PUNJABI UNIVERSITY PATIALA

B.A. Part-III (SEMESTER-V)

MATHEMATICS : PAPER-II
MATHEMATICAL METHODS-I

SECTIONS A

Lesson No. :

Section : A

Note : Student can download the syllabus from department's website www.pbidde.org

Department of Distance Education
Punjabi University, Patiala

B.A. PART - III **PAPER-II**

MATHEMATICAL METHODS-I

LESSON NO. 1.1 **Author : Dr. Chanchal**

LAPLACE TRANSFORMS–I

Structure :

I. Introduction

From our earlier studies, we are already familiar with the concept of improper integrals. Now, to define the Laplace transform, we firstly define the concept of integral transform as An improper integral of type $\int_{-\infty}^{\infty} H(s.t)\, f(t)\, dt$ is called **integral transform** of f(t) if it is convergent and is denoted by F (s) or T (f(t)).

Note :

(i) Here the function H (s, t) is known as **Kernel** of the transform.

(ii) Here x is parameter (real of complex), which is independent of t.

In particular, if H (s, t) = $\begin{cases} e^{-st}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

Then $F(s) = \int_{-\infty}^{\infty} H(s,t)\, f(t)\, dt$

$$= \int_{-\infty}^{0} H(s,t)\, f(t)\, dt + \int_{0}^{\infty} H(s,t)\, f(t)\, dt$$

$$= \int_{-\infty}^{0} 0\, f(t)\, dt + \int_{0}^{\infty} e^{-st}\, f(t)\, dt$$

$$= 0 + \int_{0}^{\infty} e^{-st} f(t)\, dt$$

i.e. $F(s) = \int_{0}^{\infty} e^{-st}\, f(t)$ dt is transform, known as **LAPLACE TRANSFORM** and it may be defined as

Let f be a real valued function of the real vairable t, defined over (–∞, ∞) such that f(t) = 0 for t < 0

Then the function F of s, defined as

$F(s) = \int_{0}^{\infty} e^{-st}\, f(t)\, dt;$ is called the **Laplace Transform** of f and is denoted as L({f(t)).

II. Existence of Laplace Transform

Firstly, we define function of exponential order and pieceuise continuous function as :

Definition : Function of exponential order

A function f (x) is said to be of exponential order $\alpha > 0$ if $\lim_{x\to\infty} e^{-\alpha x} f(x)$ exists and is finite-number.

i.e. there exists a real number K > 0 such that

$$\left|e^{-\alpha x} f(x)\right| < K \;\forall\; x \geq M$$

or $$|\, f(x)\,| < K\, e^{\alpha x} \;\forall\; x \geq M\,.$$

For Example : f (x) = x^n is of exponential order $\alpha > 0$ as $x \to \infty$, where $a \in N$.

A function f (x) is called **piecewise or sectionally continuous** on finite interval [a b] iff this interval can be divided into finite subintervals such that f(x) is continuous in each of the subinterval except at end points of these intervals.

Thus in piecewise continuity (i) L.H.L. and R.H.L. of function exists in each subinterval

(ii) Function has finite jumps at the end points of these subintervals.

Example Let $f(x)=\begin{cases}1, 0<x<2\\ 3, 2<x<3\\ 5, x>3\end{cases}$.

Now, the existence of Laplace transform can be studied through the following theorem:

Theorem 1 : If f (t) is piecewise continuous on every finite interval in its domain $t \geq 0$ and is of exponential order α as $t \to \infty$, then prove that the Laplace transform of f(t) exists for all $s > \alpha$.

