting n = 1, 2, 3, 4 ... in (8), we get,

$y_3(0) = 1^2 y_1(0) = 0$ ... (9) $[\because \text{of } (6)]$

$y_4(0) = 2^2 y_2(0) = 2.2^2$ ... (10) $[\because \text{of } (7)]$

$y_5(0) = 3^2 y_3(0) = 0$ ... (11) $[\because \text{of } (8)]$

$y_6(0) = 4^2 y_4(0) = 2 \,.\, 2^2 \,.\, 4^2$ $[\because \text{of } (10)]$

and so on.

$\therefore$ In general $y_n(0) = \begin{cases} 2.2^2.4^2...(n-2)^2 & \text{when n is even and } n \neq 2 \\ 0 & \text{when n is odd} \end{cases}$

VI. Self Check Exercise

1. If $y = e^{ax} \sinh bx$ prove that $y_2 - 2ay_1 + (a^2 - b^2)\, y = 0$.
2. If $y = \log(1 + \cos x)$, prove that $y_1 y_2 + y_3 = 0$.
3. If $x = \sin\theta$, $y = \sin m\theta$, prove that $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + m^2 y = 0$.
4. Find the n^{th} derivative of

   (i) $x \log\left(\dfrac{x-a}{x+a}\right), x > a > 0$ (ii) $e^x \cos x \cos 2x$

   (iii) $\sin x \sin 2x$ (iv) $2^x \,.\, e^x$
5. If $y = x^n \log x$, prove that $y_{n+1} = \dfrac{\lfloor n}{x}$.
6. If $y = \sin(m \sin^{-1} x)$, prove that

$$(1 - x^2)\, y_{n+2} - (2n + 1)\, x y_{n+1} - (n^2 - m^2)\, y_n = 0$$

7. If $y = (x^2 - 1)^n$, prove that $(x^2 - 1)\, y_{n+2} + 2x y_{n+1} - n\,(n + 1)\, y_n = 0$.
8. If $x = \tan(\log y)$, prove that

   $(1 + x^2)\, y_{n+2} + \{2\,(n + 1)\, x - 1\}\, y_{n+1} + n\,(n + 1)\, y_n = 0$.

VII. Suggested Readings

1.	Ahsan Akhtar & Sabita Ahsan	:	Differential Calculus
2.	UP Singh, RJ Srivastava & NH Siddiqui	:	Differential Calculus
3.	Gorakh Prasad	:	Differential Calculus
4.	Malik and Arora	:	Mathematical Analysis
5.	Thomas and Finney (Ninth Edition)	:	Calculus and Analytic Geometry

LESSON NO. 1.2 **Author : Dr. Chanchal**

SINGULAR POINTS

Structure :

I. Objectives

The prime goal of this lesson is to gain knowledge about the singular points of the curve $y = f(x)$. During the study in this lesson, our main objectives are

* To discuss about the types of singular points viz., points of inflexion and multiple points particularly double points, alongwith their classification and respective nature.
* To study about the concavity and convexity of the curve $y = f(x)$.

II. Introduction

Singular Point : A point on the curve at which the curve behaves in an extraordinary manner is called a singular point.

There are two types of singular points :

(i) Points of inflexion

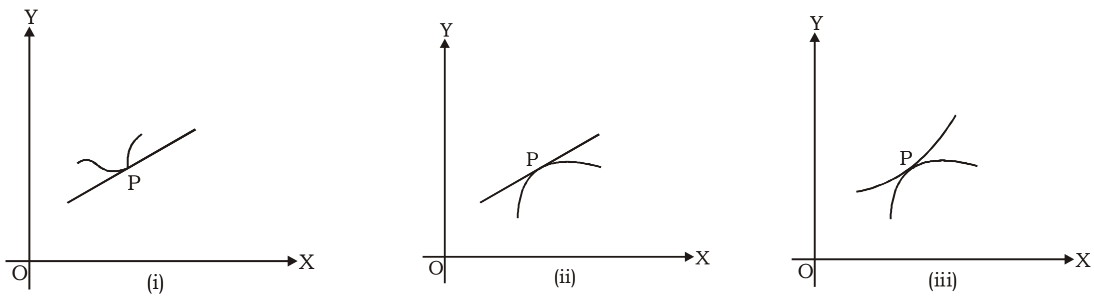
(ii) Multiple points

Firstly, we study points of inflexion for which we must be familiar with the concepts of concavity and convexity of a curve, as discussed below :

Consider the curve y = f(x) in [a, b]. Let it be continuous and possessing tangents at every point in (a, b).

Draw a tangent at any point P (c, f(c)) on the curve. Let us assume that this tangent is not parallel to Y-axis so that f'(c) is some finite number.

Now there are three mutually exclusive possibilities to consider :



(i) A portion of the curve on both side of P, however small it may be, lies above the tangent at P (i.e. towards the +ve direction of Y-axis). In this case we see that the curve is concave upwards or convex downwards at P. Such curves "hold water" [See fig. (i)].

As x-increases, f'(x) is either of the same sign and increasing or changes sign from –ve to +ve. In either case, the slope f'(x) is increasing and f"(x)>0. Such graphs are bending upwards or bulging downwards and the portion lies below chord.

(ii) A portion of the curve on both sides of P, however small it may be, lies below the tangent at P (i.e., towards the negative direction of Y-axis). In this case, we say that the curve is concave downwards or convex upwards at P [see fig. (ii)].

As x increases, f'(x) is either of the same sign and decreasing or changes sign from +ve to –ve. In either case, the slope f'(x) is decreasing and hence f"(x) < 0.

The graph in this case is bending downward or bulging upwards.

(iii) The two portions of the curve on the two sides of P lie on different sides of the tangent at P i.e., the curve crosses the tangent at P. In this case we say that P is a point of inflexion on the curve [see fig. (iii)].

So, at a point of inflexion, the curve changes from concave upwards to concave downwards or vice-versa.

So at a point of inflexion f"(x) = 0.

Concavity or Convexity of a Curve : A curve is said to be concave downwards (or convex upwards) on the interval (a, b) if all the points of the curve lie below any tangent to it on that interval. It is said to be concave upwards (or convex downwards) on the interval (a, b) if all the points of the curve lie above any tangent to it on that interval.

Note : A curve convex upwards is called a convex curve and a curve convex downwards is called a concave curve.

Point of Inflexion : A point that separates the convex part of the curve from the concave part of the curve is called a point of inflexion.

Now, we define a multiple point.

Multiple Point : A point on the curve through which more than one branches of the curve pass is called a multiple point.

III. Working Method for Concavity, Convexity and Points of Inflexion

1. Evaluate $\frac{d^2y}{dx^2}$

2. Find the interval (a, b) for which $\frac{d^2y}{dx^2} > 0$.

Then (a, b) is the interval of being convex downwards.

3. Find the interval (a, b) for which $\frac{d^2y}{dx^2} < 0$.

Then (a, b) is the interval of being convex upwards.

4. Find the values of x which satisfy $\frac{d^2y}{dx^2} = 0$, and also the values of x (if any) where $\frac{d^2y}{dx^2}$ does not exist.

Such values x = a, b, c, ... (say) are the possible points of inflexion.

5. x = a will be a point of inflexion

if (i) either $\frac{d^2y}{dx^2}$ changes sign at x = a

or (ii) $\frac{d^3y}{dx^3}$ exists and is non-zero at x = a.

Note 1. $\frac{d^2y}{dx^2} = 0$ is not a sufficient condition for graph of f to have a point of inflexion.

Note 2. If at a point, $x = c$, $f^{(n)}(c) \neq 0$ when n is even, then $x = c$ is not a point of inflexion.

Note 3. If at a point, $x = c$, $f^{(n)}(c) = 0$ for some even n and $f^{(n+1)}(c) \neq 0$, then the curve has a point of inflexion at $x = c$.

IV. Some Important Examples

Example 1 : Find the intervals in which the curve $y = (\cos x + \sin x)\, e^x$ is concave upwards or downwards in $(0, 2\pi)$. Find also the points of inflexion.

Sol. Here $y = (\cos x + \sin x)\, e^x$

$$\therefore \quad \frac{dy}{dx} = (\cos x + \sin x)\, e^x + (-\sin x + \cos x)\, e^x = 2e^x \cos x$$

$$\frac{d^2y}{dx^2} = 2\,(e^x \cos x - e^x \sin x) = 2\, e^x (\cos x - \sin x)$$

Now $\frac{d^2y}{dx^2} > 0$ when $2e^x (\cos x - \sin x) > 0$

i.e., $\cos x - \sin x > 0$ $\quad [\because 2e^x > 0]$

$$\Rightarrow \quad \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right) > 0 \Rightarrow \frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x > 0$$

$$\Rightarrow \quad \sin\frac{\pi}{4}\cos x - \cos\frac{\pi}{4}\sin x > 0 \Rightarrow \sin\left(\frac{\pi}{4} - x\right) > 0$$

$$\Rightarrow \quad \sin\left(x - \frac{\pi}{4}\right) < 0 \qquad \Rightarrow x - \frac{\pi}{4} \in (-\pi, 0) \cup (\pi, 2\pi) \text{ and } x \in (0, 2\pi)$$

$$\Rightarrow \quad x \in \left(-\frac{3\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, \frac{9\pi}{4}\right) \text{ and } x \in (0, 2\pi)$$

$$\Rightarrow \quad x \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right)$$

$\therefore$ given curve is concave upwards in $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right)$

Again $\frac{d^2y}{dx^2} < 0$ when $2e^x(\cos x - \sin x) < 0$

i.e., when $\sin\left(x - \frac{\pi}{4}\right) > 0 \Rightarrow x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

$\therefore$ given curve is concave downwards in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

Points of Inflexion

$$\frac{d^3y}{dx^3} = 2\left[e^x . \frac{d}{dx}(\cos x - \sin x) + (\cos x - \sin x)\frac{d}{dx}(e^x)\right]$$

$$= 2e^x[-\sin x - \cos x + \cos x - \sin x] = -4e^x \sin x$$

Also $\frac{d^2y}{dx^2} = 0 \quad \Rightarrow 2e^x(\cos x - \sin x) = 0$

$\Rightarrow \quad \cos x - \sin x = 0 \Rightarrow \sin x = \cos x \Rightarrow \tan x = 1$

$\therefore \quad \tan x = \tan\frac{\pi}{4}, \tan\frac{5\pi}{4} \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$

When $x = \frac{\pi}{4}, \frac{d^3y}{dx^3} = -4\, e^{\frac{\pi}{4}} \sin\frac{\pi}{4} \neq 0 \neq 0$

When $x = \frac{5\pi}{4}, \frac{d^3y}{dx^3} = -4\, e^{\frac{5\pi}{4}} \sin\frac{5\pi}{4}$

Now when $x = \frac{\pi}{4}$

$$y = \left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right)e^{\frac{\pi}{4}} = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)e^{\frac{\pi}{4}} = \sqrt{2}\, e^{\frac{\pi}{4}}$$

and when $x = \frac{5\pi}{4}$

$$y = \left(\cos\frac{5\pi}{4} + \sin\frac{5\pi}{4}\right)e^{\frac{5\pi}{4}} = \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)e^{\frac{5\pi}{4}} = \sqrt{2}\, e^{\frac{5\pi}{4}}$$

$\therefore$ given curve has points of inflexion at $\left(\frac{\pi}{4}, \sqrt{2}\, e^{\frac{\pi}{4}}\right)$ and $\left(\frac{5\pi}{4}, -\sqrt{2}\, e^{\frac{5\pi}{4}}\right)$.

Example 2 : Find the values of x for which $y = x^4 - 6x^3 + 12x^2 + 5x + 7$ is concave upwards or downwards. Also determine the points of inflexion.

Sol. Here $y = x^4 - 6x^3 + 12x^2 + 5x + 7$

$\therefore \quad \frac{dy}{dx} = 4x^3 - 18x^2 + 24x + 5$

$\therefore \quad \frac{d^2y}{dx^2} = 12x^2 - 36x + 24$

Now $\frac{d^2y}{dx^2} > 0$ iff $12x^2 - 36x + 24 > 0$

iff $x^2 - 3x + 2 > 0$

i.e., iff $x^2 - 3x > -2$

i.e., iff $x^2 - 3x + \frac{9}{4} > -2 + \frac{9}{4}$

i.e., iff $\left(x - \frac{3}{2}\right)^2 > \frac{1}{4} - 2$

i.e., iff $\left|x - \frac{3}{2}\right|^2 > \left(\frac{1}{2}\right)^2$

i.e., iff $\left|x - \frac{3}{2}\right| > \frac{1}{2}$

i.e., iff $x - \frac{3}{2} > \frac{1}{2}$ or $x - \frac{3}{2} < -\frac{1}{2}$

i.e. iff $x > 2$ or $x < 1$

$\therefore$ curve is concave upwards in $(-\infty, 1) \cup (2, \infty)$

Similarly $\frac{d^2y}{dx^2} < 0$

iff $\left|x-\frac{3}{2}\right|<\frac{1}{2}$

i.e., iff $-\frac{1}{2}<x-\frac{3}{2}<\frac{1}{2}$

if $1 < x < 2$

∴ curve is concave downwards in (1, 2)

$$\frac{d^2y}{dx^2}=0 \text{ when } x^2-3x+2=0$$

i.e., when x = 1 or x = 2

$$\frac{d^3y}{dx^3}=24x-36\neq 0 \text{ when } x = 1, \text{ or } x = 2$$

∴ x = 1, y = 19 i.e., (1, 19)
x = 2, y = 23 i.e., (2, 23)
are the points of inflexion.

V. Double Points and their Classification

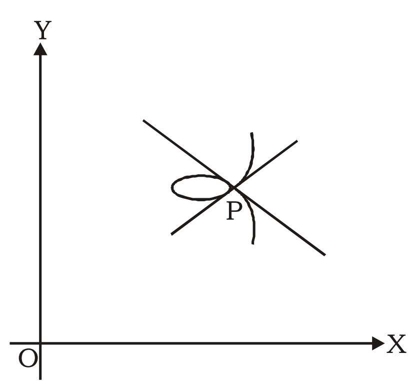
As we have already defined a multiple point, on the basis of which we can define a double point as

Double Point : A point on the curve through which two branches of the curve pass is called a double point.

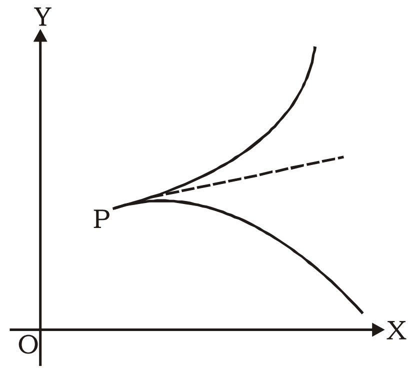
V.(a) Classification of Double Points

There are three kinds of double points.

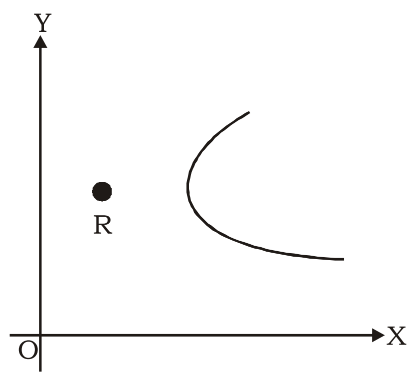
(i) Node : A node is a point on the curve through which pass two real branches of the curve and two tangents at which are real and distinct. Thus P is a node.



(ii) **Cusp :** A double point on the curve through which two real branches of the curve pass and the tangents at which are real and coincident is called a cusp. Thus P is a cusp.



(iii) **Conjugate Point or Isolated Point :** A conjugate point on a curve is a point in the neighbourhood of which there are no other real points of the curve.



The two tangents at a conjugate point are in general imaginary but sometimes they may be real.

V.(b) Working Method for Finding the Nature of Origin which is a Double Point

Find the tangents at the origin by equating to zero the lowest degree terms in x and y of the equation of the curve. If the origin is a double point, then we shall get two tangents which may by real or imaginary.

(i) If two tangents are imaginary, then origin is a conjugate point.

(ii) If two tangents real and coincident, then origin is a cusp or a conjugate point.

(iii) If the two tangents are real and distinct, then origin is a node or a conjugate point.

To be sure, examine the nature of curve in the nbd. of origin. If the curve has real branches through the origin, then it is a node, otherwise a conjugate point.

To be sure, we test the nature of curve in the nbd. of the origin as above.

Note. Test for nature of curve at origin.

If the tangents at origin are $y^2 = 0$, solve the equation of the curve for y, neglecting all terms of y containing powers above two. If the values of y, for small values of x are found to be real, the branches of the curve through the origin are real, otherwise imaginary.

If the tangents at origin are $x^2 = 0$, solve the equation for x and proceed as above.

V.(c) Working Method for Finding the Position and Nature of Double Points of the Curve f(x, y) = 0

Step I. Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}$

Step II. Solve the equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ to get possible double points.

Reject those points which do not satisfy the equation f (x, y) = 0 of the curve. Remaining are the double points

Step III. At each double point, calculate $D = \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}$

(a) If D is positive, double point is a node or conjugate point

(b) If D = 0, double point is a cusp or conjugate point.

In these cases (a) and (b), find the nature by shifting the origin to the double points and then testing the nature of tangents and existence of the curve in the nbd. of new origin.

(c) If D is negative, double point is a conjugate point.

VI. Some Important Examples

Example 3 : Prove that the curve $y^2 = (x-a)^2 (x-b)$ has at x = 0, a node if a > b, a cusp if a=b and a conjugate point if a < b.

Sol. The equation of curve is $y^2 = (x-a)^2 (x-b)$ ... (1)

When x = a, from (1), y = 0

$\therefore$ point under discussion is (a, 0)

Shifting origin to (a, 0) by transformation x = X + a, y = Y + 0 = Y

(1) becomes $Y^2 = X^2 (X + a - b)$ ... (2)

Equating to zero, the lowest degree terms, the tangents at the new origin are given by

$$Y^2 = X^2 (a - b) \quad \text{or} \quad Y = \pm X\sqrt{a-b} \qquad \dots (3)$$

Case I. When a > b

From (3), two tangents at new origin are real and different

$\therefore$ new origin (a, 0) is a node or a conjugate point

From (2), $Y = \pm X\sqrt{X + a - b}$

For small non-zero value of X, Y is real as a – b > 0

$\therefore$ new origin (a, 0) is a node

Case II. When a = b

From (3), tangents are Y = 0, Y = 0

$\therefore$ two tangents are real and coincident

$\therefore$ origin is a cusp or a conjugate point

From (2), $Y^2 = X^3$ or $Y = \pm X\sqrt{X}$

For small positive values of X, Y is real

$\therefore$ new origin (a, 0) is a cusp.

