Centre for Distance and Online Education
Punjabi University, Patiala

Class : B.A. II (Math) **Semester : III**
Paper : I (Analysis-I) **Unit–2**
Medium : English

Lesson No.

Department website : www.pbidde.org

B.A. / B.Sc. (MATHEMATICS) Semester - III

PAPER-I: ANALYSIS-I

Maximum Marks: 50 **Pass Percentage: 35%**
Maximum Time: 3 Hrs

INSTRUCTIONS FOR THE PAPER SETTER

The question paper will consist of three sections A, B and C. Sections A and 8 will have four questions each from the respective sections of the syllabus and Section C will consist of one compulsory question having eight short answer type questions covering the entire syllabus uniformly. Each question in sections A and B will be of 7.5 marks and Section C will be of 20 marks.

INSTRUCTIONS FOR THE CANDIDATES

Candidates are required to attempt five questions in all selecting two questions from each of the Section A and B and compulsory question of Section C.

Section-A

Sequence: Definition of a sequence, Bounded and monotonic sequences, Convergent sequence, Cauchy sequences, Cauchy's convergence criterion, Theorems on limits of sequences, Sub-sequence, Sequential continuity.

Infinite Series: Definition of a series, Tests of convergence, Comparison test, Cauchy's integral ratio test, Condensation test, Raabe's test, Logarithmic test, Gauss test, Cauchy's root test, Alternating series, Leibnitz's test. Absolute convergence and conditional convergence. Weierstrass M-test for uniform convergence of sequence of functions and series of functions. Simple applications. Determination of radius of convergence of power series (All tests without proofs, only applications).

Section-B

Riemann Integration: Partitions, Upper and lower Sums, Upper and lower integrals, Riemann integrability. Conditions of existence of Riemann integrability of continuous functions and monotone functions. Algebra of integrable functions.

Improper Integrals: Definitions, Statements of their conditions of existence. Tests for the convergence of improper integrals, Beta and Gamma functions and their convergence, Abel's and Dirichlet's tests.

RECOMMENDED BOOKS:

1. Tom M. Apostol : Mathematical Analysis, Second Edition, Addsion-Wesley Publishing Company, 1974.
2. W. Rudin : Principles of Mathematical Analysis, Third Edition, McGraw Hill, 2013.
3. S. C. Malik and S. Arora : Mathematical Analysis, New Age International Publishers, 1992.

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B.A. PART - II **MATHEMATICS : PAPER–I**

SEMESTER-III **ANALYSIS-I**

LESSON NO. 2.3 **Author : Dr. Chanchal**

FRULLANI'S INTEGRAL

2.3.1 Objectives

The prime objective of this lesson is to study in detail about the improper integrals, their classification and convergence accordingly.

2.3.2 Introduction

An improper integral may be defined as :-

An integral of the form $\int_a^b f(x)\,dx$ where f becomes infinite in [a, b] or in other words f has points of infinite discontinuity in [a, b] or the limits of integration become infinite i.e. a or b or both become infinite is called an improper integral.

For example, $\int_0^1 \frac{dx}{\sqrt{1-x^2}}, \int_1^2 \frac{dx}{\sqrt{x-1}}, \int_{-1}^{\infty} \frac{dx}{x}, \int_{-\infty}^{\infty} \frac{dx}{a^2+x^2}$ $(a \geq 0)$ are all improper integrals.

Further, we have the following two types of improper integrals :

1. Improper Integrals of First Kind
2. Improper Integrals of Second Kind

Firstly, we study about the improper integrals of first kind.

2.3.3 Improper Integrals of First Kind

Def : If either one or both the limits of integrations are infinite and the integrand 'f' is bounded, then $\int_a^b f(x)\,dx$ is said to be 'Improper integral of first kind'.

These are further divided into following three types, as discussed below :

Type I : When b is infinite

Let f be bounded and integrable for x ≥ a, then

$$\int_a^{\infty} f(x)\,dx = \underset{t\to\infty}{\text{Lt}} \int_a^t f(x)\,dx \qquad (t > a)$$

If the limit on the right exists, then $\int_a^{\infty} f(x)\,dx$ is said to converge, otherwise it is said to be a divergent integral.

Example 1 : Examine the converge of following integrals

(i) $\int_1^{\infty} xe^{-x}dx$ (ii) $\int_e^{\infty} \frac{dx}{x\,(\log x)^{\frac{3}{2}}}$

Sol. (i) By definition,

$$\int_1^{\infty} xe^{-x}dx = \underset{t\to\infty}{\text{Lt}} \int_1^t xe^{-x}dx,\ (t>1) = \underset{t\to\infty}{\text{Lt}} \left[\left. (x)\frac{(e^{-x})}{-1} \right\}_1^t - \int_1^t \frac{e^{-x}}{-1}\,dx \right]$$

$$= \underset{t\to\infty}{Lt}\left[(-te^{-t}+e^{-1})+\left(\frac{e^{-x}}{-1}\right)_1^t\right] = \underset{t\to\infty}{Lt}\left\{-te^{-t}+e^{-1}-(e^{-t}-e^{-1})\right\}$$

$$= \underset{t\to\infty}{Lt}\left[(-te^{-t}+e^{-1}-e^{-t}+e^{-1})\right] = \underset{t\to\infty}{Lt}\left(\frac{-t}{e^t}\right) - \underset{t\to\infty}{Lt}\left(\frac{1}{e^t}\right)+\frac{2}{e}$$

$$= \underset{t\to\infty}{Lt}\left(\frac{-1}{e^t}\right) - 0 + \frac{2}{e} = 0 - 0 + \frac{2}{e} = \frac{2}{e} \text{ which exists.}$$

$\therefore$ the given integral converges to $\frac{2}{e}$.

(ii) Let $I = \int_e^\infty \frac{dx}{x(\log x)^{\frac{3}{2}}} = \underset{t\to\infty}{Lt}\int_e^t (\log x)^{\frac{3}{2}}.\frac{1}{x}dx = \underset{t\to\infty}{Lt}\left[\frac{(\log x)^{\frac{1}{2}}}{-\frac{1}{2}}\right]_e^t$

$$= -2\underset{t\to\infty}{Lt}\left[\frac{1}{(\log t)^{\frac{1}{2}}} - \frac{1}{(\log e)^{\frac{1}{2}}}\right]$$

$= -2\left[\frac{1}{\infty}-\frac{1}{1}\right] = -2$ [0 – 1] = 2, which is a finite quantity.

$\therefore$ the given integral converges to 2.

Type II : When 'a' is infinite

$$\int_{-\infty}^b f(x)\,dx = \underset{t\to-\infty}{Lt}\int_t^b f(x)\,dx \qquad (t < b)$$

If the limit in the R.H.S. exists, then $\int_{-\infty}^b f(x)\,dx$ is said to be convergent, otherwise the integral is said to be divergent.

Example 2 : Examine the convergence of $\int_{-\infty}^0 e^{+4x}dx$

Sol. $\int_{-\infty}^{0} e^{4x}dx = \underset{t\to-\infty}{\text{Lt}} \int_{1}^{0} e^{4x}dx$ (t < 0)

$$= \underset{t\to-\infty}{\text{Lt}} \frac{1}{4}\left[e^{4x}\right]_1^0 = \frac{1}{4} \underset{t\to-\infty}{\text{Lt}} (1-e^{4t}) = \frac{1}{4}(1-0) = \frac{1}{4}$$

$\therefore$ the given integral converges to $\frac{1}{4}$.

Type III : When a and b are both infinite

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

where 'c' is any real number.

If both the integrals in (1) exist as discussed in Type I and II, then $\int_{-\infty}^{\infty} f(x)\,dx$ is said to converge otherwise, it is divergent or we can say that the limits $\underset{t_1\to-\infty}{\text{Lim}} \int_{t_1}^{c} f(x)\,dx$ and $\underset{t_2\to-\infty}{\text{Lim}} \int_{c}^{t_2} f(x)\,dx$ must exist finitely and independent of each other.

Example 3 : Examine the convergence of $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}$

Sol. $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \int_{-\infty}^{0} \frac{dx}{(x^2+1)^2} + \int_{0}^{\infty} \frac{dx}{(x^2+1)^2} = 2\int_{0}^{\infty} \frac{dx}{(x^2+1)^2}$

$$= 2\underset{t\to\infty}{\text{Lim}} \int_{0}^{t} \frac{dx}{(x^2+1)^2}$$

$$= 2\underset{t\to\infty}{\text{Lim}} \frac{1}{2}\left\{\tan^{-1}x + \frac{x}{1+x^2}\right\}_0^t$$

$$= \underset{t\to\infty}{\text{Lim}} \left\{\tan^{-1}t + \frac{t}{1+t^2}\right\}$$

Put x = tan θ

$$\int \frac{dx}{(x^2+1)^2} = \int \frac{\sec^2\theta\, d\theta}{\sec^4\theta}$$

$$= \int \cos^2\theta\, d\theta = \frac{1}{2}\int (1+\cos 2\theta)\, d\theta$$

$= \frac{\pi}{2}$

$= \frac{1}{2}\{\theta + \sin\theta\cos\theta\}$

$= \frac{1}{2}\left\{\tan^{-1} x + \frac{x}{1+x^2}\right\}$

$\therefore$ the given integral converges to $\frac{\pi}{2}$.