Proof : Given f (t) is piecewise continuous on every finite interval in its domain $t \geq 0$

$\Rightarrow$ f (t) is piecewise continuous on $[0\ t_0]$, for $t_0 > 0$

$\Rightarrow$ $e^{-st} f(t)$ is also piecewise continuous on $[0\ t_0]$

$\Rightarrow$ $e^{-st} f(t)$ is integrable on $[0\ t_0]$

(As e^{-st}, being exponented function is continuous)

$$\therefore \quad \int_0^{t_0} e^{-st} f(t)\,dt \text{ exists } \forall\ t_0 > 0$$

By def. of improper integrals

$$\int_0^{\infty} e^{-st} f(t)\,dt = \lim_{t_0 \to \infty} \int_0^{t_0} e^{-st} f(t)\,dt$$

$$\text{Now } |F(s)| = \left| \int_0^{\infty} e^{-st}\ f(t)\,dt \right|$$

$$\leq \int_0^{\infty} \left| e^{-st} f(t) \right| dt \qquad \left(\because \left| \int_0^{\infty} G(t)\,dt \right| \leq \int_0^{\infty} |G(t)|\,dt \right)$$

$$= \int_0^{\infty} e^{-st}\ |f(t)|\,dt \qquad \left(\because e^{-st} > 0 \right)$$

$$\leq \int_0^{\infty} e^{-st}\ (K\,e^{\alpha t})\,dt \qquad \left(\because f(t) \text{ is of exponential order } \alpha \right)$$

$$= K \int_0^{\infty} e^{(\alpha-s)t}\, dt$$

$$= K \left(\lim_{t_0 \to \infty} \int_0^{t_0} e^{(\alpha-s)t} dt \right) \qquad \text{(using (i))}$$

$$= K \lim_{t_0 \to \infty} \left[\frac{e^{(\alpha-s)t}}{\alpha - s} \right]_0^{t_0}$$

$$= \frac{K}{\alpha - s} \underset{t_0 \to \infty}{L\,im} \left[e^{(\alpha-s)t_0} - e^{(\alpha-s)0} \right]$$

$$= \frac{K}{\alpha - s} \underset{t_0 \to \infty}{L\,im} \left[\frac{1}{e^{(s-\alpha)t_0}} - 1 \right] \text{for } s > \alpha \cdot$$

$$= \frac{K}{\alpha - s} \underset{t_0 \to \infty}{L\,im} \left[\frac{1}{e^{(s-\alpha)t_0}} - 1 \right] \text{for } s > \alpha \cdot$$

$$= \frac{K}{\alpha - s} \left(\frac{1}{\infty} - 1 \right) = \frac{K}{\alpha - s}(0 - 1)$$

$$= \frac{K}{s - \alpha} \text{ for } s > \alpha$$

$\therefore$ $$|F(s)| \le \frac{K}{s - \alpha} \text{ for } s > \alpha$$

$\Rightarrow$ $F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$ converges for $s > \alpha$ so that Laplace Transform of f(t) exists.

III. Derivations of Some Useful Laplace Transforms

(i) $L(1) = \frac{1}{s}, s > 0$

Proof : By def. of Laplace Transform

$$L(1) = \int_0^{\infty} e^{-st}(1)\, dt$$

$$= \underset{t\to\infty}{\text{L im}} \int_0^t e^{-st}\, dt = \underset{t\to\infty}{\text{L im}} \left[\frac{e^{-st}}{-s}\right]_0^t$$

$$= -\frac{1}{s} \underset{t\to\infty}{\text{L im}} \left(e^{-st} - e^{0}\right)$$

$$= -\frac{1}{s} \underset{t\to\infty}{\text{L im}} \left(\frac{1}{e^{st}} - 1\right)$$

$$= -\frac{1}{s} \left(\frac{1}{\infty} - 1\right) \text{ if } s > 0$$

$$= -\frac{1}{s} (0 - 1) = \frac{1}{s} \text{ if } s > 0$$

$\therefore$ $\quad L(1) = \frac{1}{s}$ if $s > 0$.

(ii) $\quad L(t^{\alpha}) = \frac{\overline{|\alpha + 1}}{s^{\alpha+1}}, s > 0$ and α is any real > -1. In particular, $L(t^{n}) = \frac{\underline{|n}}{s^{n+1}}$.

Proof : By def. of Laplace Transform

$$L\,(t^{\alpha}) = \int_0^{\infty} e^{-st} t^{\alpha} dt$$

Put s t = y $\Rightarrow$ s dt = dy

$$= \int_{y=0}^{y=\infty} e^{-y} \left(\frac{y}{s}\right)^{\alpha} \left(\frac{dy}{s}\right)$$

$$= \frac{1}{s^{\alpha+1}} \int_0^{\infty} e^{-y} y^{\alpha} dy$$

$$= \frac{\overline{|(\alpha + 1)}}{x^{\alpha+1}} \text{ if } \alpha > -1$$

(By using def. of Gamma function)

$\therefore$ $\quad L(t^{\alpha}) = \frac{\overline{|(\alpha + 1)}}{x^{\alpha+1}}$ if $\alpha > -1$ and $s > 0$

In particular, when $\alpha = n = 0, 1, 2$.........