Case III. When a < b

From (2), two tangents at new origin are imaginary

$\therefore$ (a, 0) is a conjugate point.

Example 4 : Determine the position and nature of the double point on the curve $x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0$.

Sol. The equation of curve is

$f(x, y) = x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0$ ... (1)

$$\therefore \quad \frac{\partial f}{\partial x} = 3x^2 - 14x + 15, \frac{\partial f}{\partial y} = -2y + 4$$

For the double points $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, f(x, y) = 0$

Now $\frac{\partial f}{\partial x} = 0 \quad \Rightarrow 3x^2 - 14x + 15 = 0$

$\Rightarrow \qquad (x - 3)(3x - 5) = 0 \Rightarrow x = 3, \frac{5}{3}$

and $\frac{\partial f}{\partial y} = 0 \quad \Rightarrow -2y + 4 = 0 \Rightarrow y = 2$

$\therefore$ the possible double points are (3, 2), $\left(\frac{5}{3}, 2\right)$

But $\left(\frac{5}{3}, 2\right)$ does not satisfy (1)

$\therefore$ (3, 2) is the only double point

Nature of the point (3, 2) : Shifting the origin to the point (3, 2) by transformations $x = X + 3$, $y = Y + 2$

(1) becomes $(X + 3)^3 - (Y + 2)^2 - 7(X + 3)^2 + 4(Y + 2) + 15(X + 3) - 13 = 0$

or $\quad X^3 + 9X^2 + 27X + 27 - Y^2 - 4Y - 4 - 7X^2 - 42X - 63 + 4Y$
$+ 8 + 15X + 45 - 13 = 0.$

or $\quad X^3 + 2X^2 - Y^2 = 0$ ... (2)

Equating to zero, the lowest degree terms, the tangents at the new origin are given by

$$2X^2 - Y^2 = 0 \quad \text{or} \quad Y = \pm\sqrt{2}\,X$$

which are real and distinct

$\therefore$ new origin is either a node or a conjugate point

From (2), $Y = \pm X\sqrt{X + 2}$

which gives real values of Y for small values of X, positive or negative

$\therefore$ real branches of the curve exist in the nbd. of the new origin (3, 2)

$\therefore$ (3, 2) is a node.

Alter. The equation of the curve is

$$f(x, y) = x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0$$

$$\therefore \quad \frac{\partial f}{\partial x} = 3x^2 - 1x + 15, \frac{\partial f}{\partial y} = -2y + 4$$

For the double points $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, f(x, y) = 0$

Now $\frac{\partial f}{\partial y} = 0 \Rightarrow$ $3x^2 - 14x + 15 = 0$

$\Rightarrow$ $(x - 3)(3x - 5) = 0 \Rightarrow x = 3, \frac{5}{3}$

and $\frac{\partial f}{\partial y} = 0 \Rightarrow -2y + 4 = 0 \Rightarrow y = 2$

$\therefore$ the possible double points are (3, 2), $\left(\frac{5}{3}, 2\right)$

But $\left(\frac{5}{3}, 2\right)$ does not satisfy (1)

$\therefore$ (3, 2) is the only double point

Nature of the point (3, 2) :

$$\frac{\partial^2 f}{\partial x^2} = 6x - 14, \frac{\partial^2 f}{\partial y^2} = -2, \frac{\partial^2 f}{\partial x\, \partial y} = 0$$

At (3, 2)

$$\frac{\partial^2 f}{\partial x^2} = 18 - 14 = 4, \frac{\partial^2 f}{\partial y^2} = -2, \frac{\partial^2 f}{\partial x\, \partial y} = 0$$

$\therefore$ $$\left(\frac{\partial^2 f}{\partial x\, \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} = (0)^2 - (4)(-2) = 8 > 0$$

$\therefore$ (3, 2) is node.

VII. Self Check Exercise

1. Examine the curve $y = x^4 - 2x^3 + 1$ for concavity upwards, concavity downwards and points of inflexion.
2. Show that the points of inflexion of the curve $y^2 = (x - a)^2 (x - b)$ lies on the line $3x + a = 4b$.
3. If $y = ax^3 + bx^2$ has a point of inflexion (–1, 2), find a and b.
4. Show that the curve $y^2 = 2x \sin 2x$ has a node at the origin.
5. Examine the curve $x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0$ for a double point and show that it is a cusp.

6. Prove that the only singular point on the curve $(y - b)^2 = (x - a)^3$ is a cusp and find its co-ordinates.

VIII. Suggested Readings

1.	Ahsan Akhtar & Sabita Ahsan	:	Differential Calculus
2.	UP Singh, RJ Srivastava & NH Siddiqui	:	Differential Calculus
3.	Gorakh Prasad	:	Differential Calculus
4.	Malik and Arora	:	Mathematical Analysis
5.	Thomas and Finney (Ninth Edition)		Calculus and Analytic Geometry

LESSON NO. 1.3 **Author : Dr. Chanchal**

ASYMPTOTES

Structure :

I. Objectives

During the study in this particular lesson, our main objectives are

* To study the rules for finding rectangular asymptotes (horizontal and vertical asymptotes).
* To discuss the methods for finding the oblique asymptotes to the curve.

II. Introduction

We are familiar with the plane curves like parabola and hyperbola. Such types of curves, if drawn completely, will extend to infinity. Suppose that a tangent is drawn at any point of a curve which extend to infinity. Further suppose that the point of contact of the tangent moves along the curve in such a manner that its distance from origin tends to infinity. We may then find a definite straight line (a straight line at a finite distance from the origin) to which the tangent approaches. Such a straight line is called an asymptote of the curve. In other words a straight line is said to be an asymptote of a curve, if the perpendicular distance of any point P on a branch of the curve from this straight line tends to zero as the point P tends to infinity along the curve. We now give a formal definition of the asympotote.

Definition : A straight line at a finite distance from the origin to which a tangent to a curve tends as the distance from the origin of the point of contact tends to infinity, is called an asymptote of the curve.

III. Rules for Finding Asymptotes

III.(a) Rectangular Asymptotes

If an asymptote to a curve is either parallel to x–axis or parallel to y–axis, then it is called a rectrangular asymptote. An asymptote parallel to x–axis is usually called horizontal asymptote and an asymptote parallel to y–axis is called a vertical asymptote. We discuss below the rules to find these asymptotes :

1. Rule to find asymptotes parallel to x-axis.

Equate to zero the real linear factors in the coefficient of highest power of x in the equation of the given curve.

It should be noted properly that if the coefficient of highest power of x in the equation of the given curve is a constant or has no real linear factor, then the curve has no asymptote parallel to x-axis.

2. Rule to find asymptotes parallel to y-axis.

Equate to zero the real linear factors in the coefficient of highest power of y in the equation of the given curve.

It should be noted properly that if the coefficient of highest power of y in the equation of the given curve is a constant or has no real linear factor then the curve has no asymptote parallel to y-axis.

Example 1 : Find the asymptotes parallel to the axes of the curve $x^2y^2 + y^2 = 1$.

Sol. The equation of the given curve is $x^2y^2 + y^2 = 1$ ... (1)

The coefficient of highest power of x in (1) is y^2

$\therefore$ $y^2 = 0$ i.e., $y = 0$ is the only asymptote parallel to the x-axis

The coefficient of highest power of y in (1) is $x^2 + 1$. Now $x^2 + 1$ has no real linear factor.

$\therefore$ given curve has no asymptote parallel to y-axis.

III.(b) Oblique Asymptotes

An asymptote, which is neither parallel to x-axis nor parallel to y-axis is called an oblique asymptote. Such type of asymptotes can be determined under the following rule :

Rule to find oblique asymptotes

(i) Find $\underset{x\to\infty}{\text{Lt}} \frac{y}{x}$ in the equation of the curve and denote it by m.

(ii) Find $\underset{x\to\infty}{\text{Lt}} (y - mx)$ in the equation of the curve and denote it by c.

Then y = mx + c is an asymptote of the curve f (x, y) = 0.

III.(c) Asymptote of the General Rational Algebraic Curve

Let the equation of the curve be

$$x^n\phi_n\left(\frac{y}{x}\right)+x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right)+x^{n-1}\phi_{n-2}\left(\frac{y}{x}\right)+\ldots.+x\phi_1\left(\frac{y}{x}\right)+\phi_0\left(\frac{y}{x}\right)=0$$

where $\phi_n\left(\frac{y}{x}\right)$ represents a polynomial in $\frac{y}{x}$ of degree n.

Then, its asymptote can be obtained as

Rule to find oblique asymptotes of a rational algebraic curve :

Step I. Find $\phi_n(m)$, $\phi_{n-1}(m)$ by putting x = 1 and y = m in the nth degree terms and in the (n–1)th degree terms respectively of the given curve f(x, y) = 0.

Step II. Find all the real roots of $\phi_n(m) = 0$.

Step III. If m_1 is a non-repeated root of $\phi_n(m) = 0$, then the corresponding value of c is given by $c\phi'_n(m_1) + \phi_{n-1}(m_1) = 0$, provided $\phi'_n(m_1) \neq 0$.
If $\phi'_n(m_1) = 0$, then there is no asymptote to the curve corresponding to the value m_1 of m.

Step IV. If m_1 is a repeated root occurring twice, then the corresponding values of c are given by m_1 m_1 m_1

$$\frac{c^2}{\underline{|2}}\phi''_n(m)+c_1\,\phi'_{n-1}(m)+\phi_{n-2}(m)=0,\text{ provided }\phi''_n(m_1)\neq 0\cdot$$

In this case there are two parallel asymptotes to the curve.
Similarly we can proceed when m_1 is repeated three or more times.

Note : A rational algebraic curve of degree n cannot have more than n asymptotes.

Example 2 : Find all the asymptotes of the curve

$x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy + y - 1 = 0$.

Sol. Given equation is

$x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy + y - 1 = 0$ ... (1)

(1) is an equation of degree 3 in x and y

Since coefficient of x^3 is 1, which is constant

So there is no asymptotes parallel to x-axis

Similarly coefficient of y^3 is –2, which is constant

∴ there is no asymptote parallel to y-axis.

For oblique asymptotes, put y = mx + c in (1), we get,

$x^3 + 2x^2(mx + c) - x(mx + c)^2 - 2(mx + c)^3 + 4(mx + c)^2$

$+ 2x(mx + c) + (mx + c) - 1 = 0.$

or $x^3(1 + 2m - m^2 - 2m^3) + x^2(2c - 2mc - 6m^2c + 4m^2 + 2m)$
$+x(-c^2 - 6mc^2 + 8mc + 2c + m) + (-2c^3 + 4c^2 + c - 1) = 0.$

Equating the coefficient of x^3 and x^2 to zero, we get,

$1 + 2m - m^2 - 2m^3 = 0$ ... (2)

$2c - 2mc - 6m^2c + 4m^2 + 2m = 0$ ... (3)

From (2), $1(1 + 2m) - m^2(1 + 2m) = 0$

$\therefore$ $(1 - m^2)(1 + 2m) = 0$

$\therefore$ $(1 - m)(1 + m)(1 + 2m) = 0$

$\therefore$ $m = 1, -1, -\frac{1}{2}$

When $m = 1$, from (3), we have

$2c - 2c - 6c + 4 + 2 = 0$

$\therefore$ $6c = 6$ or $c = 1$.

Corresponding asymptote is $y = x + 1$

When $m = -1$, from (3), we have,

$2c + 2c - 6c + 4 - 2 = 0$

$2c = 2$ or $c = 1$

$\therefore$ corresponding asymptote is $y = -x + 1$

When $m = \frac{1}{2}$, from (3), we have,

$2c + c - \frac{3}{2}c + 1 - 1 = 0$ or $c = 0$

Corresponding asymptote is $y = -\frac{1}{2}x$.

Example 3 : Find the asymptotes of the curve

$x^3 - x^2y - xy^2 + y^3 + 2x^2 - 4y^2 + 2xy + x + y + 1 = 0.$

Sol. The equation of given curve is

$x^3 - x^2y - xy^2 + y^3 + 2x^2 - 4y^2 + 2xy + x + y + 1 = 0.$

The coefficient of highest power of x in (1) is 1, which is constant.

$\therefore$ there is no asymptote parallel to x-axis.

The coefficient of highest power of y in (1) is 1, which is constant

$\therefore$ there is no asymptote parallel to y-axis

For oblique asymptotes, we have

$\phi_3(m) = 1 - m - m^2 + m^3$ $\therefore$ $\phi'_3(m) = -1 - 2m + 3m^2$

$\phi_2(m) = 2 - 4m^2 + 2m$ $\therefore \phi''_3(m) = -2 + 6m$

$\phi_1(m) = 1 + m$ $\therefore \phi'_2(m) = -8m + 2$

$\phi_0(m) = 1$

$\phi_3(m) = 0$ gives

$1 - m - m^2 + m^3 = 0$

$\therefore$ $1(1 - m) - m^2(1 - m) = 0$

$\Rightarrow$ $(1 - m)(1 - m^2) = 0$

$\Rightarrow$ $(1 - m)(1 - m)(1 + m) = 0$

$\therefore$ $m = 1, 1, -1$

When $m = -1$, $c = -\dfrac{\phi_2(m)}{\phi_3'(m)} = -\dfrac{\phi_2(1)}{\phi_3'(1)}$

$\therefore$ $c = -\dfrac{2-4-2}{-1+2+3} = 1$

Corresponding asymptote is $y = -1x + 1$ i.e., $x + y = 1$

When $m = 1, 1$, the values of c are given by

$$\frac{c^2}{\lfloor 2}\phi_3''(m) + c\phi_2'(m) + \phi_1(m) = 0$$

$\Rightarrow$ $\dfrac{c^2}{2}(-2+6) + c(-8+2) + (1+1) = 0$

$\Rightarrow$ $2c^2 - 6c + 2 = 0$

$\Rightarrow$ $c^2 - 3c + 1 = 0$

$\Rightarrow$ $c = \dfrac{3 \pm \sqrt{9-4}}{2} = \dfrac{3 \pm \sqrt{5}}{2}$

and corresponding asymptotes are given by

$$y = 1x + \frac{3+\sqrt{5}}{2},\ y = 1x + \frac{3-\sqrt{5}}{2}$$

Hence the required asymptotes are

$$x + y = 1,\ x - y + \frac{3+\sqrt{5}}{2} = 0,\ x - y + \frac{3-\sqrt{5}}{2} = 0.$$

IV. Some Other Methods for Finding Oblique Asymptotes

Here, we discuss some special methods of finding asymptotes of $f(x, y) = 0$ when the equation $f(x, y) = 0$ is of some special types.

Method I. If the equation of the curve is of the form

$$(ax + by + c)\, f_{n-1}(x, y) + g_{n-1}(x, y) = 0$$

then the asymptote parallel to ax + by + c = 0 is given by

$$ax + by + c + \underset{\substack{x\to\infty \\ \frac{y}{x}\to\frac{a}{b}}}{\text{Lt}} \frac{g_{n-1}(x,y)}{f_{n-1}(x,y)} = 0, \text{ provided the limit exists}$$

Method II. If the equation of the curve is of the form

$$(ax + by)^2 f_{n-2}(x, y) + g_{n-2}(x, y) = 0$$

then the two asymptote parallel to ax + by = 0 are given by

$$(ax + by)^2 + \underset{\substack{x\to\infty \\ \frac{y}{x}\to\frac{a}{b}}}{\text{Lt}} \frac{g_{n-2}(x,y)}{f_{n-2}(x,y)} = 0, \text{ provided the limit exists}$$

Method III. If the equation of the curve is of the form

$$(ax + by)^2 f_{n-2}(x, y) + (ax + by)\, g_{n-2}(x, y) + h_{n-2}(x, y) = 0$$

then the two asymptotes parallel to ax + by = 0 are given by

$$(ax + by)^2 + (ax + by) \underset{\substack{x\to\infty \\ \frac{y}{x}\to\frac{a}{b}}}{\text{Lt}} \frac{g_{n-2}(x,y)}{f_{n-2}(x,y)} + \underset{\substack{x\to\infty \\ \frac{y}{x}\to\frac{a}{b}}}{\text{Lt}} \frac{h_{n-2}(x,y)}{f_{n-2}(x,y)} = 0$$

provided the limit exists

Note. Working Method

(i) Factorize the highest degree terms

(ii) Retain one linear factor and divide by the product of other factors.

(iii) Take limits when $x \to \infty$, $y \to \infty$ in the direction of the retained factor.

Note. If limits does not exist, then there is no asymptote parallel to ax + by + c = 0.

Method IV. Asymptotes by Inspection

If the equation of the curve can be written as

$$F_n(x, y) + F_{n-2}(x, y) = 0.$$

where $F_n(x, y)$ is a rational integral function in x and y of degree n and $F_{n-2}(x, y)$ of degree (n – 2) at the most then every linear factor ax + by + c of $F_n(x, y)$ equated to zero determines the asymptote of the curve, provided no two asymptotes so obtained are either parallel or coincident.

Example 4 : Find all the asymptotes of the following curve :

$$x^3 + x^2y - xy^2 - y^3 + 2xy + 2y^2 - 3x + y = 0$$

Sol. The given equation is $x^3 + x^2y - xy^2 - y^3 + 2xy + 2y^2 - 3x + y = 0$

or $x^2(x + y) - y^2(x + y) + 2xy + 2y^2 - 3x + y = 0$

or $(x + y)(x^2 - y^2) + 2xy + 2y^2 - 3x + y = 0$

or $(x - y)(x + y)^2 + 2xy + 2y^2 - 3x + y = 0$

The equation (1) can be written as

$$x - y + \frac{2xy + 2y^2 - 3x + y}{(x + y)^2} = 0$$

$\therefore$ asymptote (if it exists) parallel to $x - y = 0$ is given by

$$x - y + \underset{\substack{x\to\infty \\ y=x}}{\text{Lt}} \frac{2xy + 2y^2 - 3x + y}{(x + y)^2} = 0$$

or $$x - y + \underset{x\to\infty}{\text{Lt}} \frac{2x^2 + 2x^2 - 3x + x}{(x + x)^2} = 0$$

or $$x - y + \underset{x\to\infty}{\text{Lt}} \frac{4x^2 - 2x}{4x^2} = 0$$

or $$x - y + \underset{x\to\infty}{\text{Lt}} \frac{4 - \frac{2}{x}}{4} = 0 \quad \text{or } x - y + \frac{4 - 0}{4} = 0$$

$\therefore$ $x - y + 1 = 0$ is one asymptote.