In many of the cases, the above methods are not suitable to discuss the convergence of the integrals. So, we need to study some more methods to examine the convergence of improper integrals, as discussed in detail below.
Firstly, we state two important results :

1. Test for convergence at ∞ : When the integrand keeps the same sign, positive or negative, in [a, t], (t ≥ a), we may suppose that f is non-negative there, because of negative, it can be replaced by (–f) for testing the convergence. The case f = 0, being a trivial case, therefore, there is no loss of generality when we suppose that f is positive throughout.

2. NASC for the convergence at ∞ : A necessary and sufficient condition for the convergence at ∞ of $\int_a^\infty f(x)\,dx$ in [a, t] is that, there exists a positive number M, (independent of t), such that $\int_a^t f(x)\,dx < M$ for every t ≥ a.

2.3.3.1 Direct Comparison Test

Art 1 Statement : If f and g are both positive for every x ∈ [a, ∞) and both are integrable in [a, t], $\forall\, t \ge a$, then

(i) If f (x) ≤ g(x) and $\int_a^\infty g(x)\,dx$ is convergent at ∞, then $\int_a^\infty f(x)\,dx$ is also convergent at ∞

(ii) If f (x) ≥ g(x) and $\int_a^\infty g(x)\,dx$ is divergent at ∞, then $\int_a^\infty f(x)\,dx$ is also divergent at ∞.

Proof : (i) f and g are both bounded and integrable in [a, t], t ≥ a and
$\because$ f and g are both positive, with f (x) ≤ g (x) $\forall$ x ∈ [a, x]

$$\therefore \int_a^t f(x)\,dx \le \int_a^t g(x)\,dx \qquad \dots (1)$$

But $\int_a^\infty g(x)\,dx$ dx is given to be convergent at ∞, So $\int_a^t g(x)\,dx$ is bounded above, $\forall\ t \ge a$

$\Rightarrow \quad \exists\ M > 0$ s.t. $\int_a^t g(x)\,dx < M, \forall\ t \ge a$

$\Rightarrow \quad$ (1) gives $\int_a^t f(x)\,dx < M, \forall\ t \ge a$

$\therefore \quad \int_a^\infty f(x)\,dx$ is convergent at ∞.

(ii) $\quad \because f(x) \ge g(x)\ \forall\ x \in [a, \infty)$

$$\Rightarrow \quad \int_a^t f(x)\,dx \ge \int_a^t g(x)\,dx \qquad \ldots (1)$$

But, as $\int_a^\infty g(x)\,dx$ is divergent at ∞

$\Rightarrow \quad \int_a^t g(x)\,dx$ is unbounded

$\therefore \quad$ (1) implies that $\int_a^t f(x)\,dx$ is unbounded

$\therefore \quad \int_a^\infty f(x)\,dx$ is divergent at ∞.

2.3.3.2 Practical Comparison Test

Art 2 Statement : If f and g are positive in [a, t] and

(i) $\quad \lim_{t\to\infty} \frac{f(x)}{g(x)} = l$, where l is non-zero finite number then, the two integrals $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ converge or diverge together.

(ii) If $\underset{t\to\infty}{\text{Lim}} \frac{f(x)}{g(x)} = 0$, and $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ also converges.

(iii) If $\underset{x\to\infty}{\text{Lim}} \frac{f(x)}{g(x)} = \infty$, and $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ also diverges.

Proof : (i) Clearly $l > 0$. Take $\varepsilon \geq 0$, s.t. $l - \varepsilon > 0$.

Since $\underset{x\to\infty}{\text{Lim}} \frac{f(x)}{g(x)} = l$

$\therefore$ $\exists$ a number k (> a) however large, such that for x > k

$$\left|\frac{f(x)}{g(x)} - l\right| < \varepsilon \text{ or } (l - \varepsilon)\, g(x) < f(x) < (l + \varepsilon) g(x)$$

Now $(l-\varepsilon)$ g(x) < f(x) $\forall$ x > k > a

So, if $\int_a^\infty f(x)\,dx$ converges, then by direct comparison test $\int_a^\infty g(x)\,dx$ also converges at ∞.

Again, from (1), f (x) < (1 + ε) g(x) $\forall$ x > k > a

$\therefore$ if $\int_a^\infty fx\,dx$ diverges, then by comparison test, $\int_a^\infty g\,(x)\,dx$ also diverges at ∞.

Example 4 : Test for convergence of integrals

(i) $\int_1^\infty \frac{\log x}{x^2}\,dx$ (ii) $\int_0^\infty \left(\frac{1}{1+x} - e^{-x}\right)\frac{dx}{x}$

Sol. (i) $f(x) = \frac{\log x}{x^2} = x^{\frac{3}{2}} \cdot \frac{\log x}{x^2} = \frac{\log x}{x^{\frac{1}{2}}} \to 0, \text{ as } x \to \infty$

$\therefore$ taking $g(x) = \frac{1}{x^{\frac{3}{2}}}$, $\therefore \frac{f(x)}{g(x)} = \frac{\log x}{x^{\frac{1}{2}}}$

$\underset{x\to\infty}{\text{Lim}} \frac{f(x)}{g(x)} = 0$; but $\int_1^\infty \frac{dx}{x^{\frac{3}{2}}}$ converges at ∞

$\therefore$ $\int_1^\infty f(x)\,dx$ is also convergent at ∞.

(ii) $f(x) = \left(\frac{1}{1+x} - e^{-x}\right)\frac{1}{x} = \left(\frac{1}{1+x} - \frac{1}{e^x}\right)\frac{1}{x}$

$$= \frac{e^x - 1 - x}{x\,(1+x)\,e^x} = \frac{\left(1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots\right) - 1 - x}{x\,(1+x)\,e^x} = \frac{\frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 1} + \ldots.}{x\,(1+x)\,e^x} > 0, \forall\, x > 0$$

and $\lim_{x\to\infty} f(x) = 0$

Now, $\int_0^\infty \left(\frac{1}{1+x} - e^{-x}\right)\frac{dx}{x} = \int_0^1 \left(\frac{1}{1+x} - e^{-x}\right)\frac{dx}{x} + \int_1^\infty \left(\frac{1}{1+x} - e^{-x}\right)\frac{dx}{x}$

But first integral is proper, so we discuss the convergence of second at ∞.

Here, $f(x) = \frac{e^x - (1+x)}{e^x} \cdot \frac{1}{1+x} \cdot \frac{1}{x}$

Take $g(x) = \frac{1}{x^2}$

$$\therefore \quad \frac{f(x)}{g(x)} = \frac{e^x - (1+x)}{e^x} \cdot \left(\frac{x}{1+x}\right) = \frac{e^x - (1+x)}{e^x} \cdot \frac{1}{1 + \frac{1}{x}} = \frac{[1 - (1+x)\,e^{-x}]}{1 + \frac{1}{x}}$$

$\therefore$ $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1 \neq 0, \infty$

$\therefore$ $\int_1^\infty f(x)\,dx$ and $\int_1^\infty g(x)\,dx$ behave alike

But $\int_1^\infty g(x)\,dx$ is convergent at ∞ i.e. $\int_1^\infty f(x)\,dx$ also converges.

$\therefore$ the given integral converges.

The proof of (ii) and (iii) part is left as an exercise for the reader.

2.3.3.3 Useful Comparison Test

Art 3 Statement : Show that the improper integral $\int_a^\infty \frac{dx}{x^p}\ (a > 0)$ is convergent at ∞, if p > 1 and divergent at ∞, if p ≤ 1.

Proof : Let $I = \int_a^\infty \frac{dx}{x^p}$

When $p = 1, \int_a^t \frac{dx}{x} = [\log x]_a^t = \log \frac{t}{a}$

When $p \neq 1, \int_a^t \frac{dx}{x^p} = \left[\frac{x^{-p+1}}{-p+1}\right]_a^t = \frac{1}{(1-p)}\left\{t^{1-p} - a^{1-p}\right\}$

∴ when p = 1, $\underset{t\to\infty}{\text{Lim}} \int_a^t \frac{dx}{x} = \underset{t\to\infty}{\text{Lim}} \log \frac{t}{a} \to \infty$

∴ I diverges at ∞

When p < 1, $\underset{t\to\infty}{\text{Lim}} \int_a^t \frac{dx}{x^p} = \underset{t\to\infty}{\text{Lim}} \frac{1}{1-p}\left\{t^{1-p} - a^{1-p}\right\} \to \infty$

∴ I diverges at ∞

When p > 1, $\underset{t\to\infty}{\text{Lim}} \int_0^t \frac{dx}{x^p} = \underset{t\to\infty}{\text{Lim}} \frac{1}{1-p}\left\{\frac{1}{t^{p-1}} - \frac{1}{a^{p-1}}\right\} = \frac{1}{p-1}\left(\frac{1}{a^p - 1}\right) = \text{finite}$

∴ I converges at ∞.