The equation (1) can be written as

$$(x + y)^2 + (x + y)\,.\,\frac{2y}{x - y} - \frac{3x - y}{x - y} = 0$$

$\therefore$ asymptotes (if they exist) parallel to $x + y = 0$ are given by

$$(x + y)^2 + (x + y)\,.\,\underset{\substack{x\to\infty \\ y=-x}}{\text{Lt}} \frac{2y}{x - y} - \underset{\substack{x\to\infty \\ y=-x}}{\text{Lt}} \frac{3x - y}{x - y} = 0$$

or $$(x + y)^2 + (x + y)\,.\,\underset{x\to\infty}{\text{Lt}} \frac{-2x}{x + x} - \underset{x\to\infty}{\text{Lt}} \frac{3x + x}{x + x} = 0$$

or $(x + y)^2 - (x + y) - 2 = 0$ or $(x + y - 2)(x + y + 1) = 0$

$\therefore$ $x + y - 2 = 0$, $x + y + 1 = 0$ are the other two asymptotes.

V. Intersection of a Curve and its Asymptotes

Art 3.1 : Prove that an asymptote of a rational algebraic curve of the nth degree cuts the curve in atmost $(n - 2)$ points.

Proof : Let $y = m_1x + c_1$ ... (1) be an asymptote of the curve

$$x^n\phi_n\left(\frac{y}{x}\right)+x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right)+x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right)+\ldots=0 \qquad \ldots (2)$$

We are to find the points of intersection of (1)and (2),

From (1), $\frac{y}{x}=m_1+\frac{c_1}{x}$

Substituting the value of $\frac{y}{x}$ in (1), we get,

$$x^n\phi_n\left(m_1+\frac{c_1}{x}\right)+x^{n-1}\phi_{n-1}\left(m_1+\frac{c_1}{x}\right)+x^{n-2}\phi_{n-2}\left(m_1+\frac{c_1}{x}\right)+\ldots=0$$

Using Taylor's Theorem, we get

$$x^n\phi_n(m_1)+x^{n-1}\left[c_1\phi_n'(m_1)+\phi_{n-1}(m_1)\right]$$

$$+x^{n-2}\left[\frac{c_1^2}{2}\phi_n''(m_1)+c_1\phi_{n-1}'(m_1)+\phi_{n-2}(m_1)\right]+\ldots=0 \qquad \ldots (3)$$

Since $\phi_n(m_1) = 0$ and $c_1\phi'_n(m_1) + \phi_{n-1}(m_1) = 0$, (3) becomes

$$x^{n-2}\left[\frac{c_1^2}{2}\phi_n''(m_1)+c_1\phi_{n-1}'(m_1)+\phi_{n-2}(m_1)\right]+\ldots=0$$

which is an equation of degree (n – 2) and correspondingly (1) and (2) intersect in (n – 2) points.

∴ asymptote (1) cuts the curve (2) in at the most (n – 2) points.

Hence the result.

Cor. 1. Prove that all asymptotes of a curve of nth degree cut the curve in atmost n(n–2) points.

Proof. We know that a curve of nth degree has atmost has atmost n asymptotes and each asymptote cuts the curve in atmost (n – 2) points.

∴ all the asymptotes of a curve of nth degree cut the curve in atmost n (n – 2) points.

Cor. 2. If the equation of the curve of nth degree is of the form $F_n + F_{n-2} = 0$ and curve has no parallel asymptotes, then the points of intersection of the curve and its asymptote lie on the curve $F_{n-2} = 0$.

Proof. The equation of curve is $F_n + F_{n-2} = 0$

The equation of asymptote is $F_n = 0$

$\therefore$ the points of intersection of the asymptote and the curve satisfy the equations $F_n + F_{n-2} = 0$ and $F_n = 0$ and therefore they will satisfy

$(F_n + F_{n-2}) - F_n = 0$ i.e., $F_{n-2} = 0$.

Hence the result.

Example 5 : Find the equation of the cubic which has the same asymptotes as the curve $x^3 - 6x^2y + 11xy^2 - 6y^3 + 4x + 5y + 7 = 0$ and which passes through the points (0, 0), (–2, 0) and (0, –2).

Sol. The equation of given curve is $x^3 - 6x^2y + 11xy^2 - 6y^3 + 4x + 5y + 7 = 0$ ... (1)

It is of the form $F_3 + F_1 = 0$

$\therefore$ asymptotes are given by $F_3 = 0$

or $x^3 - 6x^2y + 11xy^2 - 6y^3 = 0$ or $(x-y)(x-2y)(x-3y) = 0$

$\therefore$ asymptotes of (1) are $x - y = 0$, $x - 2y = 0$, $x - 3y = 0$

The equation of the cubic curve which has the same asymptotes is of the type

$(x - y)(x - 2y)(x - 3y) + ax + by + c = 0$ -----

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Now (2) passes through (0, 0), $\therefore c = 0$

(2) passes through (–2, 0), $\therefore -8 - 2a = 0 \Rightarrow a = -4$

(2) passes through (0, –2), $\therefore 48 - 2b = 0 \Rightarrow b = 24$

Substituting values of a, b, c in (2), we get,

$(x - y)(x - 2y)(x - 3y) - 4x + 24y = 0$

or $x^3 - 6x^2y + 11xy^2 - 6y^3 - 4x + 24y = 0$.

VI. Self Check Exercise

1. Find all the asymptotes of the following curves :
   (i) $y^3 - 3x^2y + xy^2 - 3x^3 + 2y^2 + 2xy + 4x + 5y + 6 = 0$
   (ii) $ay^2 = x^2(a - x)$
   (iii) $y^3 + 4xy^2 + 4x^2y + 5y^2 + 15xy + 10x^2 - 2x + 1 = 0$
2. Show that the parabola $y^2 = 4ax$ has no asyptotes.
3. Find the asymptots of the curve
   $(x+ y)(x + 2y)(x + 3y) + 3x^2 + 12xy + 11y^2 + x + y + 2 = 0$
4. Find asymptotes of the curve $x^2y - xy^2 + xy + y^2 + x - y = 0$.
5. Find the asymptotes of the curve $x^2y + xy^2 + 2x^2 - 2xy - y^2 - 6x - 2y + 2 = 0$ and show that they cut the curve in almost three points which lie on the straight line $2x - 3y - 4 = 0$.
6. Find the equation of the cubic curve which has the same asymptotes as the

curve $x^3 - 6x^2y + 11\,xy^2 - 6y^3 + x + y + 1 = 0$ and which passes through the points (0, 0), (2, 0) and (0, 2).

VII. Suggested Readings

1.	Ahsan Akhtar & Sabita Ahsan	:	Differential Calculus
2.	UP Singh, RJ Srivastava & NH Siddiqui	:	Differential Calculus
3.	Gorakh Prasad	:	Differential Calculus
4.	Malik and Arora	:	Mathematical Analysis
5.	Thomas and Finney (Ninth Edition)		Calculus and Analytic Geometry

LESSON NO. 1.4 **Author : Dr. Chanchal**

CURVE TRACING AND CURVATURE

Structure :

I. Objectives

In this lesson we will deal with the graphs of the curves of given equations in Cartesian or polar systems of coordinates. The main purpose of this chapter is to point out those rules which are used in tracing the graph of a curve. After describing the main rules of curve tracing and afterwards we will use them in tracing the graph of aforesaid curves.

II. Introduction

The graph of a given function is helpful in giving a visual presentation of the behaviour of the function involving the study of symmetries of asymptotes, the intervals of rising up or falling down and of the cavity upwards and downwards etc. Curve tracing means that the equations of curves which we trace and are generally solvable for y, x or r. The case may come that some equations are not solvable for y or x, then we solve them for r by transforming from Cartesian to polar system.

III. Rules for Tracing Cartesian Curves

For tracing the curve of the equation $f(x, y) = 0$, the following important points should be considered :

I. Symmetry : Curve given by $f(x, y) = 0$ is symmetric about

(i) x-axis if it is unchanged on changing y to –y i.e., if $f(x, -y) = f(x, y)$

(ii) y-axis if it is unchanged on changing x to –x i.e., if $f(-x, y) = f(x, y)$

(iii) the origin if it is unchanged on changing x to –x and y to –y i.e., if $f(-x, -y) = f(x, y)$

(iv) the line $y = x$ if it is unchanged on changing x to y and y to x

i.e., if $f(x, y) = f(y, x)$

(v) the line $y = -x$ if it is unchanged on changing x to –y and y to –x

i.e., if $f(-y, -x) = f(x, y)$.

II. Domain and Range : Find the domain and range.

III. Origin : Check whether origin lies on the curve. If curve passes through origin, then find the tangents at the origin and also determine whether origin is node, cusp or an solated point.

IV. Asymptotes : Find all the asymptotes of the curve and the position of the curve relative to its asymptotes.

V. Points of Intersection : Find the points of intersection of the curve with co-ordinate axes and obtain the equations of the tangents at these points. If any of these is a double point, then find the nature of the double point.
Also find some other points on the curve by giving suitable values to x.

VI. Maxima and Minima : Find the points where the function has maximum value or minimum value. Also find the maximum and minimum value at each point.

VII. Points of Inflextion : (a) Find the intervals of

(i) increase and decrease of the curve

(ii) concavity and convexity of the curve.

(b) Also find the points of inflexion, if any.

VIII. Discontinuities : Find the points at which function is discontinuous. Also discuss the behaviour of the function near these points.
The method of tracing curves in cartesian co-ordinates can be made more clear with the help of following suitable examples :

Example 1 : Trace the cure $x = (y-1)(y-2)(y-3)$.

Sol. The equation of the curve is $x = (y-1)(y-2)(y-3)$ ... (1)

(i) Symmetry : The curve is neither symmetrical about axes nor about origin. Also the curve is neither symmetrical about $y = x$ nor about $y = -x$.

(ii) Origin : The curve does not pass through the origin.

(iii) Point of intersection with axis : The curve meets x-axis where $y = 0$

$\therefore$ putting $y = 0$ in (1), we get, $x = -6$

$\therefore$ curve meets x-axis in $(-6, 0)$

The curve meet y-axis where $x = 0$

$\therefore$ putting $x = 0$ in (1), we get, $(y-1)(y-2)(y-3) = 0$

$\therefore$ $y = 1, 2, 3$.

$\therefore$ curve meets y-axis in $(0, 1), (0, 2), (0, 3)$.

(iv) Asymptotes : The curve has no asymptotes.

(v) Tangents : Now $x = y^3 - 6y^2 + 11y - 6$

$\therefore \quad \frac{dx}{dy} = 3y^2 - 12y + 11$

$\therefore \quad \frac{dx}{dy} = 0 \text{ gives } 3y^2 - 12y + 11 = 0$

$\therefore \quad y = \frac{12 \pm \sqrt{144 - 132}}{6} = \frac{12 \pm 2\sqrt{3}}{6} = \frac{6 \pm 1.732}{3} = 2.6 \text{ (nearly), } 1.4 \text{ (nearly)}$

When y = 2.6, x = –0.384 (nearly)
When y = 1.4, x = 0.384 (nearly)

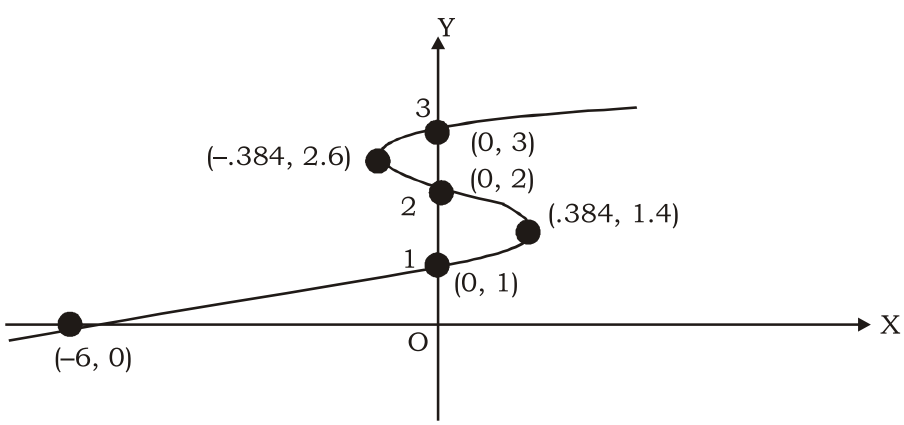
$\therefore$ tangents to the curve at (–3.84, 2.6) and (3.84, 1.4) are parallel to the y-axis.

(vi) Additional Points Now $y < 0 \Rightarrow x < 0$

$\therefore$ no portion of the curve lies in the fourth quadrant.

$0 < y < 1 \quad \Rightarrow \quad x < 0$
$1 < y < 2 \quad \Rightarrow \quad x > 0$
$2 < y < 3 \quad \Rightarrow \quad x < 0$
$3 < y \quad \Rightarrow \quad x > 0$
$x \to \infty \quad \Rightarrow \quad y \to \infty$

A rough sketch of the curve is given in the figure.



Example 2 : Trace the curve $x^3 + y^3 = 3axy$, $a \geq 0$.

Sol. The equation of the curve is $x^3 + y^3 = 3axy$, $a \geq 0$

(i) Symmetry : The given equation (1) does not change when x is changed to y and y is changed to x.

curve is symmetrical about the line y = x.

(ii) Origin : The curve passes through the origin.

The tangents at origin are given by xy = 0

i.e., x = 0, y = 0. These tangents are different.

$\therefore$ origin is a node.

(iii) Asymptotes : (1) can be written as

$$(x + y)\ (x^2 - xy + y^2) - 3axy = 0$$

Asymptote (if any) parallel to x + y = 0 is given by

$$x + y - \underset{x\to\infty}{\text{Lt}} \frac{-3axy}{x^2 - xy + y^2} = 0$$

or $$x + y - \underset{x\to\infty}{\text{Lt}} \frac{-3ax^2}{x^2 + x^2 + x^2} = 0 \text{ or } x + y + \underset{x\to\infty}{\text{Lt}} \frac{3ax^2}{3x^2} = 0$$

or $x + y + a = 0$

This is the only asymptote of the curve.

(iv) Points of intersection with axes

Putting x = 0 in (1), we get y = 0

Putting y = 0 in (1), we get x = 0

$\therefore$ curve meets axes in (0, 0) only.

Putting y = x in (1), we get,

$$x^3 + x^3 = 3ax^2 \quad \text{or } x^2\,(2x - 3\,a) = 0$$

$$\therefore \quad x = 0, \frac{3a}{2} \qquad \therefore \quad y = 0, \frac{3a}{2}$$

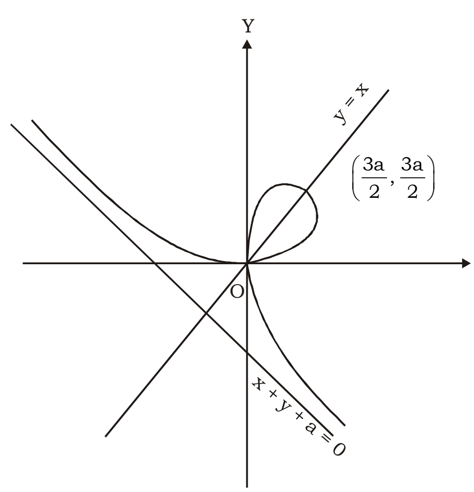
$\therefore$ line y = x meets the curve in (0, 0) and $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

(v) Region

From (1), it is clear that x and y both cannot be negative as in that case L.H.S. of (1) is negative whereas R.H.S. of (1) is positive.

$\therefore$ no portion of the curve lies in the 3rd quadrant.

A rough sketch of the curve is shown in the figure.



IV. Rules for Tracing Parametric Curves

Case I. Eliminate the parameter if possible and get the corresponding cartesian equation of the curve which can be traced as done earlier.

Case II. If the parameter cannot be easily eliminated from the given equations, then we proceed like this :

(i) Symmetry

(i) If x = f(t) is an even function of t and y = ϕ (t) an odd function of t, then the curve is symmetrical about x-axis.

(ii) If x = f(t) is an odd function of t and y = ϕ (t) an even function of t, then curve is symmetrical about y-axis.

(iii) If x = f (t) and y = ϕ(t) are both odd functions of t, then the curve is symmetrical in opposite quadrants.

(ii) Origin : If by putting x = 0, we get a real value of t, which makes y equals zero, then the curve passes through the origin.

(iii) Axes Intersection : Find the points of intersection of the curve and coordinate axes.

(iv) Limitations : If possible, find the greatest and least values of x and y which give us lines parallel to axes between which the curve lies or does not lie.

(v) Points : Find the points where $\frac{dy}{dx} = 0, \frac{dy}{dx} \to \infty.$

(vi) Region :

(i) Find the regions in which curve does not lie.

(ii) Consider the signs of $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

(iii) Consider the values of x, y, $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dy}{dx}$.

(vii) Asymptotes : Find the asymptotes, if any.

Example 3 : Trace the curve $x = a(\theta + \sin\theta)$; $y = a(1 + \cos\theta)$, $-\pi \le \theta \le \pi$.

Sol. The equations of the curve are $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$

Here the parameter θ cannot be easily eliminated.

(i) Symmetry : The curve is symmetrical about the axis of y for $(\theta + \sin\theta)$ is an odd function of θ and $(1 + \cos\theta)$ is an even function of θ.

(ii) Origin : The curve does not pass through the origin.

(iii) Intercepts : It meet the x-axis when

$y = 0$ i.e., $1 + \cos\theta = 0$

or $\cos\theta = -1$ i.e., $\theta = \pi, -\pi$

$\therefore$ the points of intersection with the x-axis are A $(a\pi, 0)$, A' $(-a\pi, 0)$.

Again it meets the y-axis when x = 0.

i.e. $\theta + \sin\theta = 0$ or $\sin\theta = -\theta$ or $\theta = 0$

$\therefore$ it meets the axis of y at B (0, 2 a).

(iv) Asymptotes : There are no asymptotes.

(v) Points : We have $\frac{dx}{d\theta} = a\,(1+\cos\theta); \frac{dy}{d\theta} = -a\sin\theta$

$$\therefore \quad \frac{dy}{dx} = \frac{-a\sin\theta}{a(1+\cos\theta)} = -\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = -\tan\frac{\theta}{2}$$

$$\therefore \quad \frac{dy}{dx} = 0 \text{ when } \theta = 0$$

i.e., at (0, 2 a), the tangent is parallel to the axis of x.

Also $\frac{dy}{dx} \to \infty$ when $\theta = \pi, -\pi$

$\therefore$ at $(a\,\pi, 0)$ and $(-a\,\pi, 0)$, the tangent is perpendicular to the axis of x.

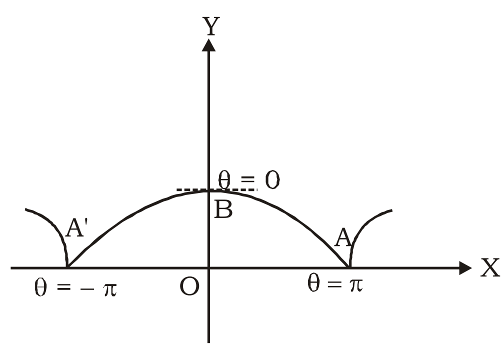
(v) Region : For all values of $\theta, \frac{dx}{d\theta}$ is + ve

$\therefore$ x always increases with θ.

Also $\frac{dx}{d\theta}$ is + ve for $- \leq \theta \leq \pi$.

Hence y increases when θ increases from $-\pi$ to 0and y decreases when θ increases from 0 to π.

Hence approximately, the shape of the curve is as shown in the diagram.



V. Rules for Tracing Polar Curves

We shall keep in mind the following points for tracing the graphs of the equation $f(r, \theta) = 0$.

1. Symmetry :

(i) Symmetry about the initial line or x-axis : If the equation of the curve remains unchanged when θ is changed to $-\theta$, the curve is symmetrical about the initial line.

(ii) Symmetry about the line $\theta = \frac{\pi}{2}$ or y-axis : If the equation of the curve remains unchanged when θ is changed to $\pi - \theta$ or when θ is changed to $-\theta$ and r to $-r$, the curve is symmetrical about the line $\theta = \frac{\pi}{2}$.

(iii) Symmetry about the line $\theta = \frac{\pi}{4}$ or y = x : If the equation of the curve remains unchanged when θ is changed to $\frac{\pi}{2} - \theta$, the curve is said to be symmetrical about the line $\theta = \frac{\pi}{4}$.

(iv) Symmetrical about the line $\theta = \frac{3\pi}{4}$ or y = –x; if the equation of the curve remains unchanged when θ is changed to $\frac{3\pi}{2} - \theta$, the curve is said to be symmetrical about the line $\theta = \frac{3\pi}{4}$.

(v) Symmetry about the pole : If the equation of the curve remains unchanged when r is changed to –r, the curve is said to be symmetrical about the pole.

II. Pole

(i) Find whether the curve passes through the pole or not. It can be done by putting r = 0 in the equation and then finding some real value of θ. If it is not possible to find a real value of θ for which r = 0, then the curve does not pass through the pole.

(ii) Find the tangents at the pole. Putting r = 0, the real values of θ give the tangents at the pole.

(iii) Find the points where the curve meets the initial line and the line $\theta = \frac{\pi}{2}$.

III. Value of ϕ

Find ϕ from the result $\tan\phi = r\dfrac{d\theta}{dr}$. Then find the points where $\phi = 0$ or $\dfrac{\pi}{2}$.

IV. Asymptotes

If $r \to \infty$ as $\theta \to \theta_1$ (any fixed number), then there is an asymptote. Find it by the method given below :

(i) Write down the given equation as $\dfrac{1}{r} = f(\theta)$, say.

(ii) Equate $f(\theta)$ to zero and solve for θ. Let the roots be θ_1, θ_2,.......

(iii) Find $f'(\theta)$ and calculate it at $\theta = \theta_1, \theta_2$,.........

(iv) Asymptotes are $r\sin(\theta - \theta_1) = \dfrac{1}{f'(\theta)}, r\sin(\theta - \theta_2) = \dfrac{1}{f'(\theta_2)}, \ldots$

V. Special Points

Find some points on the curve for convenient values of θ.

VI. Region

Solve the given equation for r or θ. Find the region in which the curve does not lie. This can be done in the following manner.

(i) No part of the curve lies between $\theta = \alpha$ and $\theta = \beta$ if for $\alpha < \theta < \beta$, r is imaginary.

(ii) If the greatest numerical value of r be a, the curve lies entirely within the circle r = a. If the least numerical value of r be b, the curve lies outside the circle r = b.

Example 4 : Trace the curve $r = a(1 + \cos\theta)$, $a > 0$.

Sol. The equation of the curve is $r = a(1 + \cos\theta)$ ... (1)

1. Symmetry : The equation of the curve remains unchanged when θ is changed to $-\theta$.

$\therefore$ curve is symmetrical about the initial line.

II. Pole : Putting r = 0 in (1), we get

$a(1 + \cos\theta) = 0$ or $\cos\theta = -1$

$\therefore$ $\theta = \pi$

$\therefore$ pole lies on the curve and tangent at the pole is $\theta = \pi$.

The curve cuts the initial line $\theta = 0$ at (2 a, 0) and the lines $\theta = \pm\dfrac{\pi}{2}$ at $\left(a, \dfrac{\pi}{2}\right), \left(a, -\dfrac{\pi}{2}\right)$.

III. Value of ϕ

$$\frac{dr}{d\theta} = -a\sin\theta$$

$\therefore \quad \tan\phi = r\frac{d\theta}{dr} = a\,(1+\cos\theta)\times\frac{1}{-a\sin\theta} = -\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

$\therefore \quad \tan\phi = \cot\frac{\theta}{2} \Rightarrow \tan\phi = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right) \Rightarrow \phi = \frac{\pi}{2}+\frac{\theta}{2}$

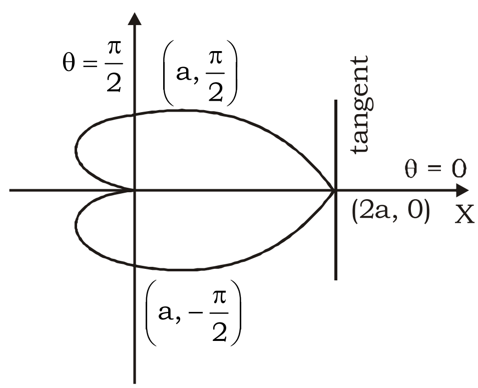
$\therefore \quad \phi = \frac{\pi}{2}$ when $\theta = 0, r = 2a$

$\therefore$ at (2a, 0), the tangent is perpendicular to initial line.

IV. Asymptotes : Since r does not tend to infinity for any finite value of θ.

$\therefore$ curve has got no asymptote

V. Special Points : We have



θ :	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π
r :	2a	$a\left(1+\frac{1}{\sqrt{2}}\right)$	a	0

VI. Region : Since $r = a\,(1 + \cos\theta)$

$\therefore$ max. value of r = 2a

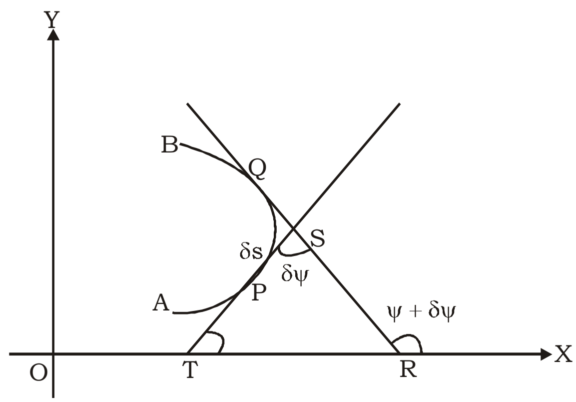
$\therefore$ curve lies entirely within the circle r = 2a

When θ increases from 0 to π, r remains positive and decreases from 2a to 0.
When θ increases from π to 2π, r remains positive and increases from 0 to 2a.
The shape of the curve is as shown in the figure.

VI. Curvature

VI.(a) Radius of Curvature

Let P and Q be any two neighbouring points on a curve AB such that are AP = s and arc AQ = s + δs so that arc PQ = δs. Let the tangents to the curve at P and Q make angles ψ and $\psi + \delta\psi$ with x-axis so that $\angle RST = \delta\psi$. Then



(i) $\delta\psi$, measured in radians, is called the total curvature or total bending of the arc PQ,

(ii) the ratio $\frac{\delta\psi}{\delta s}$ is called the average curvature of the arc PQ,

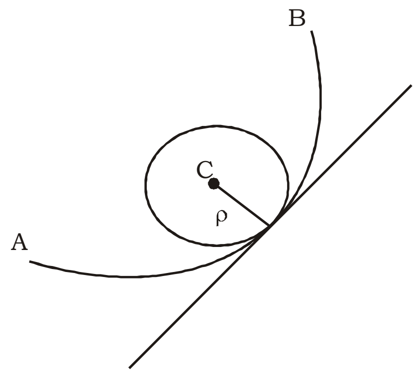
(iii) $\underset{\delta s \to 0}{\text{Lt}} \frac{\delta\psi}{\delta s}$, if it exists, is called the curvature of the curve at P and is denoted by k

(iv) The reciprocal of curvature at any point P is called the radius of curvature and is denoted by Greek letter ρ

$$\rho = \frac{1}{\frac{d\psi}{ds}} = \frac{ds}{d\psi}.$$

VI.(b) Centre of Curvature

The centre of curvature of a curve at a point P is the point C which lies on the positive direction of the normal at P and which is at a distance ρ from it.



The circle with centre C and radius CP = ρ is called circle of curvature of the curve at P.

Any chord of the circle of curvature at P passing through P is called chord of curvature through P.

VI.(c) Some Important Results of Curvature

Result I : The curvature of a circle is constant and is equal to the reciprocal of the radius.

Result II : The radius of curvature at any point of the curve y = f(x) is given by

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} \text{ where } y_1 = \frac{dy}{dx} \text{ and } y_2 = \frac{d^2y}{dx^2}$$

Result III : Rule to find the radius of curvature at the origin.

(a) Put $y = px + q\frac{x^2}{\underline{|2}} + \ldots\ldots\ldots$ in equation of curve, where

$$p = \left(\frac{dy}{dx}\right)_{(0,0)} = f'(0) \text{ and } q = \left(\frac{d^2y}{dx^2}\right)_{(0,0)} = f''(0)$$

(b) Equate the coefficients of like powers of x on both sides and find p, q.

(c) ρ (at the origin) $= \dfrac{(1+p^2)^{3/2}}{q}$

Result IV : The radius of curvature at any point of the curve x = f(t), y = g(t) is given by

$$\rho = \frac{\left[(f'(t))^2 + (g'(t))^2\right]^{\frac{3}{2}}}{f'(t)\,g''(t) - g'(t)\,f''(t)}$$

Result V : The radius of curvature at any point p(r, θ) of the curve r = f(θ) is given by

$$\rho = \frac{\left(r^2 + r_1^2\right)^{\frac{3}{2}}}{r^2 + 2r_1^2 - rr_2} \quad \text{where } r_1 = \frac{dr}{d\theta} \text{ and } r_2 = \frac{d^2r}{d\theta^2}$$

Result VI : The co-ordinates of the centre of curvature for any point P(x, y) of the curve y = f(x), are given by $\left(\bar{x}, \bar{y}\right)$ where $\bar{x} = x - \frac{y_1\left(1 + y_1^2\right)}{y_2}, \bar{y} = y + \frac{1 + y_1^2}{y_2}$.

Further, the equation of circle of curvature at P(x, y) is $\left(x - \bar{x}\right)^2 + \left(y - \bar{y}\right)^2 = \rho^2$.

Now, we clerify the above result with the help of following suitable examples:

Example 5 : Find the radius of curvature of the parabola $y^2 = 4$ ax at the point (x, y).

Sol. The equation of the parabola is $y^2 = 4ax$ ... (1)

Differentiating both sides w.r.t x, we get,

$$2y\frac{dy}{dx} = 4\text{ a or } \frac{dy}{dx} = \frac{2a}{y} \quad \text{... (2)}$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{2a}{y^2}\frac{dy}{dx} = -\frac{2a}{y^2} \times \frac{2a}{y} = -\frac{4a^2}{y^3} \quad \text{... (3)}$$

Now, $$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$\rho = \frac{\left(1 + \frac{4a^2}{y^2}\right)^{\frac{3}{2}}}{\frac{-4a^2}{y^3}} \quad [\because \text{ of (2) and (3)}]$$

$$= -\frac{(y^2+4a^2)^{\frac{3}{2}}}{4a^2} = -\frac{(4ax+4a^2)^{\frac{3}{2}}}{4a^2} \qquad [\because \text{ of } (1)]$$

$$= \frac{2}{\sqrt{a}}(x+a)^{\frac{3}{2}}.$$

Example 6 : Find the radii of curvature at the origin of the curve $y^2 - 3xy + 2x^2 - x^3 + y^4 = 0$

Sol. The equation of curve is $y^2 - 3xy + 2x^2 - x^3 + y^4 = 0$ ... (1)

Clearly (0, 0) lies on (1).

Equating to zero the lowest degree terms, we get,

$$y^2 - 3xy + 2x^2 = 0$$

or $(y-x)(y-2x) = 0 \Rightarrow y = x,\ y = 2x$

Here, neight x-axis nor y-axis is the tangent at origin

$\therefore$ putting $y = px + q\dfrac{x^2}{\lfloor 2} + \ldots\ldots\ldots$ in (1), we get,

$$\left(px+q\frac{x^2}{2}+-\right)^2 - 3x\left(px+q\frac{x^2}{2}+-\right) + 2x^2 - x^3 + \left(px+q\frac{x^2}{2}+-\right)^4 = 0$$

$$\Rightarrow \quad (p^2-3p+2)\,x^2 + \left(pq - 3\frac{q}{2} - 1\right)x^3 + \ldots\ldots\ldots = 0 \qquad \ldots (2)$$

Equating coefficients of x^2 in (2), we get,

$$p^2 - 3p + 2 = 0 \text{ or } (p-1)(p-2) = 0 \Rightarrow p = 1, 2$$

Equating coefficients of x^3 in (2), we get,

$$pq - 3\frac{q}{2} - 1 = 0$$

When $p = 1, q - 3\dfrac{q}{2} - 1 = 0 \Rightarrow q = -2$

When $p = 2, q - 3\dfrac{q}{2} - 1 = 0 \Rightarrow q = 2$

When $p = 1$, $q = -2$

$$\rho\,(\text{at origin}) = \frac{(1+p^2)^{\frac{3}{2}}}{q} = \frac{(1+1)^{\frac{3}{2}}}{-2} = \sqrt{2} \qquad (\text{in magnitude})$$

When $p = 2$, $q = 2$

$$\rho \text{ (at origin)} = \frac{(1+p^2)^{\frac{3}{2}}}{q} = \frac{(1+4)^{\frac{3}{2}}}{2} = \frac{5\sqrt{5}}{2}.$$

Example 7 : Find the circle of curvature at the point $\left(\frac{a}{4}, \frac{a}{4}\right)$ of the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$.

Sol. The equation of curve is $\sqrt{x}+\sqrt{y}=\sqrt{a}$

Differentating both sides of (1) w.r.t.x.

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

$$\frac{d^2y}{dx^2} = -\frac{\sqrt{x}\,.\,\frac{1}{2\sqrt{y}}\frac{dy}{dx} - \sqrt{y}\,.\,\frac{1}{2\sqrt{x}}}{x}$$

$$= -\frac{1}{x}\left[-\frac{\sqrt{x}}{2\sqrt{y}} \times \frac{\sqrt{y}}{\sqrt{x}} - \frac{\sqrt{y}}{2\sqrt{x}}\right]$$

$$= \frac{1}{2x}\, 1 + \frac{\sqrt{y}}{\sqrt{x}} = \frac{1}{2x}\,\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}} = \frac{1}{2x}\left(\frac{\sqrt{a}}{\sqrt{x}}\right) \qquad [\because \text{ of } (1)]$$

$$= \frac{\sqrt{a}}{2x^{\frac{3}{2}}}$$

At $P\left(\frac{a}{4}, \frac{a}{4}\right)$, $y_1 = -1$, $y_2 = \frac{\sqrt{a}}{2\,\frac{a}{4}^{\frac{3}{2}}} = \frac{4}{a}$

$$\therefore \quad \rho = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2} = \frac{(1+1)^{\frac{3}{2}}}{\frac{4}{a}} = 2\sqrt{2} \times \frac{a}{4} = \frac{a}{\sqrt{2}}$$

Let $(\bar{x}, \bar{y})$ be the centre of curvature

$$\therefore \qquad \bar{x} = x - \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2} = \frac{a}{4} - \frac{(-1)\,(1+1)}{\frac{4}{a}} = \frac{a}{4} + \frac{a}{2} = \frac{3a}{4}$$

$$\bar{y} = y + \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2} = \frac{a}{4} + \frac{(1+1)}{\frac{4}{a}} = \frac{3a}{4}$$

$\therefore$ centre of curvature is $\left(\frac{3a}{4}, \frac{3a}{4}\right)$

$\therefore$ equation of circle of curvature at $P\left(\frac{a}{4}, \frac{a}{4}\right)$ is

$$\left(x - \frac{3a}{4}\right)^2 + \left(y - \frac{3a}{4}\right)^2 = \left(\frac{a}{\sqrt{2}}\right)^2$$

or $$\left(x - \frac{3a}{4}\right)^2 + \left(y - \frac{3a}{4}\right)^2 = \frac{a^2}{2}.$$

VII. Self Check Exercise

1. Trace the curve $y = \frac{x^2+1}{x+1}$
2. Trace the curve x = a(θ + sinθ), y = a (1–cosθ)
3. Trace the curve r = a (1+sinθ)
4. Find the radius of curvature for the parabola $\frac{2a}{r} = 1 + \cos\theta$.

VIII. Suggested Readings

1.	Ahsan Akhtar & Sabita Ahsan	:	Differential Calculus
2.	UP Singh, RJ Srivastava & NH Siddiqui	:	Differential Calculus
3.	Gorakh Prasad	:	Differential Calculus
4.	Malik and Arora	:	Mathematical Analysis
5.	Thomas and Finney (Ninth Edition)		Calculus and Analytic Geometry

LESSON NO. 2.1

Author: Dr. Chanchal

LIMIT, CONTINUITY AND PARTIAL DIFFERENTIATION OF FUNCTIONS OF TWO VARIABLES-I

Structure:

Objectives

The prime goal of this lesson is to enlighten the basic concepts of real valued functions of two variables $f(x,y)$. During the study in this particular lesson, our main objectives are

- To discuss the limit of function $f(x,y)$ and how this limit can be classified.
- To discuss the continuity of function $f(x,y)$.

I. Introduction

From our previous study, we are already familiar with the concepts of limit, continuity and differentiability of the real valued functions $f(x)$. In this unit, we have introduced the concept of real valued functions of two real variables. In this lesson, we start with the study of limit and continuity of the function $f(x,y)$ as already highlighted under the objectives of this lesson. Before starting the main part of this lesson, we define some basic concepts below :

$\Re^2$ **:** Mathematically, $\Re^2 = \Re \times \Re = \{(x,y) : x \in \Re, y \in \Re\}$ and geometrically, $\Re^2$ represents two dimensional plane.

Here $\Re$ represents the set of real numbers.

Square Neighborhood of a Point : A square neighborhood of a point (a,b) in $\Re^2$ is the set of points (x,y) that lie inside an open square region with centre at (a,b) and sides parallel to the co-ordinate axes such that

$|x-a| < \delta$ and $|y-b| < \delta$ for some $\delta > 0$.