Example 5 : Test for the convergence of the integral $\int_{e^2}^\infty \frac{dx}{x \log(\log x)}$

Sol. Let, $\int_{e^2}^\infty \frac{dx}{x \log(\log x)} = \int_2^\infty \frac{dt}{\log t}$ $\left[\text{Put } \log x = t \Rightarrow \frac{1}{x}dx = dt\right]$

Take $g(t) = \frac{1}{t^m}, 0 < m \leq 1$

$\therefore \quad \frac{f(t)}{g(t)} = \frac{\frac{1}{\log t}}{t^m} = \frac{t^m}{\log t}$

$$\operatorname*{Lim}_{t\to\infty} \frac{f(t)}{g(t)} = \operatorname*{Lim}_{t\to\infty} \frac{t^m}{\log t} = \operatorname*{Lim}_{t\to\infty} \frac{mt^{m-1}}{\frac{1}{t}} = \operatorname*{Lim}_{t\to\infty} mt^m \to \infty$$

But $\int_2^\infty g(t)\,dt$ diverges by useful compersion test

$\therefore$ I diverges at ∞.

2.3.3.4 Cauchy's Test

Art 4 Statement : The NASC for the convergence of the improper integral $\int_a^\infty f(x)\,dx$ at ∞, is that to each $\in > 0$, there exsits a positive number k, such that

$$\left|\int_{t_1}^{t_2} f(x)\,dx\right| < \in \ \forall\ t_1, t_2 > k.$$

Proof : The proof is left as an exercise for the reader.

Example 6 : Show that $\int_0^\infty \frac{\sin x}{x}\,dx$ is convergent.

Sol. $\because \frac{\sin x}{x} \to 1$, as $x \to 0$, So, 0 is not a point of infinite discontinuity

$$\therefore \quad \int_0^\infty \frac{\sin x}{x}\,dx = \int_0^1 \frac{\sin x}{x}\,dx + \int_1^\infty \frac{\sin x}{x}\,dx$$

We discuss the convergence of $\int_1^\infty \frac{\sin x}{x}\,dx$ at ∞

Let $\in > 0$ be given; let t_1, t_2, be two numbers, both greater than $\frac{4}{\in}$

$\therefore \quad t_1 > \frac{4}{\in}$ or $\frac{1}{t_1} < \frac{\in}{4}$ and $t_2 > \frac{4}{\in}$ or $\frac{1}{t_2} < \frac{\in}{4}$

$$\therefore \quad \int_{t_1}^{t_2} \frac{\sin x}{x}\, dx = \int_{t_1}^{t_2} \left(\frac{1}{x}\right)(\sin x)\, dx = \left[\left(\frac{1}{x}\right)(-\cos x)\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\cos x}{x^2}\, dx$$

$$= \left(\frac{\cos t_1}{t_1} - \frac{\cos t_2}{t_2}\right) - \int_{t_1}^{t_2} \frac{\cos x}{x^2}\, dx$$

$$\Rightarrow \quad \left|\int_{t_1}^{t_2} \frac{\sin x}{x}\, dx\right| \le \left|\frac{\cos t_1}{t_1} - \frac{\cos t_2}{t_2}\right| + \left|\int_{t_1}^{t_2} \frac{\cos x}{x^2}\, dx\right| \le \frac{1}{t_1} + \frac{1}{t_2} + \left|\int_{t_1}^{t_2} \frac{dx}{x^2}\right|$$

$$(\because |\cos x| \le 1)$$

$$= \left(\frac{1}{t_1} + \frac{1}{t_2}\right) + \left(\frac{1}{t_1} + \frac{1}{t_2}\right) < \frac{\in}{4} + \frac{\in}{4} + \frac{\in}{4} + \frac{\in}{4} = \in$$

$$\therefore \quad \left|\int_{t_1}^{t_2} \frac{\sin x}{x}\, dx\right| < \in$$

$\therefore$ by Cauchy's test $\int_1^{\infty} \frac{\sin x}{x}\, dx$ converges and consequently $\int_0^{\infty} \frac{\sin x}{x}\, dx$ is convergent.

2.3.3.5 Abel's Test

Art 5 Statement : If g is bounded and monotonic in (a, ∞) and $\int_a^{\infty} f(x)\, dx$ is convergent at ∞; then f (x) g(x) is convergent at ∞.

Proof : Since g (x) is monotonic in (a, ∞), it is integrable in (a, t) $\forall$ $t \ge a$.

Also f (x) is integrable in (a, t)

So, by second mean value theorem

$$\int_{t_1}^{t_2} f(x)\, g(x)\, dx = g(t_1) \int_{t_1}^{\xi} f(x)\, dx + g(t_1) \int_{\xi}^{t_2} f(x)\, dx, \text{ for } a < t_1 \le \xi \le t_2 \qquad \dots (1)$$

Let $\varepsilon > 0$ be given

Since, g is bounded in (a, ∞)

$\therefore$ $\exists$ a positive number k s.t. $|g(t_1)| \le k \ \forall\ x \ge a$

In particular, $|g(t_1)| \le k$ and $|g(t_2)| \le k$ ... (2)

Since $\int_a^{\infty} f(x)\, dx$ is convergent,

$\therefore$ by Cauchy's test, $\exists$ a number M such that

$$\left|\int_{t_1}^{t_2} f(x)\right| < \frac{\varepsilon}{2M}\ \forall\ t_1, t_2 \ge M \qquad \dots (3)$$

Let t_1, t_2 of (1) be $\ge$ M, so that ξ lying between t_1, t_2 is also $\ge$ M.

$\therefore$ from (3), we have

$$\left|\int_{t_1}^{\xi} f(x)\right| < \frac{\varepsilon}{2k}, \left|\int_{\xi}^{t_2} f(x)\right| < \frac{\varepsilon}{2k} \qquad \dots (4)$$

i.e. from (1), (2) and (4), it follows that $\exists$ M such that for $t_1, t_2 \ge M$

$$\left|\int_{t_1}^{t_2} f(x)\, g(x)\right| \le |g(t_1)| \left|\int_{t_1}^{\xi} f(x)\right| + |g(t_2)| \left|\int_{\xi}^{t_2} f(x)\right| < k \cdot \frac{\varepsilon}{2k} + k \cdot \frac{\varepsilon}{2k} = \varepsilon$$

Hence, by Cauchy's test, $\int_a^{\infty} f(x)\, g(x)$ is convergent at ∞.

2.3.3.6 Dirichlet's Test

Art 6 Statement : If g (x) is bounded, monotonic and tends to 0, as $x \to \infty$ and $\int_a^{t} f(x)$ is bounded for $t \ge a$ then $\int_a^{\infty} f(x)\, g(x)$ is convergent at ∞.

Proof : The proof of Dirichlet's test is important and left as an exercise for the reader.

Example 7 : Discuss the convergence or otherwise of the integral :

$$\int_a^{\infty} \frac{\cos ax - \cos bx}{x}\, dx \ (a > 0)$$

Sol. $I = \int_a^{\infty} \frac{\cos ax - \cos bx}{x}\, dx$

Take $f(x) = \cos ax - \cos bx$, $g(x) = \frac{1}{x}$

Now, $\left|\int_a^t f(x)\, dx\right| = \left|\int_a^t (\cos ax - \cos bx)\, dx\right| = \left|\left(\frac{\sin ax}{a} - \frac{\sin bx}{b}\right)\right|_a^t$

$$= \left|\frac{\sin at - \sin a^2}{a} - \frac{\sin bt - \sin ab}{b}\right|$$

$$\leq \frac{|\sin at|}{|a|} + \frac{|\sin a^2|}{|a|} + \frac{|\sin bt|}{|b|} + \frac{|\sin ab|}{|b|} \leq \frac{1}{|a|} + \frac{1}{|a|} + \frac{1}{|b|} + \frac{1}{|b|} = 2\left(\frac{1}{|a|} + \frac{1}{|b|}\right)$$

$\therefore$ $\int_a^t f(x)\, dx$ is bounded and $g(x) = \frac{1}{x}$ is monotonic decreasing and $\to 0$, as $x \to \infty$

$\therefore$ by Dirichlet's test, I converges at ∞.