In other words, it may be represented as $\{(x,y): |x-a| < \delta, |y-b| < \delta\}$.

Circular Neighborhood of a Point : A circular neighborhood of a point (a,b) in $\Re^2$ is the set of points (x,y) that lie inside a circle with centre at (a,b) such that

$(x-a)^2 + (y-b)^2 < \delta^2$ for some $\delta > 0$.

It may also be represented as $\{(x,y): (x-a)^2 + (y-b)^2 < \delta^2\}$.

Functions of Two Variables : A real valued function of two variables x and y is a rule which associates a unique real number $f(x,y)$ to every possible ordered pair (x,y) of real numbers.

Note : Usually, we write $z = f(x,y)$ where x and y are independent variables and z is the dependent variable.

II. Limit of a Function of Two Variables

A function $f(x,y)$ is said to tend to a limit l as the point (x,y) tends to a point (a,b) if for any pre-assigned positive number $\in > 0$, however small, we can find a number δ such that

$|f(x,y) - l| < \in$

for all points (x,y) other than (a,b) for which $a-\delta < x < a+\delta, b-\delta < y < b+\delta$ i.e. $|x-a| < \delta$ and $|y-b| < \delta$.

The above definition of limit is based on square neighborhood of a point. It may also be defined as :

A function $f(x,y)$ is said to tend to a limit l as the point (x,y) tends to a point (a,b) if for any pre-assigned positive number $\in > 0$, however small, we can find a number δ such that

$|f(x,y) - l| < \in$

for all points (x,y) other than (a,b) for which

$|(x,y) - (a,b)| < \delta$.

This definition is based on circular neighborhood of a point.

Note : 1. If the limit $\underset{(x,y)\to(a,b)}{lt} f(x,y)$ exists finitely, then it is unique.

2. If $\underset{(x,y)\to(a,b)}{lt} f(x,y)$ exists, then the limit is independent of the path along which we approach the point (a,b).

Example 1 : By using definition, prove that $\underset{\substack{x\to 0\\ y\to 0}}{lt}\left(x^2+y^2\right)=0.$

Sol. Here, $f:\Re^2\to\Re$ is defined by $f(x,y)=x^2+y^2$

Let $\in>0$ and take $\delta=\sqrt{\in}$

$\therefore\left|(x,y)-(0,0)\right|<\delta$ i.e. $\sqrt{x^2+y^2}<\delta$

$\Rightarrow\left|f(x,y)-0\right|=\left|x^2+y^2\right|=x^2+y^2<\delta^2=\in$

$\therefore$ for $\in>0$, there exists $\delta>0$ such that $\left|(x,y)-(0,0)\right|<\delta \Rightarrow \left|f(x,y)-0\right|<\in$

so, $\underset{(x,y)\to(0,0)}{lt} f(x,y)=0$.

Hence, the result is proved.

Example 2 : Let $f:\Re^2\to\Re$ be defined as : $f(x,y)=1$ if x is rational and $f(x,y)=0$ if x is irrational. Show that $\underset{(x,y)\to(x_0,y_0)}{lt} f(x,y)$ does not exist.

Sol. If possible, suppose that $\underset{(x,y)\to(x_0,y_0)}{lt} f(x,y)=l$

So, there exist reals $\delta_1>0,\delta_2>0$ such that

$0<\left|x-x_0\right|<\delta_1, 0<\left|y-y_0\right|<\delta_2$, where $(x,y)\neq(x_0,y_0)$

$\Rightarrow\left|f(x,y)-l\right|<\frac{1}{2}$ (say)

Let x_1 be any irrational number and x_2 be any rational number in $(x_0-\delta_1,x_0+\delta_1)$ and let y be any real number different from y_0 in $(y_0-\delta_2,y_0+\delta_2)$.

$\therefore\left|f(x_1,y)-l\right|<\frac{1}{2}$ and $\left|f(x_2,y)-l\right|<\frac{1}{2}$ (1)

Since x_1 is irrational and x_2 is rational,

$f(x_1,y)=0$ and $f(x_2,y)=1$ (2)

From (1) and (2), $\left|-l\right|<\frac{1}{2}$ and $\left|1-l\right|<\frac{1}{2}$

Or $\left|l\right|<\frac{1}{2}$ and $\left|1-l\right|<\frac{1}{2}$ (3)

Further, $1=\left|(1-l)+l\right|\leq\left|1-l\right|+\left|l\right|$

$<\frac{1}{2}+\frac{1}{2}=1$ [using (3)]

$\therefore 1<1$, which is absurd and our supposition is wrong.

Hence, $\underset{(x,y)\to(x_0,y_0)}{lt} f(x,y)$ does not exist.

III. Simultaneous and Iterated (or Repeated) Limits

If $f(x,y)$ is a function of two variables x and y and then (a,b) is the limiting point of a set of values on two dimensional space, then we have

Simultaneous Limit :

I. $\underset{\substack{x\to a\\ y\to b}}{lt} f(x,y)$ or $\underset{(x,y)\to(a,b)}{lt} f(x,y)$

Iterated or Repeated Limits :

II. $\underset{x\to a}{lt}\left[\underset{y\to b}{lt} f(x,y)\right]$ or $\underset{x\to a}{lt}\ \underset{y\to b}{lt} f(x,y)$

III. $\underset{y\to b}{lt}\left\lfloor\underset{x\to a}{lt} f(x,y)\right\rfloor$ or $\underset{y\to b}{lt}\ \underset{x\to a}{lt} f(x,y)$

Note: **1.** The two iterated limits may exist but may not be equal.

2. The two iterated limits may exist and may be equal but the simultaneous limit may not exist.

3. If simultaneous limit exists and two iterated limits also exist, then they must be equal.

4. Criterion for Non-Existence of Simultaneous Limit :

If we find two functions $y=\phi(x)$ and $y=\psi(x)$ such that $\underset{x\to a}{lt}\phi(x)=\underset{x\to a}{lt}\psi(x)=b$

and $\underset{x\to a}{lt} f(x,\phi(x))\neq\underset{x\to b}{lt} f(x,\psi(x))$,

then, $\underset{\substack{x\to a\\ y\to b}}{lt} f(x,y)$ does not exist.

Example 3 : If a function f be defined by $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2},(x,y)\neq(0,0)$, then show that the two iterated limits $\underset{x\to 0}{lt}\left[\underset{y\to 0}{lt} f(x,y)\right]$ and $\underset{y\to 0}{lt}\left\lfloor\underset{x\to 0}{lt} f(x,y)\right\rfloor$ exist but the simultaneous limit $\underset{(x,y)\to(0,0)}{lt} f(x,y)$ does not exist.

Sol. $\underset{x\to 0}{lt}\left[\underset{y\to 0}{lt} f(x,y)\right]=\underset{x\to 0}{lt}\left[\underset{y\to 0}{lt}\dfrac{x^2-y^2}{x^2+y^2}\right]=\underset{x\to 0}{lt}\left(\dfrac{x^2}{x^2}\right)=1$

and $\underset{y\to 0}{lt}\left[\underset{x\to 0}{lt} f(x,y)\right]=\underset{y\to 0}{lt}\left[\underset{x\to 0}{lt}\dfrac{x^2-y^2}{x^2+y^2}\right]=\underset{x\to 0}{lt}\left(\dfrac{-y^2}{y^2}\right)=-1$

Therefore, two iterated limits $\underset{x\to 0}{lt}\left[\underset{y\to 0}{lt} f(x,y)\right]$ and $\underset{y\to 0}{lt}\left[\underset{x\to 0}{lt} f(x,y)\right]$ exist.

Let $(x,y)\to(0,0)$ along the line $y=mx$, then

$$\underset{(x,y)\to(0,0)}{lt} f(x,y)=\underset{(x,y)\to(0,0)}{lt}\frac{x^2-y^2}{x^2+y^2}=\underset{x\to 0}{lt}\frac{x^2-m^2x^2}{x^2+m^2x^2}=\underset{x\to 0}{lt}\frac{1-m^2}{1+m^2}=\frac{1-m^2}{1+m^2},$$

which is not unique as it takes different values for different values of m.

$\therefore$ simultaneous limit $\underset{(x,y)\to(0,0)}{lt} f(x,y)$ does not exist.

IV. Continuity of Functions of Two Variables

A function $f(x,y)$ is said to be continuous at (x_0,y_0) if for every $\in>0$, there exists $\delta>0$ such that

$$|x-x_0|<\delta,\ |y-y_0|<\delta$$

$$\Rightarrow |f(x,y)-f(x_0,y_0)|<\in$$

Or $f(x,y)$ is said to be continuous at (x_0,y_0) if the simultaneous limit $\underset{(x,y)\to(x_0,y_0)}{lt} f(x,y)$ exists and is equal to the functional value $f(x_0,y_0)$ at (x_0,y_0).

Example 4 : Discuss the continuity of $f(x,y)$ at (0,0) where

$$f(x,y)=\begin{cases}\dfrac{2xy^2}{x^3+y^3},(x,y)\neq(0,0)\\ 0,(x,y)=(0,0)\end{cases}$$

Sol. Here,

$$f(x,y)=\begin{cases}\dfrac{2xy^2}{x^3+y^3},(x,y)\neq(0,0)\\ 0,(x,y)=(0,0)\end{cases}$$

Let $(x,y)\to(0,0)$ along the line $y=mx$, then

$$\underset{(x,y)\to(0,0)}{lt} f(x,y)=\underset{(x,y)\to(0,0)}{lt}\frac{2xy^2}{x^3+y^3}=\underset{x\to 0}{lt}\frac{2m^2x^3}{x^3+m^3x^3}=\underset{x\to 0}{lt}\frac{2m^2}{1+m^3}=\frac{2m^2}{1+m^3}$$

which is not unique as it takes different values for different values of m.

$\therefore$ simultaneous limit $\underset{(x,y)\to(0,0)}{lt} f(x,y)$ does not exist.

Hence, the function $f(x,y)$ is not continuous at (0,0).

Example 5 : Examine the function $f(x,y)=xy\sin\left(\frac{1}{x}\right), x\neq 0, y\neq 0$ and $f(0,0)=0$ for continuity at the origin.

Sol. For the given $f(x,y)$, $D_f = \Re^2$

Let $\varepsilon > 0$ be arbitrary, then $|x-0| < \sqrt{\varepsilon}, |y-0| < \sqrt{\varepsilon}$, where $(x,y) \neq (0,0)$.

$$\Rightarrow |f(x,y)-0| = \left|xy\sin\frac{1}{x}\right| = |xy|\left|\sin\frac{1}{x}\right| \leq |x||y| \qquad \left[\because \left|\sin\frac{1}{x}\right| \leq 1\right]$$

$$< \sqrt{\varepsilon}\sqrt{\varepsilon} = \varepsilon$$

$$\therefore |f(x,y)-0| < \varepsilon$$

$$\Rightarrow \underset{(x,y)\to(0,0)}{lt} f(x,y) = 0 = f(0,0)$$

Hence, f is continuous at (0,0).

Example 6 : Let $f:\Re^2 \to \Re$ be a continuous function. Define $g:\Re^2 \to \Re$ as $g(x,y) = f(x,y), x \neq 0, y \neq 0$ and $g(x,y) = f(x,y)+1, x=0, y=0$. Show that the function g is not continuous at the origin.

Sol. Being a continuous function, f is continuous at the origin (0,0).

$\Rightarrow \underset{(x,y)\to(0,0)}{lt} f(x,y)$ and $f(0,0)$ both exist and $\underset{(x,y)\to(0,0)}{lt} f(x,y) = f(0,0)$

Now $g(0,0) = f(0,0)+1$ (1)

Also, $\underset{(x,y)\to(0,0)}{lt} g(x,y) = \underset{(x,y)\to(0,0)}{lt} f(x,y) = f(0,0)$ (2)

From (1) and (2), $\underset{(x,y)\to(0,0)}{lt} g(x,y) \neq g(0,0)$.

So, g is not continuous at the origin.

V. Self Check Exercise

1. By using definition, prove that

   (i) $\underset{\substack{x\to1\\y\to2}}{lt}(3xy+5x-4) = 7$ (ii) $\underset{\substack{x\to3\\y\to-1}}{lt}(3x+y^2) = 10$

2. Let $f:\Re^2 \to \Re$ be defined as $f(x,y)=1$ if x is irrational and $f(x,y)=0$ if x is rational. Show that for any point (x_0,y_0), $\underset{(x,y)\to(x_0,y_0)}{lt} f(x,y)$ does not exist.

3. Show that for the function f defined by $f(x,y) = \dfrac{x^2y^2}{x^2y^2+(x-y)^2}$, the two repeated limits exist and are equal but the simultaneous limit does not exist.

4. Evaluate the following limits (if they exist):

   (i) $\underset{(x,y)\to(0,0)}{lt} \dfrac{x^4}{x^4+y^4-x}$ (ii) $\underset{(x,y)\to(0,0)}{lt} \dfrac{xy^2}{x^3+2y-3x}$

5. Examine the function $f(x,y)=\dfrac{x^3+y^3}{x^2+y^2}, x\neq 0, y\neq 0$ and $f(0,0)=0$ for continuity at $(0,0)$.

6. Show that $f(x,y)=x^2+y-1$ is continuous at (1,-2).

Suggested Readings

1. RK Jain, SRK Lyenger — Advanced Engineering Mathematics
2. JR Sharma — Advanced Calculus
3. Malik and Arora — Mathematical Analysis
4. Shanti Narayan — Mathematical Analysis
5. Thomas and Finney — Calculus and Analytical Geometry

LESSON NO. 2.2 **Author: Dr. Chanchal**

LIMIT, CONTINUITY AND PARTIAL DIFFERENTIATION OF FUNCTIONS OF TWO VARIABLES-II

Objectives

- To learn about the first order and second order partial derivatives.
- To study the conditions for the interchange of order of differentiation.

I. Partial Derivative

Let $z = f(x, y)$ be a function of two independent variables x and y. Then, the partial derivative of z with respect to x is the ordinary derivative of z when y is regarded as a constant. Similarly, the the partial derivative of z with respect to y is the ordinary derivative of z when x is regarded as a constant.

For example : If $z = 3x^3y^2 + 5x^2y^3$

then, partial derivative of z w.r.t. x is equal to

$3y^2(3x^2) + 5y^3(2x) = 9y^2x^2 + 10y^3x$

Similarly, partial derivative of z w.r.t. y is equal to

$3x^3(2y) + 5x^2(3y^2) = 6x^3y + 15x^2y^2$

I.(a) First Order Partial Derivatives

Let $z = f(x, y)$ be a real valued function of two independent variables with an open domain $D_f \subset \Re^2$, then $\underset{\delta x \to 0}{lt} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$ if it exists, is called the partial derivative of z w.r.t. x. It is denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or f_x or $f_x(x, y)$ or $D_1 f$.

$$\therefore f_x = \frac{\partial f}{\partial x} = \underset{\partial x \to 0}{lt} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

Also, the partial derivative of f w.r.t. x at any point $(a,b) \in D_f$ is denoted by $\left(\frac{\partial z}{\partial x}\right)_{(a,b)}$ or $\left(\frac{\partial f}{\partial x}\right)_{(a,b)}$ or $f_x(a,b)$.

Thus, $f_x(a,b) = \underset{h \to 0}{lt} \frac{f(a+h, b) - f(a,b)}{h}$

On similar lines, $\underset{\delta y \to 0}{lt} \frac{f(x, y+\delta y) - f(x, y)}{\delta y}$, if it exists, is called the partial derivative of f w.r.t. y. It is denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or f_y or $f_y(x, y)$ or $D_2 f$.

Thus, $f_y = \frac{\partial f}{\partial y} = \underset{\partial y \to 0}{lt} \frac{f(x, y+\delta y) - f(x, y)}{\delta y}$

The partial derivative of f w.r.t. y at any point $(a,b) \in D_f$ is denoted by $\left(\frac{\partial z}{\partial y}\right)_{(a,b)}$ or $\left(\frac{\partial f}{\partial y}\right)_{(a,b)}$ or $f_y(a,b)$ and it may be expressed as

$$f_y(a,b) = \underset{k \to 0}{lt} \frac{f(a, b+k) - f(a,b)}{k}$$

I.(b) Second Order Partial Derivatives

If $z = f(x, y)$ and the first order partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist, then they are themselves functions of x and y. The partial derivatives of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, if they exist, are called second order partial derivatives of $f(x, y)$. It is denoted as

I. $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}$ or $f_x^{\,2}$ or f_{xx} or $\frac{\partial^2 z}{\partial x^2}$

II. $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\partial x}$ or f_{xy} or $\frac{\partial^2 z}{\partial y\partial x}$

III. $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\partial y}$ or f_{yx} or $\frac{\partial^2 z}{\partial x\partial y}$

IV. $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}$ or $f_y^{\,2}$ or f_{yy} or $\frac{\partial^2 z}{\partial y^2}$

These second order partial derivatives may be further expressed as

$$\frac{\partial^2 f}{\partial x^2}=\underset{\partial x\to 0}{lt}\frac{f_x(x+\delta x,y)-f_x(x,y)}{\delta x}$$

$$\frac{\partial^2 f}{\partial y\partial x}=\underset{\partial y\to 0}{lt}\frac{f_x(x,y+\delta y)-f_x(x,y)}{\delta y}$$

$$\frac{\partial^2 f}{\partial x\partial y}=\underset{\partial x\to 0}{lt}\frac{f_y(x+\delta x,y)-f_y(x,y)}{\delta x}$$

$$\frac{\partial^2 f}{\partial y^2}=\underset{\partial y\to 0}{lt}\frac{f_y(x,y+\delta y)-f_y(x,y)}{\delta y}$$

Example 1 : If $f(x,y)=\begin{cases}\sin\left(\frac{xy}{x^2+y^2}\right),(x,y)\neq(0,0)\\ 0,(x,y)=(0,0)\end{cases}$,

then evaluate $f_x(0,0)$ and $f_y(0,0)$.

Sol. For the given function $f(x,y)$, we have

$$f_x(0,0)=\underset{h\to 0}{lt}\frac{f(0+h,0)-f(0,0)}{h}=\underset{h\to 0}{lt}\frac{\sin 0-0}{h}=\underset{h\to 0}{lt}\frac{0-0}{h}=0$$

and $f_y(0,0)=\underset{k\to 0}{lt}\frac{f(0,0+k)-f(0,0)}{k}=\underset{k\to 0}{lt}\frac{\sin 0-0}{k}=\underset{k\to 0}{lt}\frac{0-0}{k}=0$

Example 2 : Let $f(x,y)=\log\left(xy+2y^2-2x\right)$. Find $f_x(2,3)$ and $f_y(2,3)$.

Sol. Here $f(x,y)=\log\left(xy+2y^2-2x\right)$

So, $f_x=\frac{\partial}{\partial x}\left[\log\left(xy+2y^2-2x\right)\right]=\frac{1}{xy+2y^2-2x}\times\left(y-2\right)=\frac{y-2}{xy+2y^2-2x}$

$$\therefore f_x(2,3) = \frac{3-2}{(2)(3)+(2)(3)^2-2(2)} = \frac{1}{6+18-4} = \frac{1}{20}$$

Also, $f_y = \frac{\partial}{\partial y}\left[\log\left(xy+2y^2-2x\right)\right] = \frac{1}{xy+2y^2-2x}\times\left(x+4y\right) = \frac{x+4y}{xy+2y^2-2x}$

$$\therefore f_y(2,3) = \frac{2+12}{6+18-4} = \frac{14}{20} = \frac{7}{10}$$

II. Change of Independent Variables

Consider the two relations : $x = r\cos\theta, y = r\sin\theta$ connecting the four variables x, y, r and θ. Here, each variable can be expressed in terms of two of the remaining three variables.

For example : x can be expressed in terms of (i) r,θ (ii) θ, y (iii) r, y

Now, if we have to find $\frac{\partial x}{\partial r}$, then it is meaningful in (i) and (iii) while it has no meaning in (ii).

Note : $\frac{\partial x}{\partial r}$ in (i) keeping θ constant is not equal to $\frac{\partial x}{\partial r}$ in (iii) keeping y constant. So, it is necessary to distinguish between the two values of $\frac{\partial x}{\partial r}$ which can be done by denoting them as $\left(\frac{\partial x}{\partial r}\right)_\theta$ and $\left(\frac{\partial x}{\partial r}\right)_y$ respectively.

Thus, $\left(\frac{\partial x}{\partial r}\right)_\theta$ = partial derivative of x where r and θ are independent variables and

$\left(\frac{\partial x}{\partial r}\right)_y$ = partial derivative of x where r and y are the independent variables.

Example 3 : If $x = r\cos\theta, y = r\sin\theta$, the prove that

$$\frac{\partial^2\theta}{\partial x\partial y} = \frac{\partial^2}{\partial x^2}\left(\log r\right) = -\frac{\partial^2}{\partial y^2}\left(\log r\right) = -\frac{1}{r^2}\cos 2\theta$$

Sol. Given: $x = r\cos\theta, y = r\sin\theta$

on dividing y by x, we get $\tan\theta = \frac{y}{x}$

which gives $\theta = \tan^{-1}\frac{y}{x}$

$$\therefore \frac{\partial\theta}{\partial y}=\frac{1}{1+\left(\frac{y}{x}\right)^2}.\frac{1}{x}=\frac{x}{x^2+y^2}$$

$$\Rightarrow \frac{\partial^2\theta}{\partial x\partial y}=\frac{(x^2+y^2).1-x.2x}{\left(x^2+y^2\right)^2}=\frac{y^2-x^2}{\left(x^2+y^2\right)^2}=\frac{r^2\sin^2\theta-r^2\cos^2\theta}{\left(r^2\right)^2}=-\frac{\cos^2\theta-\sin^2\theta}{r^2}$$ which

gives $\frac{\partial^2\theta}{\partial x\partial y}=-\frac{1}{r^2}\cos 2\theta$ ....(1)

Now, squaring and adding x and y, we have $r^2=x^2+y^2$

$\Rightarrow \log r^2=\log\left(x^2+y^2\right) \quad \Rightarrow 2\log r=\log\left(x^2+y^2\right)$

$\Rightarrow \log r=\frac{1}{2}\log\left(x^2+y^2\right)$

$$\therefore \frac{\partial}{\partial x}(\log r)=\frac{1}{2}.\frac{1}{x^2+y^2}.2x=\frac{x}{x^2+y^2}$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}(\log r)=\frac{\left(x^2+y^2\right).1-x.2x}{\left(x^2+y^2\right)^2}=\frac{y^2-x^2}{\left(x^2+y^2\right)^2}=-\frac{1}{r^2}\cos 2\theta \quad ....(2)$$

Now, $\frac{\partial}{\partial y}(\log r)=\frac{1}{2}.\frac{1}{x^2+y^2}.2y=\frac{y}{x^2+y^2}$

$$\Rightarrow \frac{\partial^2}{\partial y^2}(\log r)=\frac{\left(x^2+y^2\right).1-y.2y}{\left(x^2+y^2\right)^2}=\frac{x^2-y^2}{\left(x^2+y^2\right)^2}=\frac{1}{r^2}\cos 2\theta \quad ....(3)$$

From (1), (2) and (3), we get

$$\frac{\partial^2\theta}{\partial x\partial y}=\frac{\partial^2}{\partial x^2}(\log r)=-\frac{\partial^2}{\partial y^2}(\log r)=-\frac{1}{r^2}\cos 2\theta .$$

III. Interchange of Order of Differentiation

For some functions, f_{xy} and f_{yx} both exist and equal i.e. we can change the order of differentiation. But this result is not always true. For all classes of functions, we cannot interchange the order of differentiation

i.e. $f_{xy}(a,b)\neq f_{yx}(a,b)$ (In general)

Here, $f_{xy}(a,b)=\lim_{k\to 0}\frac{f_x(a,b+k)-f_x(a,b)}{k}$

where $f_x(a,b+k)=\lim_{h\to 0}\frac{f(a+h,b+k)-f(a,b+k)}{h}$ and

$$f_x(a,b) = \underset{h\to 0}{lt} \frac{f(a+h,b)-f(a,b)}{h}$$

$\therefore$ on substituting, we get

$$f_{xy}(a,b) = \underset{k\to 0}{lt}\,\underset{h\to 0}{lt} \frac{f(a+h,b+k)-f(a,b+k)-f(a+h,b)+f(a,b)}{hk} = \underset{k\to 0}{lt}\,\underset{h\to 0}{lt} \frac{\phi(h,k)}{hk}$$

where $\phi(h,k) = f(a+h,b+k)-f(a,b+k)-f(a+h,b)+f(a,b)$

Similarly, $f_{yx}(a,b) = \underset{h\to 0}{lt}\,\underset{k\to 0}{lt} \frac{\phi(h,k)}{hk}$

Thus, $f_{xy}(a,b)$ and $f_{yx}(a,b)$ appear as repeated limits of the same function and we know that these limits may or may not be equal. Therefore, the equality of the two derivatives may not be taken surely.

III.(a) Sufficient Condition for the Interchange of Order of Differentiation

The below stated theorems give the sufficient condition for the equality of f_{xy} and f_{yx}

A. Statement of Schwarz's Theorem : If (a,b) is a point of the domain $D_f \subset \Re^2$ of a function f such that

i. f_x, f_y, f_{xy} all exist in a certain neighborhood of (a,b),

ii. f_{xy} is continuous at (a,b),

then $f_{yx}(a,b)$ exists and $f_{yx}(a,b) = f_{xy}(a,b)$.

B. Statement of Young's Theorem : If (a,b) is a point of the domain $D_f \subset \Re^2$ of a function f such that

i. f_x, f_y both exist in a certain neighborhood of (a,b),

ii. f_x, f_y are differentiable at (a,b),

then $f_{yx}(a,b) = f_{xy}(a,b)$.

We will prove these theorems in the next lesson of this unit.

Example 4 : Let $f(x,y) = xy\left(\frac{x^2-y^2}{x^2+y^2}\right)$, where $(x,y) \neq (0,0)$ and $f(0,0)=0$. Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Sol. We have $f_{xy}(0,0) = \underset{k\to 0}{lt} \frac{f_x(0,0+k)-f_x(0,0)}{k}$ ...(1)

Now, $f_x(0,k) = \underset{h\to 0}{lt}\dfrac{f(0+h,k)-f(0,k)}{h} = \underset{h\to 0}{lt}\dfrac{hk\left(\dfrac{h^2-k^2}{h^2+k^2}\right)-0}{h} = k\left(\dfrac{0-k^2}{0+k^2}\right)$

$\therefore f_x(0,k) = -k$ ...(2)

Also, $f_x(0,0) = \underset{h\to 0}{lt}\dfrac{f(0+h,0)-f(0,0)}{h} = \underset{h\to 0}{lt}\dfrac{0-0}{h}$

$\therefore f_x(0,0) = 0$ ...(3)

From (1), (2) and (3), we get $f_{xy}(0,0) = \underset{k\to 0}{lt}\dfrac{-k-0}{k} = -1$ ...(4)

Similarly,

$f_{yx}(0,0) = \underset{h\to 0}{lt}\dfrac{f_y(0+h,0)-f_y(0,0)}{h} = \underset{h\to 0}{lt}\dfrac{h-0}{h} = 1$, ...(5)

where $f_y(h,0) = \underset{k\to 0}{lt}\dfrac{f(h,0+k)-f(h,0)}{k} = \underset{k\to 0}{lt}\dfrac{hk\left(\dfrac{h^2-k^2}{h^2+k^2}\right)-0}{k} = h\left(\dfrac{h^2-0}{h^2+0}\right) = h$,

and $f_y(0,0) = \underset{k\to 0}{lt}\dfrac{f(0,0+k)-f(0,0)}{k} = \underset{k\to 0}{lt}\dfrac{0-0}{k} = 0$

It is clear from (4) and (5) that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

IV. Self Check Exercise

1. Find the first order partial derivatives for $z =$

   (i) $\log(x^2+y^2)$ (ii) $\sin(x^2+y^2)$ (iii) $\sin^{-1}\left(\dfrac{x}{y}\right)$

2. Let $f(x,y) = \sqrt{x^4+y^4+1}$. Evaluate $f_x(1,2)$ and $f_y(1,2)$.

3. For the following $f(x,y)$, find the second order partial derivatives

   (i) $\sin\left(\dfrac{y}{x}\right)$ (ii) $x^2\sin(x+y)$ (iii) e^{x-y}

4. Verify that $\dfrac{\partial^2 f}{\partial y\partial x} = \dfrac{\partial^2 f}{\partial x\partial y}$ where $f = \log\left(\dfrac{x^2+y^2}{xy}\right)$.

5. If $x = r\cos\theta, y = r\sin\theta$, prove that

   (i) $\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2 = 1$ (ii) $\dfrac{\partial^2 r}{\partial x^2}\times\dfrac{\partial^2 r}{\partial y^2} = \left(\dfrac{\partial^2 r}{\partial x\partial y}\right)^2$

6. Let $f(x,y) = xy\left(\frac{x-y}{x+y}\right)$, where $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

7. For the function, $f(x,y) = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right), xy \neq 0$ and $f(x,y) = 0$ if $xy = 0$, show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Suggested Readings

1. RK Jain, SRK Lyenger	Advanced Engineering Mathematics
2. JR Sharma	Advanced Calculus
3. Malik and Arora	Mathematical Analysis
4. Shanti Narayan	Mathematical Analysis
5. Thomas and Finney	Calculus and Analytical Geometry

LESSON NO. 2.3 **Author: Dr. Chanchal**

SOME BASIC THEOREMS ON DIFFERENTIABILITY OF $f(x,y)$

Structure:

Objectives

The prime goal of this lesson is to understand the concept of differentiability of real valued functions of two or more variables and to study some basic theorems concerning these such as

- Schwarz's theorem and Young's theorem that illustrates the sufficient conditions for the interchange of order of differentiation and already stated in the previous lesson.
- Euler's theorem of homogeneous functions of the form $z = x^n f\left(\frac{y}{x}\right)$ and

  Taylor's theorem for function of two variables.

I. Introduction

As we have already discussed about the first order and second order partial derivatives of the function $f(x,y)$ in the previous lesson. So at this level, we have the enough knowledge to discuss about the differentiability or derivability of the function $f(x,y)$ and to understand various important results and theorems which are concerned with the partial derivatives and differentiability of $f(x,y)$. Before starting the main part of this lesson, it is required to define the differentiability of $f(x,y)$.

Def : Differentiability of $f(x,y)$: A function $f(x,y)$ with domain $D_f \subset \Re^2$ is known as differentiable at a point $(x_0, y_0) \in D_f$, if in a neighborhood of (x_0, y_0), it can be represented as $f(x_0+h, y_0+k) = f(x_0, y_0) + Ah + Bk + \in_1 h + \in_2 k$

Here, A, B are independent of variables h, k and $\in_1, \in_2$ tend to zero as h, k tend to zero independently.

II. Some Important Results

A. If a function $f(x,y)$ is differentiable at a point $(x_0, y_0) \in D_f \subset \Re^2$, then prove that it is continuous at that point (the proof is left for the reader).

B. If a function $f(x,y)$ is differentiable at a point (x_0, y_0) , then prove that $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ both exist (the proof is left for the reader).

C. **Sufficient Condition for Differentiability :** If $f_x(x,y)$ and $f_y(x,y)$ are defined in a neighborhood of (x_0, y_0) and are continuous at (x_0, y_0), then f is differentiable at (x_0, y_0) (the proof is left for the reader).

Example 1 : Show that $f(x,y) = \sin x + \cos y$ is differentiable at every point of $\Re^2$.

Sol. Here, $f(x,y) = \sin x + \cos y$

$\therefore f_x = \cos x, f_y = -\sin y$

$\Rightarrow f(x+h, y+k) - f(x,y) - hf_x(x,y) - kf_y(x,y)$

$= \sin(x+h) + \cos(y+k) - \sin x - \cos y - h\cos x + k\sin y$

$$= h\left[\frac{\sin(x+h)-\sin x}{h} - \cos x\right] + k\left[\frac{\cos(y+k)-\cos y}{k} + \sin y\right]$$

$h \in_1 + k \in_2$

where $\in_1 = \dfrac{\sinh(x+h)-\sin x}{h} - \cos x$ and $\in_2 = \dfrac{\cos(y+k)-\cos y}{k} + \sin y$

Now, $\underset{h,k\to 0}{lt} \in_1 = \underset{h\to 0}{lt}\left[\dfrac{\sin(x+h)-\sin x}{h} - \cos x\right] = \underset{h\to 0}{lt}\left[\dfrac{2\cos\left(x+\dfrac{h}{2}\right)\sin\dfrac{h}{2}}{h} - \cos x\right]$

$$= \underset{h\to 0}{lt}\left[\cos\left(x+\frac{h}{2}\right)\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right) - \cos x\right] = \cos x - \cos x = 0$$

Also, $\underset{h,k\to 0}{lt} \in_2 = \underset{k\to 0}{lt}\left[\dfrac{\cos(y+k)-\cos y}{k} + \sin y\right] = \underset{k\to 0}{lt}\left[\dfrac{-2\sin\left(y+\dfrac{k}{2}\right)\sin\dfrac{k}{2}}{k} + \sin y\right]$

$$= -\underset{k\to 0}{lt}\left[\sin\left(y+\frac{k}{2}\right)\left(\frac{\sin\frac{k}{2}}{\frac{k}{2}}\right) + \sin y\right] = -\sin y + \sin y = 0$$

So, both $\in_1, \in_2$ tend to zero as h, k tend to zero independently. Hence, f is differentiable at every point of $(x, y) \in \Re^2$.

III. Some Important Theorems

Schwarz's Theorem

Statement : If (a,b) is a point of the domain $D_f \subset \Re^2$ of a function f such that

i. f_x, f_y, f_{xy} all exist in a certain neighborhood of (a,b),

ii. f_{xy} is continuous at (a,b),

then $f_{yx}(a,b)$ exists and $f_{yx}(a,b) = f_{xy}(a,b)$.

Proof : From the given conditions, it is clear that there exists a certain nhd. of (a,b) at every point of (x, y) of which f_x, f_y, f_{xy} .

Let $(a+h, b+k)$ be any point of the nbd.

Consider $\phi(x) = f(x, b+k) - f(x, b)$ (1)

$\therefore \phi(a+h) - \phi(a) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b) = \Delta^2 f$ (say)

By Lagrange's mean value theorem

$\phi(a+h)-\phi(a)=h\phi'(a+\theta h), 0<\theta<1$ (2)

From (1), $\phi'(x)=f_x(x,b+k)-f_x(x,b)$

$\therefore \phi'(a+\theta h)=f_x(a+\theta h,b+k)-f_x(a+\theta h,b)$

So, from (2), we have $\phi(a+h)-\phi(a)=h\left[f_x(a+\theta h,b+k)-f_x(a+\theta h,b)\right]$ (3)

Again by Lagrange's mean value theorem, we get

$f_x(a+\theta h,b+k)-f_x(a+\theta h,b)=kf_{yx}(a+\theta h,b+\theta' k), 0<\theta'<1$

Now, from (3), we get

$\phi(a+h)-\phi(a)=hkf_{yx}(a+\theta h,b+\theta' k)=hk\left[f_{yx}(a,h)+\varepsilon_1\right]$, where $\varepsilon_1\to 0$ as $h,k\to 0$.

$\therefore \Delta^2 f=hk\left[f_{yx}(a,b)+\varepsilon_1\right]$

$\Rightarrow \dfrac{\Delta^2 f}{hk}=f_{yx}(a,b)+\varepsilon_1$ (4)

Similarly, $\dfrac{\Delta^2 f}{hk}=f_{xy}(a,b)+\varepsilon_2$ (5)

where $\varepsilon_2\to 0$ as $h,k\to 0$.

From (4) and (5), we get, $f_{yx}(a,b)+\varepsilon_1=f_{xy}(a,b)+\varepsilon_2$

Taking limits as $h,k\to 0$, we get, $f_{yx}(a,b)+0=f_{xy}(a,b)+0$

$\Rightarrow f_{yx}(a,b)=f_{xy}(a,b)$

Young's Theorem

Statement : If (a,b) is a point of the domain $D_f\subset\Re^2$ of a function f such that

i. f_x, f_y both exist in a certain neighborhood of (a,b),

ii. f_x, f_y are differentiable at (a,b),

then $f_{yx}(a,b)=f_{xy}(a,b)$.

Proof : Since f_x, f_y are differentiable, so $f_{xx}, f_{xy}, f_{yx}, f_{yy}$ exist at (a,b).

Let $(a+h,b+h)$ be any point of the nbd.

Consider $\phi(x)=f(x,b+h)-f(x,b)$ (1)

$\therefore \phi(a+h)-\phi(a)=f(a+h,b+h)-f(a+h,b)-f(a,b+h)+f(a,b)=\Delta^2 f$ (say)

By Lagrange's mean value theorem

$\phi(a+h)-\phi(a)=h\phi'(a+\theta h), 0<\theta<1$ (2)

From (1), $\phi'(x)=f_x(x,b+h)-f_x(x,b)$

$\therefore \phi'(a+\theta h) = f_x(a+\theta h, b+h) - f_x(a+\theta h, b)$

So, from (2), we have $\phi(a+h)-\phi(a) = h\left[f_x(a+\theta h, b+h) - f_x(a+\theta h, b)\right]$ (3)

Since f_x, f_y are differentiable,

$f_x(a+\theta h, b+h) - f_x(a,b) = \theta h f_{xx} + h f_{yx} + h\varepsilon'$, (4)

where $\varepsilon' \to 0$ as $h \to 0$.