2.3.4 Improper Integrals of Second Kind

If f (x) has a point of infinite discontinuity on [a, b], then $\int_a^b f(x)\, dx$ is called an improper integral of second kind. These are further divided into four types :

Type I : Convergence at the left end point

Here we define $\int_a^b f(x)\, dx = \lim_{\varepsilon \to 0+0} \int_{a+\varepsilon}^b f(x)\, dx$ provided the limit on the right exists, (finitely).

Otherwise, it is called divergent.

Type II : Convergence at the right-end point

Define $\int_a^b f(x)\, dx = \lim_{\varepsilon \to 0+0} \int_a^{b-\varepsilon} f(x)\, dx,$ provided the limit on the right exists (finitely), otherwise it is called divergent.

Type III : Convergence at both ends.

If a and b are only points of infinite discontinuity of f,

$$\text{then } \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \qquad \dots (1)$$

$$= \lim_{\varepsilon \to 0+0} \int_{a+\varepsilon}^{c} f(x)\,dx + \lim_{\epsilon^1 \to 0+0} \int_{c}^{b-\epsilon^1} f(x)\,dx$$

$$= \lim_{\varepsilon \to 0+0} \int_{a+\varepsilon}^{b-\varepsilon'} f(x)\,dx,$$ provided the limit on the right exists, independent of ε, ε'.

Type IV : Convergence at interior points

If an interior point c (a < c < b) is the only point of discontinuity of f, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = I_1 + I_2$$

The improper integral $\int_a^b f(x)\,dx$ is convergent, if both the improper integrals I_1 and I_2 on the right converge as in accordance with the definition given above.

As discussed in case of improper integrals of first kind, the above discussed types may not be appropriate for some of the cases. So, we need to study some other tests to discuss the convergence of improper integrals of second kind. Firstly, we state two important results :

1. Test for convergence at 'a' of $\int_a^b f(x)\,dx$

When a, the left end point is the only point of infinite discontinuity of f in [a, b], we assume that f is positive on [a, b]. In case f is negative, we can replace it by (–f).

The case when b is the only point of infinite discontinuity can be treated in the same manner.

2. NASC for convergence at 'a' of $\int_a^b f(x)\,dx$

NASC for the convergence of the improper integral $\int_a^b f(x)\,dx$ at 'a', where f is positive on [a, b], is that there exists a positive constant M, independently of $\in > 0$ such that

$$\int_{a+\varepsilon}^{b} f(x)\,dx \le M, \forall\, \varepsilon \text{ belonging to } (0, b-a).$$

2.3.4.1 Comparison Test (First Form)

Art 7 Statement : If f and g are two positive functions such that $f(x) \le g(x)\ \forall\, x \in [a, b]$; then

(i) $\int_a^b f(x)\,dx$ converges, if $\int_a^b g(x)\,dx$ converges

(ii) $\int_a^b g(x)\,dx$ diverges, if $\int_a^b f(x)\,dx$ diverges

Proof : Let f and g be both bounded and integrable in $[a + \varepsilon, b]$, $0 < \varepsilon < (b - a)$ and 'a' is the only point of infinite discontinuity of f in [a, b].

Since, f and g are positive and $f(x) \le g(x)\ \forall\, x \in [a, b]$

$$\therefore \quad \int_{a+\varepsilon}^{b} f(x)\,dx \le \int_{a+\varepsilon}^{b} g(x)\,dx \qquad \ldots (1)$$

(i) Let $\int_a^b g(x)\,dx$ be convergent

$\therefore$ by def. $\exists$ a positive integer M such that $\int_{a+\varepsilon}^{b} g(x)\,dx \le M$ for $0 < \in < (b-a)$

$\therefore$ from (1) $\int_{a+\varepsilon}^{b} f(x)\,dx \le M$ for $0 < \varepsilon < (b-a)$

$\therefore$ $\int_a^b f(x)\,dx$ is convergent at a

(ii) It is left as an exercise for the reader.

2.3.4.2 Practical Comparison Test (Limit-Method)

Art 8 Statement : If f (x) > 0, g (x) > 0, $\forall$ x $\in$ (a, b] and 'a' is the only point of infinite discontinuity of both f(x) and g(x) and

I. $\lim_{x\to a}\frac{f(x)}{g(x)} = l,$ where l, is neither zero, nor infinite, then the two integrals

$\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ either both converge or both diverge to ∞, at 'a'.

II. If $\lim_{x\to\infty}\frac{f(x)}{g(x)} = 0$ and $\int_a^b g(x)\,dx$ is convergent at 'a', then $\int_a^b f(x)\,dx$ is also convegent at 'a'.

III. If $\lim_{x\to a}\frac{f(x)}{g(x)} = \infty$ and $\int_a^b g(x)\,dx$ is divergent at 'a', then $\int_a^b f(x)\,dx$ is also divergent at 'a'.

Proof : Since, $\lim_{x\to a}\frac{f(x)}{g(x)} = l;$ where l is non-zero, finite.

$\therefore$ $\forall\ \varepsilon > 0, \exists$ a +ve number δ, such that

$$l - \varepsilon < \frac{f(x)}{g(x)} < l + \varepsilon \text{ whenever, } a < x < a + \delta$$

Take ε, so small that $l - \varepsilon$ is +ve

$\therefore$ $(l - \varepsilon)\, g(x) < f(x) < (1 + \varepsilon)\, g(x)\ \forall\ a < x < a + \delta$

(i) If $\int_a^b g(x)\,dx$ is convergent at 'a', then $\int_a^{a+\delta} g(x)\,dx$ is also convergent at a

As f (x) < (l + $\in$) g(x) and $\int_a^{a+\delta} (l + \varepsilon)\, g(x)\,dx$ is convergent

$\Rightarrow$ $\int_a^{a+\delta} f(x)\,dx$ is convergent at 'a' and hence, $\int_a^b f(x)\,dx$ is also convergent at 'a'.

(ii) If $\int_a^b g(x)\,dx$ is divergent at 'a', then $\int_a^{a+\delta} g(x)\,dx$ is also divergent

$\because$ $f(x) > (l-\varepsilon)\, g(x)$ and $\int_a^{a+\delta} (l-\varepsilon)\, g(x)\,dx$ is divergent at 'a'

$\Rightarrow$ $\int_a^{a+\delta} f(x)\,dx$ is divergent at 'a' and hence, $\int_a^b f(x)\,dx$ is divergent at 'a'.

(iii) If $\int_a^b f(x)\,dx$ is convergent at 'a', then $\int_a^{a+\delta} f(x)\,dx$ is also convergent at 'a' But

$(l-\varepsilon)\, g(x) < f(x)$ and $\int_a^{a+\delta} f(x)\,dx$ is convergent at 'a'

$\Rightarrow$ $\int_a^{a+\delta} (l-\varepsilon)\, g(x)\,dx$ is also convergent at 'a'

$\Rightarrow$ $\int_a^{a+\delta} g(x)\,dx$ is also convergent and have, $\int_a^b g(x)\,dx$ is also convergent.

(iv) If $\int_a^b f(x)\,dx$ is divergent at 'a', then $\int_a^{a+\delta} f(x)\,dx$ is divergent at 'a'. But as

$(l+\varepsilon)\, g(x) > f(x)$ and $\int_a^{a+\delta} f(x)\,dx$ is divergent at 'a'.

$\Rightarrow$ $\int_a^{a+\delta} (l+\varepsilon)\, g(x)\,dx$ is also divergent at 'a'

$\Rightarrow$ $\int_a^{a+\delta} g(x)\,dx$ is divergent at 'a' and hence, $\int_a^b g(x)\,dx$ is divergent at 'a'.

Hence $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$, behave alike.

The proof of part II and part III left as an exercise for the reader.

2.3.4.3 Useful Comparison Test

Art 9 Statement : The improper integral $\int_a^b \frac{dx}{(x-a)^p}$ converges, if only if p < 1.

Proof : The proof is left as an exercise for the reader.

2.3.4.4 General Test for Convergence

Art 10 Statement : The NASC for the convergence of improper integral $\int_a^b f(x)\,dx$ at 'a' is that for every given η > 0, there corresponds δ > 0, s.t.

$$\left|\int_{a-\epsilon_1}^{a+\epsilon_2} f(x)\,dx\right| < \eta, \forall \text{ positive number } \varepsilon_1,\ \varepsilon_2 \le \delta$$

Proof : The proof is left as an exercise for the reader.

Example 8 : Discuss the convergence of the following integrals :

(i) $\int_1^2 \frac{dx}{(x-1)^{1/2}(2-x)^{1/3}}$ (ii) $\int_1^2 \frac{\sqrt{x}}{\log x}dx$ (iii) $\int_0^1 \frac{\sin x}{x^m}dx$

Sol. (i) Let $I = \int_1^2 \frac{dx}{(x-1)^{\frac{1}{2}}(2-x)^{\frac{1}{3}}} = \int_1^{\frac{3}{2}} \frac{dx}{(x-1)^{\frac{1}{2}}(2-x)^{\frac{1}{3}}} + \int_{\frac{3}{2}}^2 \frac{dx}{(x-1)^{\frac{1}{2}}(2-x)^{\frac{1}{3}}} = I_1 + I_2$

For $I_1 = \int_1^{\frac{3}{2}} \frac{1}{(x-1)^{\frac{1}{2}}(2-x)^{\frac{1}{3}}}dx, 1$ is the only point of infinite discontinuity.