Similarly, $f_x(a+\theta h, b) - f_x(a,b) = \theta h f_{xx} + h\varepsilon''$, (5)

where $\varepsilon'' \to 0$ as $h \to 0$.

Subtracting (5) from (4), we get

$f_x(a+\theta h, b+h) - f_x(a+\theta h, b) = h f_{yx} + h(\varepsilon' - \varepsilon'')$

So, from (3), we get, $\phi(a+h)-\phi(a) = h\left[h f_{yx} + h(\varepsilon' - \varepsilon'')\right]$

$\Rightarrow \Delta^2 f = h^2 f_{yx} + h^2 \varepsilon_1$ where $\varepsilon_1 = \varepsilon' - \varepsilon'' \to 0$ as $h \to 0$.

$\Rightarrow \dfrac{\Delta^2 f}{h^2} = f_{yx} + \varepsilon_1$ (6)

Let $\psi(y) = f(a+h, y) - f(a, y)$

Proceeding as above, we get, $\dfrac{\Delta^2 f}{h^2} = f_{xy} + \varepsilon_2$ (7)

Where $\varepsilon_2 \to 0$ as $h \to 0$.

From (6) and (7), we get, $f_{yx}(a,b) + \varepsilon_1 = f_{xy}(a,b) + \varepsilon_2$

Taking limits as $h \to 0$, we get, $f_{yx}(a,b) + 0 = f_{xy}(a,b) + 0$

$\Rightarrow f_{yx}(a,b) = f_{xy}(a,b)$

Theorem on Total Differentials

Statement : If $z = f(x,y)$ possesses continuous partial derivatives of the first order, the total differential of z is given by

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Proof : The proof is left as an exercise for the reader.

IV. Homogeneous Function

If any function z can be expressed in the form $x^n f\left(\dfrac{y}{x}\right)$, then it is said to be homogeneous function of x and y. Here n represents the degree of the function.

For example : If $z=\dfrac{x^3-y^3}{x+y}=\dfrac{x^3\left(1-\dfrac{y^3}{x^3}\right)}{x\left(1+\dfrac{y}{x}\right)}=x^2 f\left(\dfrac{y}{x}\right)$

then, z is homogeneous function of degree 2.

Euler's Theorem of Homogeneous Functions

Statement : If z is a homogeneous function of x and y of degree n, then

$x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=nz$; $\forall x, y \in$ the domain of the function.

Proof : Since z is a homogeneous function of x and y of degree n.

$$\therefore z=x^n f\left(\frac{y}{x}\right) \qquad \ldots(1)$$

Partial differentiating w.r.t x, we get

$$\frac{\partial z}{\partial x}=nx^{n-1}f\left(\frac{y}{x}\right)+x^n f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$\Rightarrow \frac{\partial z}{\partial x}=nx^{n-1}f\left(\frac{y}{x}\right)-yx^{n-2}f'\left(\frac{y}{x}\right)$$

Multiplying both sides of the above equation by x, we get

$$x\frac{\partial z}{\partial x}=nx^n f\left(\frac{y}{x}\right)-yx^{n-1}f'\left(\frac{y}{x}\right)$$

$$\Rightarrow x\frac{\partial z}{\partial x}=nz-yx^{n-1}f'\left(\frac{y}{x}\right) \qquad \ldots(2)$$

Now, partial differentiating (1) w.r.t. y, we have

$$\frac{\partial z}{\partial y}=x^n f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)=x^{n-1}f'\left(\frac{y}{x}\right)$$

Multiplying both sides of the above equation by y, we get

$$y\frac{\partial z}{\partial y}=yx^{n-1}f'\left(\frac{y}{x}\right) \qquad \ldots(3)$$

Now, adding (2) and (3), we obtain the required result as

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$$

Corollary : If z is a homogeneous function of x and y of degree n, then prove that

i. $x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\partial y}=(n-1)\frac{\partial z}{\partial x}$

ii. $x\frac{\partial^2 z}{\partial y\partial x}+y\frac{\partial^2 z}{\partial y^2}=(n-1)\frac{\partial z}{\partial y}$

iii. $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y\frac{\partial^2 z}{\partial y^2}=n(n-1)z$

Proof : It is given that z is a homogeneous function of x and y of degree n, so by Euler's theorem

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz \qquad \ldots(1)$$

i. Differentiating (1) partially w.r.t. x, we get

$$x\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}.1+y\frac{\partial^2 z}{\partial x\partial y}=n\frac{\partial z}{\partial x}$$

$$\Rightarrow x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\partial y}=(n-1)\frac{\partial z}{\partial x} \qquad \ldots(2)$$

ii. Differentiating (1) partially w.r.t. y, we get

$$x\frac{\partial^2 z}{\partial y\partial x}+y\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial y}.1=n\frac{\partial z}{\partial y}$$

$$\Rightarrow x\frac{\partial^2 z}{\partial y\partial x}+y\frac{\partial^2 z}{\partial y^2}=(n-1)\frac{\partial z}{\partial y} \qquad \ldots(3)$$

iii. Multiplying (2) by x and (3) by y and adding, we get

$$x\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y\frac{\partial^2 z}{\partial y^2}=(n-1)\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)=n(n-1)z.$$

Example 2 : Verify Euler's theorem for the function $z=\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{4}}+y^{\frac{1}{4}}}$.

Sol. Here, $z = \dfrac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{4}} + y^{\frac{1}{4}}} = \dfrac{x^{\frac{1}{3}}\left(1+\left(\dfrac{y}{x}\right)^{\frac{1}{3}}\right)}{x^{\frac{1}{4}}\left(1+\left(\dfrac{y}{x}\right)^{\frac{1}{4}}\right)} = x^{\frac{1}{12}} f\left(\dfrac{y}{x}\right)$

$\therefore z$ is a homogeneous function of x and y of degree $\dfrac{1}{12}$.

So, we have to verify that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = \dfrac{1}{12}z$

Now, $z = \dfrac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{4}} + y^{\frac{1}{4}}}$

Taking log on both sides

$$\log z = \log\frac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{4}} + y^{\frac{1}{4}}} = \log\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right) - \log\left(x^{\frac{1}{4}} + y^{\frac{1}{4}}\right) \quad \ldots(1)$$

Differentiating (1) partially w.r.t. x

$$\frac{1}{z}\frac{\partial z}{\partial x} = \frac{\frac{1}{3}x^{\frac{-2}{3}}}{x^{\frac{1}{3}} + y^{\frac{1}{3}}} - \frac{\frac{1}{4}x^{\frac{-3}{4}}}{x^{\frac{1}{4}} + y^{\frac{1}{4}}}$$

$$\Rightarrow x.\frac{1}{z}\frac{\partial z}{\partial x} = \frac{\frac{1}{3}x^{\frac{1}{3}}}{x^{\frac{1}{3}} + y^{\frac{1}{3}}} - \frac{\frac{1}{4}x^{\frac{1}{4}}}{x^{\frac{1}{4}} + y^{\frac{1}{4}}} \quad \ldots(2)$$

Now, differentiating (1) partially w.r.t. y, we have

$$\frac{1}{z}\frac{\partial z}{\partial y} = \frac{\frac{1}{3}y^{\frac{-2}{3}}}{x^{\frac{1}{3}} + y^{\frac{1}{3}}} - \frac{\frac{1}{4}y^{\frac{-3}{4}}}{x^{\frac{1}{4}} + y^{\frac{1}{4}}}$$

$$\Rightarrow y.\frac{1}{z}\frac{\partial z}{\partial y} = \frac{\frac{1}{3}y^{\frac{1}{3}}}{x^{\frac{1}{3}} + y^{\frac{1}{3}}} - \frac{\frac{1}{4}y^{\frac{1}{4}}}{x^{\frac{1}{4}} + y^{\frac{1}{4}}} \quad \ldots(3)$$

Adding (2) and (3), we get

$$\frac{1}{z}\left[x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right]=\frac{1}{3}\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}-\frac{1}{4}\frac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{x^{\frac{1}{4}}+y^{\frac{1}{4}}}=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}$$

$\Rightarrow x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{1}{12}z$ and hence the result is verified.

V. Implicit Function

Let $f(x,y)$ be a function of two variables and $y=\phi(x)$ be a function of x such that $f(x,\phi(x))$ vanishes identically, then $y=\phi(x)$ is an implicit function defined by the functional equation $f(x,y)=0$.

Art. : Assuming that $f(x,y)=0$ satisfies the conditions under which y is defined as a derivable function of x, show that

i. $\frac{dy}{dx}=\frac{-f_x}{f_y}, f_y \neq 0$

ii. $\frac{d^2y}{dx^2}=\frac{f_x{}^2(f_y)^2-2f_{xy}f_xf_y+f_y{}^2(f_x)^2}{(f_y)^3}, f_y \neq 0$

provided $f_{xy}=f_{yx}$.

Proof : The proof is left as an exercise for the reader.

V.(a) Statement of Implicit Function Theorem

Let $f(x,y)$ be a function of two variables x and y and (a,b) be a point in its domain of definition such that (i) $f(a,b)=0$, (ii) f_x, f_y both exist and are continuous in a certain neighborhood of (a,b), (iii) $f_y(a,b)\neq 0$, then there exists a rectangle $(a-h,a+h;b-k,b+k)$ about (a,b) such that for every x in interval $[a-h,a+h]$, $f(x,y)=0$ determines one and only one value $y=\phi(x)$, lying in the interval $[b-k,b+k]$, with the following properties:

(1) $b=\phi(a)$,

(2) $f(x,\phi(x))=0$ for every x in interval $[a-h,a+h]$ and

(3) $\phi(x)$ is derivable, and both $\phi(x)$ and $\phi'(x)$ are continuous in $[a-h,a+h]$.

V.(b) Statement of Inverse Function Theorem

Let A be an open subset of $\Re^n$, f is a continuously differentiable mapping of A into $\Re^n$, $f'(a)$ is invertible for some $a\in A$ and $b=f(a)$. Then

(1) There exists open sets G and H in $\Re^n$ such that $a \in G, b \in H$. f is one-one on G and $f(G) = H$.

(2) If g is the inverse of f defined in H by $g(f(x)) = x, (x \in G)$, then g is continuously differentiable mapping in H.

VI. Composite Functions and their Differentiation

Composite Function : If z is a function of x, y and x, y are themselves functions of t , then z is said to be composite function of t . Similarly, if z is a function of x, y and x, y are themselves functions of u, v , then z is said to be composite function of u, v .

Differentiation of Composite Function : If a function $z = f(x, y)$ has continuous partial derivatives w.r.t. x and y and x, y have derivatives w.r.t. t , then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Example 3 : If $z = x^3 - xy + y^3$ and $x = r\cos\theta, y = r\sin\theta$, then find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$.

Sol. $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = (3x^2 - y)(\cos\theta) + (-x + 3y^2)(\sin\theta)$

$= \left[3(r\cos\theta)^2 - r\sin\theta\right]\cos\theta + \left[-r\cos\theta + 3(r\sin\theta)^2\right]\sin\theta$

$= 3r^2\cos^3\theta - r\sin\theta\cos\theta - r\sin\theta\cos\theta + 3r^2\sin^3\theta$

$= 3r^2\left(\cos^3\theta + \sin^3\theta\right) - 2r\sin\theta\cos\theta$

Now, $\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = (3x^2 - y)(-r\sin\theta) + (-x + 3y^2)(r\cos\theta)$

$= \left(3r^2\cos^2\theta - r\sin\theta\right)\left(-r\sin\theta\right) + \left(-r\cos\theta + 3r^2\sin^2\theta\right)\left(r\cos\theta\right)$

$= -3r^3\cos^2\theta\sin\theta + r^2\sin^2\theta - r^2\cos^2\theta + 3r^3\sin^2\theta\cos\theta$

$= -3r^3\cos\theta\sin\theta\left(\cos\theta - \sin\theta\right) + r^2\left(\sin^2\theta - \cos^2\theta\right)$

VII. Taylor's Theorem for Functions of Two Variables

Statement : If $f(x, y)$ and all its partial derivatives upto order n be continuous in all neighborhoods of the point (x, y), then

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{\angle 2}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)$$

$$+\frac{1}{\angle 3}\left(h^3\frac{\partial^3 f}{\partial x^3}+3h^2k\frac{\partial^3 f}{\partial x^2\partial y}+3hk^2\frac{\partial^3 f}{\partial x\partial y^2}+k^3\frac{\partial^3 f}{\partial y^3}\right)+\ldots\ldots..$$

Proof : Applying Taylor's theorem to $f(x+h, y+k)$, where $y+k$ is regarded as constant, we have,

$$f(x+h,y+k)=f(x,y+k)+h\frac{\partial}{\partial x}f(x,y+k)+\frac{h^2}{\angle 2}\frac{\partial^2}{\partial x^2}f(x,y+k)+\frac{h^3}{\angle 3}\frac{\partial^3}{\partial x^3}f(x,y+k)+\ldots.\,(1)$$

Again, applying Taylor's theorem to $f(x, y+k)$, where x is regarded as constant,

$$f(x,y+k)=f(x,y)+k\frac{\partial}{\partial y}f(x,y)+\frac{k^2}{\angle 2}\frac{\partial^2}{\partial y^2}f(x,y)+\frac{k^3}{\angle 3}\frac{\partial^3}{\partial y^3}f(x,y)+\ldots. \qquad (2)$$

From (1) and (2), we get

$$f(x+h,y+k)=\left[f(x,y)+k\frac{\partial}{\partial y}f(x,y)+\frac{k^2}{\angle 2}\frac{\partial^2}{\partial y^2}f(x,y)+\frac{k^3}{\angle 3}\frac{\partial^3}{\partial y^3}f(x,y)+\ldots.\right]$$

$$+h\frac{\partial}{\partial x}\left[f(x,y)+k\frac{\partial}{\partial y}f(x,y)+\frac{k^2}{\angle 2}\frac{\partial^2}{\partial y^2}f(x,y)+\ldots.\right]$$

$$+\frac{h^2}{\angle 2}\frac{\partial^2}{\partial x^2}\left[f(x,y)+k\frac{\partial}{\partial y}f(x,y)+\ldots\right]+\frac{h^3}{\angle 3}\frac{\partial^3}{\partial x^3}\left[f(x,y)+\ldots\right]+\ldots\ldots$$

$$=\left[f(x,y)+k\frac{\partial}{\partial y}f(x,y)+\frac{k^2}{\angle 2}\frac{\partial^2}{\partial y^2}f(x,y)+\frac{k^3}{\angle 3}\frac{\partial^3}{\partial y^3}f(x,y)+\ldots.\right]+\left[h\frac{\partial f}{\partial x}+hk\frac{\partial^2 f}{\partial x\partial y}+\frac{hk^2}{\angle 2}\frac{\partial^3 f}{\partial x\partial y^2}+\ldots\right]$$

$$+\left[\frac{h^2}{\angle 2}\frac{\partial^2 f}{\partial x^2}+\frac{h^2k}{\angle 2}\frac{\partial^3 f}{\partial x^2\partial y}+\ldots\right]+\left[\frac{h^3}{\angle 3}\frac{\partial^3 f}{\partial x^3}+\ldots\ldots\right]$$

$$=f(x,y)+\left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right)+\left(\frac{h^2}{\angle 2}\frac{\partial^2 f}{\partial x^2}+hk\frac{\partial^2 f}{\partial x\partial y}+\frac{k^2}{\angle 2}\frac{\partial^2 f}{\partial y^2}\right)$$

$$+\left(\frac{h^3}{\angle 3}\frac{\partial^3 f}{\partial x^3}+\frac{h^2k}{\angle 2}\frac{\partial^3 f}{\partial x^2\partial y}+\frac{hk^2}{\angle 2}\frac{\partial^3 f}{\partial x\partial y^2}+\frac{k^3}{\angle 3}\frac{\partial^3 f}{\partial y^3}\right)+\ldots..$$

So, $f(x+h,y+k)=f(x,y)+\left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right)+\frac{1}{\angle 2}\left(h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\partial y}+k^2\frac{\partial^2 f}{\partial y^2}\right)$

$$+\frac{1}{\angle 3}\left(h^3\frac{\partial^3 f}{\partial x^3}+3h^2k\frac{\partial^3 f}{\partial x^2\partial y}+3hk^2\frac{\partial^3 f}{\partial x\partial y^2}+k^3\frac{\partial^3 f}{\partial y^3}\right)+\ldots\ldots..$$

Corollary 1 : Put $a=0, b=0, h=x, k=y$, in the above Taylor's theorem, we obtain the **Maclaurin's theorem** as

$$f(x,y)=f(0,0)+\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)f(0,0)+\frac{1}{\angle 2}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 f(0,0)+......+$$

$$\frac{1}{\angle n-1}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^{n-1} f(0,0)+\frac{1}{\angle n}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^{n} f(tx,ty), 0<t<1$$

Corollary 2 : Put $a+h=x, b+k=y$, we obtain the Taylor's expansion of $f(x,y)$ in powers of $x-a$ and $y-b$ as

$$f(x,y)=f(a,b)+\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right)f(a,b)+\frac{1}{\angle 2}\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right)^2 f(a,b)+......+$$

$$\frac{1}{\angle n-1}\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right)^{n-1} f(a,b)+\frac{1}{\angle n}\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right)^{n} f(a+t(x-a), b+t(y-b)),$$

Where $0<t<1$.

VIII. Summary

In this lesson, while discussing about the differentiability of the function $f(x,y)$, we have studied about several functions such as homogeneous function, implicit function and composite function and their important results. During the study, it is also learned about the differentiation of composite functions or about the chain rule. Further, through this lesson, we are now familiar with the generalization of the Taylor's theorem on function of two variables.

IX. Self Check Exercise

1. Show that $f(x,y)=|x|+|y|$ is continuous at (0,0) but not differentiable at (0,0).
2. Discuss the differentiability of the function $f(x,y)=(xy)^{\frac{1}{3}}$ at (0,0).
3. Discuss the differentiability at (0,0) of $f(x,y)=\dfrac{xy^2}{x^2+y^2}$ when $f(x,y)\neq(0,0)$ and $f(0,0)=0$.
4. Verify Euler's theorem

(i) $z = \dfrac{x^{\frac{1}{4}} + y^{\frac{1}{4}}}{x^{\frac{1}{5}} + y^{\frac{1}{5}}}$ (ii) $z = x^n \log \dfrac{y}{x}$

5. If $u = \sin^{-1}\left(\dfrac{x^2 + y^2}{x + y}\right)$, then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$.

6. If $u = x^2 - y^2$, $x = 2r - 3s + 4$, $y = -r + 8s - 5$, then find $\dfrac{\partial u}{\partial r}$.

7. z is a function of x and y. Prove that if $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$, then $\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v} = x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y}$.