Take $g(x) = \frac{1}{(x-1)^{\frac{1}{2}}}$ such that $\frac{f(x)}{g(x)} = \frac{1}{(2-x)^{\frac{1}{3}}} \to 1 \text{ as } x \to 1+0$

But $\int_{1}^{\frac{3}{2}} \frac{1}{(x-1)^{\frac{1}{2}}} dx$ converges at 1 and $\therefore$, 1, converges at 1.

For $I_2 = \int_{\frac{3}{2}}^{2} \frac{1}{(x-1)^{\frac{1}{2}}(2-x)^{\frac{1}{3}}} dx,$ is the only point of infinite discontinuity

Take $g(x) = \frac{1}{(2-x)^{\frac{1}{3}}}$ such that $\frac{f(x)}{g(x)} = \frac{1}{(x-1)^{\frac{1}{2}}} \to 1 \text{ as } x \to 2-0$

But $\int_{\frac{3}{2}}^{2} \frac{1}{(2-x)^{\frac{1}{3}}} dx$ or I_2 converges at 2 and $\therefore$, I_2 converges at 2.

Hence the given integral converges.

(ii) $I = \int_{1}^{2} \frac{\sqrt{x}}{\log x} dx$

1 is the only point of infinite discontinuity of f

$\therefore$ take $g(x) = \frac{1}{x-1}$ such that $\frac{f(x)}{g(x)} = \frac{\sqrt{x}}{\log x}(x-1)$

$$\operatorname*{Lim}_{x\to 1+0} \frac{f(x)}{g(x)} = \operatorname*{Lim}_{x\to 1+0} \frac{x^{\frac{3}{2}} - x^{\frac{1}{2}}}{\log x} = \operatorname*{Lim}_{x\to 1+0} \frac{\frac{3}{2}x^{\frac{1}{3}} - \frac{1}{2}x^{-\frac{1}{2}}}{\frac{1}{x}}$$

$$= \operatorname*{Lim}_{x\to 1+0} \left(\frac{3}{2}x^{\frac{3}{2}} - \frac{1}{2}x^{\frac{1}{2}}\right) = \frac{3}{2} - \frac{1}{2} = 1 \neq 0, \infty$$

But $\int_{1}^{2} \frac{dx}{(x-1)}$ diverges at 1. and therefore, $\int_{1}^{2} \frac{\sqrt{x}}{\log x} dx$ also diverges at 1.

(iii) $I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{x^m} dx$

When m > 0, the only point of infinite discontinuity is 0

Now, $\frac{\sin x}{x^m} = \frac{\sin x}{x} \cdot \frac{1}{x^{m-1}} \le \frac{1}{x^{m-1}}$ $\left(\because \frac{\sin x}{x} \le 1\right)$

But $\int_0^{\frac{\pi}{2}} \frac{dx}{x^{m-1}}$ is convergent for m–1 < 1 i.e. for m < 2

$\therefore$ $\int_0^{\frac{\pi}{2}} \frac{\sin x}{x^m} dx$ is convergent for m < 2 and is divergent for $m \ge 2$.

2.3.5 Absolute and Conditional Convergence

Def : (i) The improper integral $\int_a^b f(x)\,dx$ is said to be absolutely convergent if $\int_a^b |f(x)|\,dx$ is convergent.

(ii) An improper integral $\int_a^\infty f(x)\,dx$ is said to be absolutely convergent, if $\int_a^\infty |f(x)|\,dx$ is convergent.

Some Useful Results :

Result (1) : $\int_a^b |f(x)|\,dx$ exist $\Rightarrow \int_a^b f(x)\,dx$ exists.

or Every absolutly convergent integral is convergent.

Result (2) : If $\int_a^\infty |f(x)|\,dx$ converges, then $\int_a^\infty f(x)\,dx$ also converges.

Result (3) : Let ϕ be bounded in $[a, \infty)$ and integrable in $[a, t]$, $\forall\, t \ge a$.

Let $\int_a^\infty f(x)\,dx$ converge absolutely at ∞, then $\int_a^\infty f(x)\,\phi(x)\,dx$ is absolutely convergent.

Result (4) : A convergent integral not absolutely convergent, is **conditionally convergent.**

Example 9 : Test the convergence of $\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}\,dx$

Sol. $f(x) = \frac{\sin\frac{1}{x}}{\sqrt{x}}$

$\forall\ x \to (0, 1], \left|\frac{\sin\frac{1}{x}}{\sqrt{x}}\right| = \frac{\left|\sin\frac{1}{x}\right|}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$ and $\int_0^1 \frac{dx}{\sqrt{x}}\,dx$ is convergent

$\Rightarrow$ $\int_0^1 \left|\frac{\sin\frac{1}{x}}{\sqrt{x}}\right| dx$ is convergent and hence, $\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}\,dx$ is absolutely convergent.

2.3.6 Summary

In this lesson, we have elaborated the concept of improper integrals of first kind (one or both of the limits are infinite) and second kind (limits are finite but the function f(x) contains a point of discontinuity). For both of these improper integrals, we have discussed various useful tests to check their convergence. Moreover, the concept of absolute and conditional convergence is also explained. The concepts are made more clear with the help of various suitable examples.

2.3.7 Key Concepts

Convergence, Improper Integrals, Comparison Tests, Cauchy's Tests, Abel's Tests, Dirichlet's Tests, Absolute Convergence, Conditional Convergence

2.3.8 Long Questions

1. Examine for the convergence of $\int_0^\infty \left(\frac{1}{x} - \frac{1}{\sinh x}\right)\frac{dx}{x}$

2. Discuss the convergence of $\int_0^\infty \sin(x^2)\,dx$

3. Test the convergence of the integrals :

(a) $\int_{-x}^{a} \frac{x}{\sqrt{a^2 - x^2}}\,dx$ (b) $\int_0^1 \frac{\log x}{\sqrt{x}}\,dx$ (c) $\int_0^{\frac{\pi}{2}} \frac{\sin^m x}{x^n}\,dx$

for $n \geq 1$.

4. Show that $\int_0^\infty \frac{\sin x}{x}\,dx$ is not absolutely convergent.

2.3.9 Short Questions

1. Test for the convergence of $\int_1^\infty \frac{x}{(1+x)^5}\,dx$.
2. Define improper integral of first kind.
3. State direct comparison test for the convergence of improper integral of first kind.
4. State Cauchy's test for the convergence of improper integral of first kind. 5. State Dirichlet's test for the convergence of improper integral of first kind. 6. Write a note on absolute and conditional convergence.

2.3.10 Suggested Readings

1. Malik and Arora : Mathematical Analysis
2. Thomas and Finney (Ninth Edition) Calculus and Analytic Geometry

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B.A. PART - II **MATHEMATICS : PAPER–I**

SEMESTER-III **ANALYSIS-I**

LESSON NO. 2.4 **Author : Dr. Chanchal**

BETA AND GAMMA FUNCTIONS AND FRULLANI'S INTEGRAL

2.4.1 Objectives

The prime objective of this lesson is to discuss in detail about the two important improper integrals known as Beta functions and Gamma functions.

2.4.2 Introduction

Firstly, we introduce the beta and gamma functions as :

Beta Functions : The integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ where m > 0, n > 0 is called a Beta Function and is denoted by B (m, n). The quantities m, n are positive but not necessarily integers.

For Example : $\int_0^1 x^3(1-x)^5 dx$ is a Beta Function and is denoted by B (4, 6).

Gamma Function : The integral $\int_0^\infty e^{-x}x^{n-1}dx, n > 0$ is called a Gamma function and is denoted by $\Gamma(n)$. The quantity n is positive but not necessarily integer.

For Example : $\int_0^\infty x^3 e^{-x}dx$ is a Gamma function and is denoted by $\Gamma(4)$.

2.4.3 Convergence of Beta and Gamma Functions

Art 1 : Show that $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ exists, if and only if m and n are both positive.

Proof : $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

The integral is proper, if $m \geq 1$ and $n \geq 1$.
The number 0 is a point of infinite discontinuity if $m < 1$ and the number 1 is a point of infinite discontinuity if $n < 1$.

We take a number (say) $\frac{1}{2}$, between 0 and 1 to examine the convergence of the improper integrals $\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}dx$ and $\int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}dx$ at 0 and 1 respectively.