8. If $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, prove that $\dfrac{d^2y}{dx^2} = \dfrac{-b^4}{a^2y^3}$.

9. State Taylor's theorem for functions of two variables. Use it to expand $xy^2 + 3x - 2$ in powers of $x + 2$ and $y - 1$.

10. Expand $x^4 + x^2y^2 - y^4$ about the point $(1,1)$ upto the terms of the second degree.

11. Obtain Taylor's expansion for $f(x, y) = e^{xy}$ at $(1,1)$ upto the third term.

Suggested Readings

1. RK Jain, SRK Lyenger — Advanced Engineering Mathematics
2. JR Sharma — Advanced Calculus
3. Malik and Arora — Mathematical Analysis
4. Shanti Narayan — Mathematical Analysis
5. Thomas and Finney — Calculus and Analytical Geometry

LESSON NO. 2.4 **Author: Dr. Chanchal**

SOME BASIC FUNCTIONS CONCERNING PARTIAL DERIVATIVES

Structure:

Objectives

During the study in this particular lesson, our main purpose is to study the rules and methods such as Lagrange's method, under which the maximum and minimum of the function $f(x,y)$ can be obtained. Further, an important function known as Jacobian and its important properties are also elaborated under the same.

I. Introduction

Before introducing the main part of this lesson, we firstly define the extreme values of the function $f(x,y)$, below:

A. Maximum Value : A function $f(x,y)$ is said to have a maximum value at $x=a, y=b$ if $f(a,b) > f(a+h, b+k)$ for small values of h and k, positive or negative.

B. Minimum Value : A function $f(x,y)$ is said to have a maximum value at $x=a, y=b$ if $f(a,b) < f(a+h, b+k)$ for small values of h and k, positive or negative.

C. Extreme Value : A maximum or minimum value of a function is called an extreme value.

II. Working Method for Maxima and Minima of a Function $f(x,y)$

Let $f(x,y)$ be the given function

Step I : Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Step II : Solve $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$ simultaneously for x and y. Let $(x_1,y_1),(x_2,y_2)$........ be the points.

Step III : Calculate the values of $A=\frac{\partial^2 f}{\partial x^2}, B=\frac{\partial^2 f}{\partial x\partial y}, C=\frac{\partial^2 f}{\partial y^2}$ for each point.

Step IV : (i) If for a point (x_1,y_1), $AC-B^2>0$ and $A<0$, then $f(x,y)$ has a maxima for this pair and maximum value is $f(x_1,y_1)$.

(ii) If for a point (x_1,y_1), $AC-B^2>0$ and $A>0$, then $f(x,y)$ has a minima for this pair and minimum value is $f(x_1,y_1)$.

(iii) If for a point (x_1,y_1), $AC-B^2<0$, then there is neither maximum nor minimum of $f(x,y)$ and $f(x,y)$ is said to have a saddle point at (x_1,y_1).

(iv) If $AC-B^2=0$ for some point (a,b), then we have the following cases:

(a) if $f(a,b)-f(a+h,b+k)>0$ for small values of h and k, positive or negative, then f has maxima at (a,b).

(b) if $f(a,b)-f(a+h,b+k)<0$ for small values of h and k, positive or negative, then f has minima at (a,b).

(c) if $f(a,b)-f(a+h,b+k)$ does not keep the same sign for small values of h and k, then there is neither maxima nor minima.

Example 1 : Find all the maxima and minima of the function $f(x,y)=x^3+y^3-63(x+y)+12xy.$

Sol. Step I. $\frac{\partial f}{\partial x}=3x^2+12y-63$ and $\frac{\partial f}{\partial y}=3y^2+12x-63$

Step II. Let us solve $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$

i.e., $3x^2+12y-63=0$ and $3y^2+12x-63=0$

or $x^2+4y-21=0$ (1)

and $y^2+4x-21=0$ (2)

Subtracting (2) from (1), we get

$x^2-y^2+4y-4x=0 \Rightarrow (x-y)(x+y)-4(x-y)=0 \Rightarrow (x-y)(x+y-4)=0$

$\Rightarrow$ either $x-y=0$ which gives $x=y$, or $x+y-4=0$ which gives $x=4-y$

For $x=y$, (1) becomes

$x^2+4x-21=0 \Rightarrow (x+7)(x-3)=0 \Rightarrow x=3,-7$

As $y=x$, so $y=3,-7$

For $x=4-y$, (1) becomes

$(4-y)^2+4y-21=0$

or $y^2-8y+16+4y-21=0$

or $y^2-4y-5=0 \Rightarrow (y-5)(y+1)=0 \Rightarrow y=5,-1$

Further, from $x=4-y$, $x=-1,5$.

so, the four critical points are (3,3), (-7,-7), (-1,5) and (5,-1).

Step III. $A=\dfrac{\partial^2 f}{\partial x^2}=6x, B=\dfrac{\partial^2 f}{\partial x \partial y}=12, C=\dfrac{\partial^2 f}{\partial y^2}=6y$

$\therefore AC-B^2=36xy-144.$

Step IV. At (3,3), $AC-B^2=36(3)(3)-144=180>0$ and $A=6(3)=18>0$

$\therefore f(x,y)$ is minimum at (3,3) and the minimum value is given by

$f(3,3)=27+27-63(3+3)+12(3)(3)=-216$

At (-7,-7), $AC-B^2=36(-7)(-7)-144=1764-144=1620>0$ and $A=6(-7)=-42<0$

$\therefore f(x,y)$ is maximum at (-7,-7) and the maximum value is given by

$f(-7,-7)=-343-343+882+588=784$

At (-1,5), $AC-B^2=36(-1)(5)-144=-180-144=-324<0$

$\therefore f(x,y)$ has neither maximum nor minimum at (-1,5) and therefore, (-1,5) is a saddle point.

At (5,-1), $AC-B^2=36(5)(-1)-144=-180-144=-324<0$

$\therefore f(x,y)$ has neither maximum nor minimum at (5,-1) and therefore, (5,-1) is a saddle point.

Example 2 : Find the extreme value (if any) of $f(x,y)=2x^4-3x^2y+y^2$.

Sol. Step I. $\frac{\partial f}{\partial x} = 8x^3 - 6xy$ and $\frac{\partial f}{\partial y} = -3x^2 + 2y$

Step II. Let us solve $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

or $8x^3 - 6xy = 0$ (1)

and $-3x^2 + 2y = 0$ (2)

From (1), $2x(4x^2 - 3y) = 0 \Rightarrow x = 0, x^2 = \frac{3y}{4}$

For $x = 0$, from (2), $0 + 2y = 0 \Rightarrow y = 0$.

So, the point is (0,0).

For $x^2 = \frac{3y}{4}$, from (2), $3 \times \frac{9y}{16} - 2y = 0 \Rightarrow y = 0$

$\therefore (0,0)$ is the only critical point.

Step III. $A = \frac{\partial^2 f}{\partial x^2} = 24x^2 - 6y, B = \frac{\partial^2 f}{\partial x \partial y} = -6x, C = \frac{\partial^2 f}{\partial y^2} = 2$

$\therefore AC - B^2 = (24x^2 - 6y)(2) - (-6x)^2$.

Step IV. At (0,1), $AC - B^2 = (0-0)(2) - 0 = 0$.

So, at (0,0) further investigation is required.

Consider

$$f(a,b) - f(a+h,b+k) = f(0,0) - f(h,k) = 0 - (2h^4 - 3h^2k + k^2) = \frac{1}{8}k^2 - 2\left(h^2 - \frac{3k}{4}\right)^2$$

which does not keep the same sign for small values of h and k, so there is neither maximum nor minimum at (0,0).

III. Lagrange's Method of Undetermined Multipliers

Let $f(x,y,z)$ be a function of x,y,z which is to be examined for maximum or minimum values and let the variables be connected by the relation

$\phi(x,y,z) = 0$ (1)

Since $f(x,y,z)$ is to have maximum or minimum value,

$\therefore \frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$ and $\frac{\partial f}{\partial z} = 0$

$\Rightarrow \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0$ (2)

Differentiating (1), we get

$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0$$

(3)

Now, adding λ times of (3) into (2), we get

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0$$

In order to satisfy this equation identically, coefficients of dx, dy, dz should be zero separately

i.e., $\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0$ (4)

and $\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0$ (5)

and $\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0$ (6)

Equations (1), (4), (5) and (6) will give the values of x, y, z, λ for which the function $f(x, y, z)$ has maximum and minimum values.

Example 3 : Find the maximum and minimum value of the function $x^2 + y^2$ subject to the condition $3x^2 + 4xy + 6y^2 = 140$.

Sol. Let $f(x, y) = x^2 + y^2$

The constraint is

$$3x^2 + 4xy + 6y^2 = 140$$

(1)

Let $F(x, y) = x^2 + y^2 + \lambda\left(3x^2 + 4xy + 6y^2 - 140\right)$ where λ is Lagrange's multiplier.

For extreme points,

$$\frac{\partial F}{\partial x} = 2x + \lambda(6x + 4y) = 0$$

and $\frac{\partial F}{\partial y} = 2y + \lambda(4x + 12y) = 0$

or $(1 + 3\lambda)x + 2\lambda y = 0$ (2)

and $2\lambda x + (1 + 6\lambda)y = 0$ (3)

Since x, y are both non-zero,

$$\therefore \begin{vmatrix} 1+3\lambda & 2\lambda \\ 2\lambda & 1+6\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1+3\lambda)(1+6\lambda) - 4\lambda^2 = 0$$

$$\Rightarrow 14\lambda^2 + 9\lambda + 1 = 0$$

$$\Rightarrow \lambda = \frac{-9 \pm \sqrt{81-56}}{28} = \frac{-9 \pm 5}{28} = -\frac{1}{2}, -\frac{1}{7}$$

Taking $\lambda = -\frac{1}{2}$

From (2), $x = -2y$

And from (1), $12y^2 - 8y^2 + 6y^2 - 140 = 0 \Rightarrow y^2 = 14$

$\therefore x^2 = 4y^2 = 56$

Which gives $x^2 + y^2 = 56 + 14 = 70$

Taking $\lambda = -\frac{1}{7}$

From (2), $y = 2x$

And from (1), $3x^2 + 8x^2 + 24x^2 - 140 = 0 \Rightarrow x^2 = 4$

$\therefore y^2 = 4x^2 = 16$

Which gives $x^2 + y^2 = 16 + 4 = 20$

Hence, the maximum value of $x^2 + y^2$ is 70 and the minimum value is 20.

IV. Jacobian of n-Function

If $f_1, f_2, \ldots\ldots\ldots\ldots, f_n$ be n functions of n variables $x_1, x_2, \ldots\ldots\ldots\ldots, x_n$ possessing partial derivatives of the first order at every point of the domain of definition of the function, then the determinant

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \ldots\ldots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \ldots\ldots & \frac{\partial f_2}{\partial x_n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \ldots\ldots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}$$

Is called the Jacobian of $f_1, f_2, \ldots\ldots\ldots\ldots, f_n$ w.r.t. $x_1, x_2, \ldots\ldots\ldots\ldots, x_n$. It is denoted by $\frac{\partial(f_1, f_2, \ldots\ldots\ldots, f_n)}{\partial(x_1, x_2, \ldots\ldots\ldots, x_n)}$ or $J(f_1, f_2, \ldots\ldots\ldots\ldots, f_n)$.

Example 4 : Find $\frac{\partial(f,g)}{\partial(x,y)}$ if $f = x^2 - x\sin y$ and $g = x^2y^2 + x + y$.

Sol. Here, $f = x^2 - x\sin y$ and $g = x^2y^2 + x + y$.

$\therefore \quad \frac{\partial f}{\partial x} = 2x - \sin y, \qquad \frac{\partial f}{\partial y} = -x\cos y$

And $\frac{\partial g}{\partial x} = 2xy^2 + 1, \qquad \frac{\partial g}{\partial y} = 2x^2y + 1$

Now, $$\frac{\partial(f,g)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x - \sin y & -x\cos y \\ 2xy^2 + 1 & 2x^2y + 1 \end{vmatrix}$$

$$= (2x - \sin y)(2x^2y + 1) + (2xy^2 + 1)(x\cos y)$$

V. Some Important Articles Concerning Jacobian

Art. 1 : Jacobian of Composite Functions

Statement : If $f : \Re^n \to \Re^n$ and $g : \Re^n \to \Re^n$ differentiable functions, then $J_I(x) = J_g(f(x))J_f(x)$, where $I = gof$.

Art. 2 : Let D be an open subset of $\Re^n$ and $f : D \to \Re^n$ be differentiable at every point of D. Suppose that f is invertible on D and let f^{-1} be differentiable at every point of the range of f, then

$$J_{f^{-1}}(f(x)) = \frac{1}{J_f(x)} \forall x \in D.$$

Art. 3 : Jacobian of Implicit Functions

Statement : If $u_1, u_2, \ldots\ldots\ldots\ldots, u_n$ are functions of $x_1, x_2, \ldots\ldots\ldots\ldots, x_n$ defined implicitly by n equations

$F_1(u_1, u_2, \ldots\ldots\ldots u_n, x_1, x_2, \ldots\ldots\ldots\ldots, x_n) = 0$,

$F_2(u_1, u_2, \ldots\ldots\ldots u_n, x_1, x_2, \ldots\ldots\ldots\ldots, x_n) = 0$,

………………………………………………

$F_n(u_1, u_2, ...u_n, x_1, x_2, ..., x_n) = 0$,

then, $$\frac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)} = (-1)^n \frac{\dfrac{\partial(F_1, F_2, ..., F_n)}{\partial(x_1, x_2, ..., x_n)}}{\dfrac{\partial(F_1, F_2, ..., F_n)}{\partial(u_1, u_2, ..., u_n)}}$$

Art. 4 : Functional Dependence (Necessary and Sufficient Condition for a Jacobian to Vanish)

Statement : Let $u_1, u_2, ..., u_n$ be n functions of n independent variables $x_1, x_2, ..., x_n$. In order that there may exist between these functions a relation $F(u_1, u_2, ..., u_n) = 0$, it is necessary and sufficient that the Jacobian $\dfrac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)}$ should vanish identically.

Note : The proof of above articles is easy and left as an exercise for the reader.

Example 5 : Prove that $J_{f^{-1}}(\xi, n) = \xi$ for any (ξ, n) belonging to the range of f, where $f(x, y) = \left(\sqrt{x^2 + y^2}, \tan^{-1}\dfrac{y}{x}\right)$.

Sol. Here $f(x, y) = \left(\sqrt{x^2 + y^2}, \tan^{-1}\dfrac{y}{x}\right) = (f_1, f_2)$

Where $f_1(x, y) = \sqrt{x^2 + y^2}$ and $f_2(x, y) = \tan^{-1}\dfrac{y}{x}$

$$\therefore \quad \frac{\partial f_1}{\partial x} = \frac{2x}{2\sqrt{x^2 + y^2}} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial f_2}{\partial x} = \frac{2y}{2\sqrt{x^2 + y^2}} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial g_1}{\partial x} = \frac{1}{1 + \left(\dfrac{y}{x}\right)^2} \cdot \left(\frac{-y}{x^2}\right) = \frac{-y}{x^2 + y^2}$$

$$\frac{\partial g_2}{\partial x} = \frac{1}{1 + \left(\dfrac{y}{x}\right)^2} \cdot \left(\frac{1}{x}\right) = \frac{x}{x^2 + y^2}$$

Now, $J_f(x,y)=\dfrac{\partial(f_1,f_2)}{\partial(x,y)}=\begin{vmatrix}\dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y}\\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y}\end{vmatrix}=\begin{vmatrix}\dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}}\\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2}\end{vmatrix}$

$$=\frac{x^2}{(x^2+y^2)^{\frac{3}{2}}}+\frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}=\frac{1}{\sqrt{x^2+y^2}}$$

$$=\frac{1}{\xi} \qquad \left[\because (\xi,\eta)=f(x,y)\Rightarrow \xi=\sqrt{x^2+y^2}\right]$$

Now, $J_{f^{-1}}(\xi,n)=\dfrac{1}{J_f(x,y)}=\xi$

Example 6 : Show that the functions $u=x-2y+z, v=x^2+2xy-xz, w=3x+2y-z$ are not independent of one another. Also find the relation between them.

Sol. For the given functions, we have

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z}\\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z}\\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z}\end{vmatrix}=\begin{vmatrix}1 & -2 & 1\\ 2x+2y-z & 2x & -x\\ 3 & 2 & -1\end{vmatrix}$$

$$=\begin{vmatrix}1 & -2 & 1\\ 2x+2y-z & 2x & -x\\ 4 & 0 & 0\end{vmatrix}, \quad \text{by } R_3+R_1$$

$$=4(2x-2x)=0$$

$\therefore u,v$ and w are not independent of one another.

Further,

$$w^2-u^2=(3x+2y-z)^2-(x-2y+z)^2=(3x+2y-z+x-2y+z)(3x+2y-z-x+2y-z)$$

$$=4x(2x+4y-2z)=8x(x+2y-z)=8(x^2+2xy-zx)=8v$$

So, $w^2-u^2=8v$ is the required relation between u,v and w.

VI. Self Check Exercise

1. Find all the critical points of the function $f(x,y)=x^2+xy$ and examine for maxima, minima or neither.
2. Find the minimum value of $f(x,y,z)=x^2+y^2+z^2$ subject to the condition $yz+zx+xy=3a^2$.
3. Find the point on the plane $2x+3y-z=5$ which is nearest to the origin in $\Re^3$.
4. If $x=r\cos\theta, y=r\sin\theta, z=z$, then evaluate $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$.
5. Prove that $J_f(x,y)=e^{x+y}\sin(x+y)$ where $f(x,y)=(e^x\sin y, e^y\cos x)$.
6. Evaluate $J_{f^{-1}}(\xi,n)$, where $f(x,y)=(x-y,x+y)$.
7. If $u^3+v^3=x+y, u^2+v^2=x^3+y^3$, then show that $\frac{\partial(u,v)}{\partial(x,y)}=\frac{1}{2}.\frac{y^2-x^2}{uv(u-v)}$.
8. Show that the functions $u=x+y-z, v=x-y+z, w=x^2+y^2+z^2-2yz$ are not independent of one another. Also find the relation between them.

Suggested Readings

1. RK Jain, SRK Lyenger — Advanced Engineering Mathematics
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