Convergence at 0 :- Consider $\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}dx$

Here $f(x) = x^{m-1}(1-x)^{n-1} = \frac{(1-x)^{n-1}}{x^{1-m}}$

Take $g(x) = \frac{1}{x^{1-m}}$

$\therefore \quad \lim_{x \to 0+0} \frac{f(x)}{g(x)} = \lim_{x \to 0+0} \frac{(1-x)^{n-1}}{x^{1-m}} \times x^{1-m} = 1$ which is non-zero finite.

$\therefore \quad \int_0^{\frac{1}{2}} f(x)\,dx$ and $\int_0^{\frac{1}{2}} g(x)\,dx$ behave alike.

But $\int_0^{\frac{1}{2}} \frac{1}{x^{1-m}}\,dx$ is convergent if 1 – m < 1 or if m > 0

$\therefore \quad \int_0^{\frac{1}{2}} f(x)\,dx$ converges, for m > 0.

Convergence at 1 : Consider $\int_{\frac{1}{2}}^{1} x^{m-1}(1-x)^{n-1}\,dx$

Here, $f(x) = x^{m-1}(1-x)^{n-1} = \frac{x^{m-1}}{(1-x)^{1-n}}$

Take $g(x) = \frac{1}{(1-x)^{1-n}}$

$$\lim_{x \to 1-0} \frac{f(x)}{g(x)} = \lim_{x \to 1-0} \frac{1}{x^{1-m}} = 1$$ which is non-zero finite

$\therefore \quad \int_{\frac{1}{2}}^{1} f(x)\,dx$ and $\int_{\frac{1}{2}}^{1} g(x)\,dx$ behave alike

But $\int_{\frac{1}{2}}^{1} \frac{1}{(1-x)^{1-n}}\,dx$ converges for 1 – n < 1, i.e., if n > 0.

and $\therefore \int_{\frac{1}{2}}^{1} f(x)\,dx$ converges for n > 0 ...(2)

From (1)and (2), we get

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx \text{ converges for positive values of m and n only.}$$

Cor. Using β-function, discuss the convergence of the integral

$$\int_0^{\frac{\pi}{2}} \sin^{p-1} x \cos^{q-1} x\, dx$$

Proof : The proof is left as an exercise for the reader.

Art 2 : Show that, the integral $\int_0^{\infty} x^{n-1}e^{-x}\,dx$ is convergent, if and only n > 0.

Proof : Let $f(x) = x^{n-1}e^{-x} = \dfrac{e^{-x}}{x^{1-n}}$

Clearly f has infinite discontinuity at 0, if 1 – n > 0 i.e. if n < 1.

∴ we have to discuss convergence at 0 and ∞ both.

Now $\int_0^{\infty} x^{n-1}e^{-x}dx = \int_0^1 x^{n-1}e^{-x}dx + \int_1^{\infty} x^{n-1}e^{-x}dx = I_1 + I_2$

Convergence at 0 of I_1 (m < 1)

Take $g(x) = \dfrac{1}{x^{1-n}}$ so that $\dfrac{f(x)}{g(x)} = e^{-x} \to 1, \text{as } x \to 0$

But $\int_0^1 g(x)\,dx = \int_0^1 \dfrac{dx}{x^{1-n}}$, which converges iff 1–n < 1 i.e. iff n > 0.

∴ $\int_0^1 x^{n-1}e^{-x}dx$ converges iff n > 0.

Convergence at ∞ of I_2

Take $g(x) = \frac{1}{x^2}$ so that $\frac{f(x)}{g(x)} = \frac{x^{n+1}}{e^x} \to 0$, as $x \to \infty \ \forall \ n$

But $\int_1^\infty \frac{dx}{x^2}$ converges $\Rightarrow \int_1^\infty x^{n-1}e^{-x}$ dx also converges for $\forall$ n

Hence $\int_1^\infty x^{n-1}e^{-x}dx$ is convergent, iff n > 0

Cor.1 : Prove that $\Gamma(1) = 1$

Proof : Put n = 1

$$\therefore \quad \Gamma(1) = \int_0^\infty e^{-x}dx = \lim_{t\to\infty}\int_0^t e^{-x}dx = \lim_{t\to\infty}\left[\frac{e^{-x}}{-1}\right]_0^t = \lim_{t\to\infty}(1-e^{-t}) = 1$$

Cor.2 : If n > 1 is any real number, then $\Gamma(n) = (n-1)\Gamma(n-1)$

Proof : $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx = \lim_{t\to\infty}\int_0^t x^{n-1}e^{-x}dx \ (n \geq 1)$

$$= \lim_{t\to\infty}\left\{(x^{n-1})(-e^{-x})\right\}_0^t + \int_0^t (n-1)\,x^{n-2}e^{-x}\,dx$$

(Integrating by parts)

$$= \lim_{t\to\infty}\left(0 - e^{-t}t^{n-1}\right) + \lim_{t\to\infty}\int_0^t (n-1)\,x^{n-2}e^{-x}dx$$

$$= 0 + \int_0^\infty (n-1)\,x^{n-2}e^{-x}dx = (n-1)\,\Gamma\,(n-1)$$

$\therefore \quad \Gamma(n) = (n-1)\Gamma(n-1)$

Cor. 3 : If n is a+ve integer, then $\Gamma(n) = \underline{|n-1}$

Proof : The proof is left as an exercise for the reader.

Note : The formulal deduced under Cor. 2 and Cor. 3 are known as Recurrence Formulae for Gamma function.

2.4.4 Relation Between Beta and Gamma Functions

Art 3 : Prove that $B(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ where m > 0, n > 0.

Proof : We have $\Gamma(n)=\int_0^\infty x^{n-1}e^{-x}dx$

Put x = tz, $\therefore$ dx = t dz and x = 0 $\Rightarrow$ z = 0, x $\to\infty\Rightarrow$ z $\to\infty$

$$\therefore \qquad \Gamma(n)=\int_0^\infty (tz)^{n-1}e^{-tz}\,.t\,dz=\int_0^\infty t^n z^{n-1}e^{-tz}\,dz$$

Multiplying both sides by $e^{-t}\,t^{m-1}$, we get,

$$e^{-t}t^{m-1}\Gamma(n)=e^{-t}t^{m-1}\int_0^\infty t^n z^{n-1}e^{-tz}\,dz$$

or $$\Gamma(n)\,.\,e^{-t}t^{m-1}=\int_0^\infty t^{m+n-1}z^{n-1}e^{-t(z+1)}dz$$

Integrating both sides w.r.t. t between the limits 0 to ∞, we get,

$$\Gamma(n)\int_0^\infty e^{-t}t^{m-1}dt=\int_0^\infty\left[\int_0^\infty t^{m+n-1}\,z^{n-1}\,e^{-t(z+1)}\,dz\right]dt$$

$$\Rightarrow \qquad \Gamma(n)\int_0^\infty e^{-t}t^{m-1}dt=\int_0^\infty z^{n-1}\left[\int_0^\infty t^{m+n-1}\,e^{-t(z+1)}\,dt\right]dz \qquad \ldots (1)$$

Put t (z + 1) = y $\quad\therefore\ t=\dfrac{y}{z+1}\quad\Rightarrow\ dt=\dfrac{dy}{z+1}$

Now t = 0 $\Rightarrow$ y = 0 and t $\to\infty\Rightarrow$ y $\to\infty$

$\therefore$ from (1), we get, $$\Gamma(n)\int_0^\infty e^{-t}t^{m-1}\,dt=\int_0^\infty z^{n-1}\left[\int_0^\infty\left(\frac{y}{z+1}\right)^{m+n-1}e^{-y}\,\frac{dy}{z+1}\right]dz$$

$$\therefore \qquad \Gamma(n)\,\Gamma(m)=\int_0^\infty\frac{z^{n-1}}{(z+1)^{m+n}}\left[\int_0^\infty y^{m+n-1}\,e^{-y}\,dy\right]dz$$

$$\Rightarrow \qquad \Gamma(m)\,\Gamma(n) = \int_0^\infty \frac{z^{n-1}}{(z+1)^{m+n}}\,\Gamma(m+n)\,dz = \Gamma(m+n)\int_0^\infty \frac{z^{n-1}}{(z+1)^{m+n}}\,dz$$

$$\Rightarrow \qquad \Gamma(m)\,\Gamma(n) = \Gamma(m+n)\,B\,(m, n) \text{ i.e. } B\,(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Cor 1. Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof : The proof is left as an exercise for the reader.

$\left[\text{Hint: Put } m=\frac{1}{2}, n = \frac{1}{2}\right]$.

2.4.5 Properties of Beta Function

Property I : B (m, n) = B (n, m) (Symmetry of Beta Function)

Property II : $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx;\ m, n > 0$

Property III : $B\,(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx;\ m, n > 0$

Property IV : B (m, n) = B (m, n + 1) + B (m + 1, n)

Property V : $\frac{B\,(m, n+1)}{n} = \frac{B\,(m+1, n)}{m}$

Property VI : $\frac{B\,(m, n+1)}{n} = \frac{B\,(m, n)}{m+n} = \frac{B\,(m+1, n)}{m}$

Property VII : $B\,(m,n) = \frac{\lfloor m-1 \,\lfloor n-1}{\lfloor m+n-1};\ m, n > 0 \text{ and } m, n \in Z$

Property VIII : $B(m, n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta,\ m, n > 0$

Now, we will prove the properties II, VII and VIII, while the proof of rest of the properties is an easy exercise for the reader.

Proof of Property II : $B(m,n)=\int_0^{\infty}\frac{x^{m-1}}{(1+x)^{m+n}}dx=\int_0^{\infty}\frac{x^{n-1}}{(1+x)^{m+n}}dx;\ m,n>0$.

As $B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx;\ m,n>0$

Put $x=\frac{t}{1+t},\ \therefore dx=\left[\frac{(1+t)-t(1)}{(1+t)^2}\right]dt\Rightarrow dx=\frac{1}{(1+t)^2}dt$

Now $x(1+t)=t\Rightarrow x+tx=t\Rightarrow x=t-tx\Rightarrow x=t(1-x)\Rightarrow t=\frac{x}{1-x}$

When x = 0, t = 0 and when x = 1, $t\to\infty$

$$\therefore\quad B(m,n)=\int_0^{\infty}\left(\frac{t}{1+t}\right)^{m-1}\left(1-\frac{t}{1+t}\right)^{n-1}\frac{1}{(1+t)^2}dt=\int_0^{\infty}\frac{t^{m-1}}{(1+t)^{m-1}}\cdot\frac{1}{(1+t)^{n-1}}\cdot\frac{1}{(1+t)^2}dt$$

$$=\int_0^{\infty}\frac{t^{m-1}}{(1+t)^{m-1+n-1+2}}dt=\int_0^{\infty}\frac{t^{m-1}}{(1+t)^{m+n}}dt$$

$$\therefore\quad B(m,n)=\int_0^{\infty}\frac{x^{m-1}}{(1+x)^{m+n}}dx\qquad\ldots(1)$$

[Since variable of integration can be changed in definite integration]

$$\therefore\quad B(m,n)=\int_0^{\infty}\frac{x^{n-1}}{(1+x)^{m+n}}dx\ \ldots(2)\quad[\because B(m,n)=B(n,m)]$$

$[\because$ of (1)$]$

From (1)and (2), we get,

$$B(m,n)=\int_0^{\infty}\frac{x^{m-1}}{(1+x)^{m+n}}dx=\int_0^{\infty}\frac{x^{n-1}}{(1+x)^{m+n}}dx.$$

Proof of Property VII : $B(m,n)=\frac{\underline{|m-1}\,\underline{|n-1}}{\underline{|m+n-1}};\ m,n>0$ and $m,n\in Z$

As $B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx;\ m,n>0$

$$= \left[x^{m-1} \frac{(1-x)^{n-1+1}}{(n-1+1)(-1)} \right]_0^1 - \int_0^1 (m-1)\, x^{m-2} \,.\, 1 \frac{(1-x)^{n-1+1}}{(n-1+1)(-1)} dx$$

$$= [0-0] + \frac{m-1}{n} \int_0^1 x^{m-2} (1-x)^n dx = \frac{m-1}{n} B(m-2+1, n+1)$$

$\therefore$ $$B(m,n) = \frac{m-1}{n} B(m-1, n+1) \qquad \ldots (1)$$

Changing m to (m–1) and n to (n + 1) on both sides of (1), we get

$$B(m-1, n+1) = \frac{m-2}{n+1} B(m-2, n+2)$$

$\therefore$ from (1), we get

$$B(m,n) = \left(\frac{m-1}{n}\right)\left(\frac{m-2}{n+1}\right) B(m-2, n+2) \qquad \ldots (2)$$

Changing m to (m–2) and n to (n + 2) in (1), we get

$$B(m-2, n+2) = \frac{m-3}{n+2} B(m-3, n+3)$$

$\therefore$ from (2), we get

$$B(m,n) = \left(\frac{m-1}{n}\right)\left(\frac{m-2}{n+1}\right)\left(\frac{m-3}{n+2}\right) B(m-3, n+3) \text{ and so on}$$

$\therefore$ $$B(m,n) = \frac{(m-1)(m-2)\ldots\ldots 2.1}{n(n+1)(n+2)\ldots\ldots(m+n-3)(m+n-2)} B(1, m+n-1)$$

$$= \frac{[(m-1)(m-2)\ldots\ldots 2.1]\,[(n-1)(n-2)\ldots\ldots\ldots 2.1]}{[1.2.3\ldots.(n-1)]\,[n(n+1)(n+2)\ldots\ldots.(m+n-3)(m+n-2)]}$$

$$\times \int_0^1 x^{1-1} (1-x)^{m+n-1-1} dx$$

$\therefore$ $$B(m,n) = \frac{\lfloor m-1 \, \lfloor n-1}{\lfloor m+n-2} \int_0^1 (1-x)^{m+n-2} dx \qquad \ldots (3)$$

$\Rightarrow \quad B(m,n) = \dfrac{\lfloor m-1 \lfloor n-1}{\lfloor m+n-2} \cdot \dfrac{1}{(m+n-1)}$

$\therefore \quad B(m,n) = \dfrac{\lfloor m-1 \lfloor n-1}{\lfloor m+n-1}$ where m, n are +ve integers.

$\because \quad \int_0^1 (1-x)^{m+n-2} dx = \left[\dfrac{(1-x)^{m+n-1}}{(m+n-1)(-1)}\right]_0^1 = -\left[0 - \dfrac{1}{m+n-1}\right] = \dfrac{1}{m+n-1}.$

Proof of Property VIII : $B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

As $\quad B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx; \; m, n > 0$

Put $\quad x = \sin^2\theta, \; \therefore \; dx = 2 \sin\theta \cos\theta \, d\theta$

Now $x = 0 \Rightarrow \theta = 0$ and $x = 1 \Rightarrow \theta = \dfrac{\pi}{2}$

$\therefore \quad B(m,n) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1} (1-\sin^2\theta)^{(n-1)} . 2 \sin\theta \cos\theta \, d\theta$

$= 2\int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1} (\cos^2\theta)^{n-1} . \sin\theta \cos\theta \, d\theta = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-2+1} (\cos\theta)^{2n-2+1} \, d\theta$

$\therefore \quad B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n+1}\theta \, d\theta.$

Art 4 : Prove that $\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta \, d\theta = \dfrac{1}{2} B\left(\dfrac{p+1}{2}, \dfrac{q+1}{2}\right).$

Proof : Let $I = \int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta \, d\theta = \int_0^{\frac{\pi}{2}} (\sin^2 \theta)^{\frac{p}{2}} (\cos^2 \theta)^{\frac{q}{2}} \, d\theta$

$\therefore \quad I = \int_0^{\frac{\pi}{2}} (\sin^2 \theta)^{\frac{p}{2}} (1 - \sin^2 \theta)^{\frac{q}{2}} \, d\theta$ ... (1)

Put $\sin^2\theta = t$, $\therefore$ $2 \sin \theta \cos\theta \, d\theta = dt$

$\therefore \quad d\theta = \frac{1}{2\sin\theta\cos\theta} dt = \frac{1}{2\sin\theta\sqrt{1-\sin^2\theta}} dt = \frac{1}{2\sqrt{t}} \frac{dt}{\sqrt{1-t}}$

When $\theta = 0$, $t = 0$ and when $\theta = \pi/2$, $t = 1$

$\therefore \quad I = \int_0^1 t^{\frac{p}{2}} (1-t)^{\frac{q}{2}} \cdot \frac{1}{2\sqrt{t}\sqrt{1-t}} dt = \frac{1}{2} \int_0^1 t^{\frac{p}{2}-\frac{1}{2}} (1-t)^{\frac{q}{2}-\frac{1}{2}} dt$

$= \frac{1}{2} B\left(\frac{p}{2} + \frac{1}{2}, \frac{q}{2} + \frac{1}{2}\right) = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$\therefore \quad \int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right).$

Example 1 : Evaluate : $\int_0^1 x^5 (1-x^3)^3 dx$

Sol. Let $I = \int_0^1 x^5 (1-x^3)^3 dx = \int_0^1 x^3 (1-x^3)^3 \cdot x^2 dx$

Put $x^3 = t, \therefore 3x^2 dx = dt \Rightarrow x^2 dx = \frac{1}{3} dt$

Now $x = 0 \Rightarrow t = 0$ and $x = 1 \Rightarrow t = 1$

$\therefore \quad I = \int_0^1 t (1-t)^3 \frac{1}{2} dt = \frac{1}{3} \int_0^1 t (1-t)^3 dt$

$$= \frac{1}{3} B(1+1, 3+1) = \frac{1}{3} B(2, 4) = \frac{1}{3} \frac{\underline{|2-1}\,\underline{|4-1}}{\underline{|6-1}}$$

$$= \frac{1}{3} \cdot \frac{\underline{|1}\,\underline{|3}}{\underline{|5}} = \frac{1}{3} \cdot \frac{\underline{|3}}{5 \times 4 \times \underline{|3}} = \frac{1}{60}.$$

Example 2 : Show that $\int_0^\infty \frac{x^3}{(1+x)^7} dx = \frac{1}{60}$

Sol. Let $I = \int_0^\infty \frac{x^3}{(1+x)^7} dx = \int_0^\infty \frac{x^{4-1}}{(1+x)^{3+4}} dx$

$$= B(4, 3) \qquad \left[\because B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx\right]$$

$$= \frac{\underline{|4-1}\,\underline{|3-1}}{\underline{|4+3-1}} = \frac{\underline{|3} \times \underline{|2}}{\underline{|6}} = \frac{(3 \times 2 \times 1) \times (2 \times 1)}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{1}{60}$$

2.4.6 Properties of Gamma Functions

Art 5 : Prove that $\int_0^{\frac{\pi}{2}} \sin^p x \cos^q x\, dx = \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+1}{2} + \frac{q+1}{2}\right)}$ $p > -1, q > -1.$

Proof : Let $I = \int_0^{\frac{\pi}{2}} \sin^p x \cos^q x\, dx = \int_0^{\frac{\pi}{2}} \sin^{p-1} x \cos^{q-1} x\, (\sin x \cos x)\, dx$

$$= \int_0^{\frac{\pi}{2}} (\sin^2 x)^{\frac{p-1}{2}} (\cos^2 x)^{\frac{q-1}{2}} (\sin x \cos x)\, dx$$

$$\therefore \qquad I = \frac{1}{2} \int_0^{\frac{\pi}{2}} (\sin^2 x)^{\frac{p-1}{2}} (1 - \sin^2 x)^{\frac{q-1}{2}} (2 \sin x \cos x)\, dx$$

Put $\sin^2 x = t \therefore 2 \sin x \cos x\, dx = dt$

When x = 0, t = 0 and when x = π/2, t = 1

$$\therefore \quad I = \frac{1}{2}\int_0^1 t^{\frac{p-1}{2}}(1-t)^{\frac{q-1}{2}}\,dt = \frac{1}{2}B\left(\frac{p-1}{2}+1, \frac{q-1}{2}+1\right)$$

$$= \frac{1}{2}B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)}$$

$$\therefore \quad \int_0^{\frac{\pi}{2}} \sin^p x \cos^q x dx = \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)}.$$

2.4.6.1 Duplication Formula

Art 6 : Show that $\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m)$

Proof : We have $B(m, m) = \int_0^1 x^{m-1}(1-x)^{m-1}dx$

Put $x = \sin^2\theta$, $\therefore$ $dx = 2\sin\theta\cos\theta\,d\theta$

When x = 0, θ = 0 and When x = 1, $\theta = \frac{\pi}{2}$

$$\therefore \quad B(m, m) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{m-1}\,.\,2\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1}(\cos^2\theta)^{m-1}\,.\,2\sin\theta\cos\theta\,d\theta = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2m-1}\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \left(\frac{2\sin\theta\cos\theta}{2}\right)^{2m-1} d\theta = 2\int_0^{\frac{\pi}{2}} \left(\frac{\sin 2\theta}{2}\right)^{2m-1} d\theta$$

$\therefore \quad B(m, m) = \frac{2}{2^{2m-1}} \int_0^{\frac{\pi}{2}} \sin^{2m-1} 2\theta \, d\theta$

Put $\quad 2\theta = z$ i.e. $\theta = \frac{1}{2} z \quad \Rightarrow \quad d\theta = \frac{1}{2} dz$

When $\theta = 0$, $z = 0$ and When $\theta = \frac{\pi}{2}, z = \pi$

$\therefore \quad B(m, m) = \frac{2}{2^{2m-1}} \int_0^{\pi} \sin^{2m-1} z \cdot \frac{1}{2} \, dz$

$\Rightarrow \quad \frac{\Gamma(m)\,\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \int_0^{\pi} \sin^{2m-1} z \, dz = \frac{2}{2^{2m-1}} \int_0^{\frac{\pi}{2}} \sin^{2m-1} z \, dz$

$$\left[\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \text{ if } f(2a-x) = f(x)\right]$$

$$= \frac{2}{2^{2m-1}} \int_0^{\frac{\pi}{2}} \sin^{2m-1} \cos^0 z \, dz = \frac{2}{2^{2m-1}} \cdot \frac{1}{2} \frac{\Gamma\left(\frac{2m-1+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{2m-1+1}{2} + \frac{0+1}{2}\right)}$$

$\therefore \quad \frac{\Gamma(m)\,\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \cdot \frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2m+1}{2}\right)} \Rightarrow \frac{\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m + \frac{1}{2}\right)}$

$\Rightarrow \quad \frac{\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \cdot \frac{\sqrt{\pi}}{\Gamma\left(m + \frac{1}{2}\right)} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$

$\Rightarrow \quad \Gamma(m)\,\Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma(2m)$

Example 3 : Evaluate $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^{\frac{5}{2}} x \, dx$

Sol. Let $I = \int_0^{\frac{\pi}{2}} \sin^3 x \cos^{\frac{5}{2}} x \, dx$ $\left[p = 3, q = \frac{5}{2}\right]$

$$= \frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{\frac{5}{2}+1}{2}\right)}{2\Gamma\left(\frac{3+1}{2}+\frac{\frac{5}{2}+1}{2}\right)} = \frac{\Gamma(2)\,\Gamma\left(\frac{7}{4}\right)}{2\Gamma\left(\frac{15}{4}\right)} = \frac{1\,.\,\Gamma\left(\frac{7}{4}\right)}{2\,.\,\frac{11}{4}\,.\,\frac{7}{4}\,.\,\Gamma\left(\frac{7}{4}\right)} = \frac{8}{77}\,.$$

Example 4 : Show that $\int_0^{\infty} \frac{x^4(1+x^5)}{(1+x)^{15}} dx = \frac{1}{5005}$

Sol. Let $I = \int_0^{\infty} \frac{x^4(1+x^5)}{(1+x)^{15}} dx = \int_0^{\infty} \frac{x^4 + x^9}{(1+x)^{15}} dx = \int_0^{\infty} \frac{x^4}{(1+x)^{15}} dx + \int_0^{\infty} \frac{x^9}{(1+x)^{15}} dx$

$$= \int_0^{\infty} \frac{x^{5-1}}{(1+x)^{5+10}} dx + \int_0^{\infty} \frac{x^{10-1}}{(1+x)^{10+5}} dx$$

$= B(5, 10) + B(10, 5) = B(5, 10) + B(5, 10)$ $[\because B(m,n) = B(n,m)]$

$$= 2B(5,10) = 2\,\frac{\Gamma(5)\,\Gamma(10)}{\Gamma(5+10)} = 2\,\frac{\Gamma(5)\,\Gamma(10)}{\Gamma(15)} = 2\,\frac{\underline{|4}\,\underline{|9}}{\underline{|14}} = \frac{1}{5005}\,.$$

2.4.7 Summary

In this lesson, we have discussed about the convergence and properties of two special improper integrals called Beta and Gamma functions. The concepts are made more elaborative with the help of some suitable examples.

2.4.8 Key Concepts

Convergence, Improper Integrals, Beta Function, Gamma Function, Duplication Formula

2.4.9 Long Questions

1. Show that $\int_0^a x^{m-1}(a-x)^{n-1}dx = a^{m+n-1}B(m,n)$

2. Show $2^n \Gamma\left(n+\frac{1}{2}\right) = 1.3.5..............(2n-1)\sqrt{\pi}$

3. Show that $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta = \frac{\pi}{\sqrt{2}}$

4. Prove that $B(m,m) . B\left(m+\frac{1}{2}, m+\frac{1}{2}\right) = \frac{\pi m^{-1}}{2^{4m}-1}$

5. Prove that $\int_0^{\infty} \frac{\log(1+a^2x^2)}{1+b^2x^2}dx = \frac{\pi}{b}\left(1+\frac{a}{b}\right)$

6. State and prove Duplication formula.

2.4.10 Short Questions

1. Define beta function.
2. Define gamma function.
3. Discuss the convergence of beta function.
4. Discuss the convergence of gamma function.
5. State duplication formula.

2.4.11 Suggested Readings

1. Malik and Arora : Mathematical Analysis
2. Thomas and Finney (Ninth Edition) Calculus and Analytic Geometry

Mandatory Student Feedback Form

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Note: Students, kindly click this google form link, and fill this feedback form once.