Then $L(t^{\alpha}) = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{\lfloor n}{s^{n+1}}$ if $n > -1, s > 0$.

(iii) $L(e^{\alpha t}) = \frac{1}{s-\alpha}$ if $s > \alpha$

Proof : By def. of Laplace Transform

$$L(e^{\alpha t}) = \int_0^{\infty} e^{-st} e^{\alpha t}\, dt = \int_0^{\infty} e^{(\alpha - s)t}\, dt$$

$$= \lim_{t\to\infty} \int_0^{t} e^{(\alpha-s)t} dt$$

$$= \lim_{t\to\infty} \left[\frac{e^{(\alpha-s)t}}{\alpha - s}\right]_0^t$$

$$= \frac{1}{\alpha - s} \lim_{t\to\infty} \left(e^{(\alpha-s)t} - e^{(\alpha-s)(0)}\right)$$

$$= \frac{1}{\alpha - s} \lim_{t\to\infty} \left(\frac{1}{e^{(s-\alpha)t}} - 1\right)$$

$$= \frac{1}{\alpha - s} \left(\frac{1}{\infty} - 1\right) \text{ if } s > \alpha$$

$$= \frac{1}{\alpha - s} (0 - 1) = \frac{1}{s - \alpha} \text{ if } s > \alpha$$

$\therefore$ $L(e^{\alpha t}) = \frac{1}{s-\alpha}$ if $s > \alpha$.

(iv) $L(\sinh \alpha t) = \frac{\alpha}{s^2 - \alpha^2}$ if $s > |\alpha|$

Proof : By def. $L(\sinh \alpha t) = L\left(\frac{e^{\alpha t} - e^{-\alpha t}}{2}\right)$

$$= \int_0^\infty e^{-st}\left(\frac{e^{\alpha t} - e^{-\alpha t}}{2}\right)dt$$

$$= \frac{1}{2}\int_0^\infty (e^{(\alpha - s)t} - e^{-(\alpha + s)t})\,dt$$

$$= \frac{1}{2}\underset{t\to\infty}{L\,im}\int_0^t (e^{(\alpha - s)t} - e^{-(\alpha + s)})dt$$

$$= \frac{1}{2}\underset{t\to\infty}{L\,im}\left(\frac{e^{(\alpha - s)t}}{\alpha - s} - \frac{e^{-(\alpha + s)t}}{-(\alpha + s)}\right)_0^t$$

$$= \frac{1}{2}\underset{t\to\infty}{L\,im}\left(\frac{1}{\alpha - s}(e^{(\alpha - s)t} - 1) + \frac{1}{\alpha + s}(e^{-(\alpha + s)t} - 1)\right)$$

$$= \frac{1}{2}\underset{t\to\infty}{L\,im}\left[\frac{1}{\alpha - s}\left(\frac{1}{e^{(s-\alpha)^t}} - 1\right) + \frac{1}{\alpha + s}\left(\frac{1}{e^{(s+\alpha)\,t}} - 1\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{\alpha - s}\left(\frac{1}{\infty} - 1\right) + \frac{1}{\alpha + s}\left(\frac{1}{\infty} - 1\right)\right] \text{ if } s > \alpha.\ s > -\alpha$$

$$= \frac{1}{2}\left[\frac{1}{\alpha - s}(0 - 1) + \frac{1}{\alpha + s}(0 - 1)\right] \text{ if } s >|\,\alpha\,|$$

$$= \frac{1}{2}\left[\frac{1}{s - \alpha} - \frac{1}{s + \alpha}\right] = \frac{\alpha}{s^2 - \alpha^2} \text{ if } s > |\,\alpha\,|$$

$\therefore$ $L\,(\sinh \alpha t) = \dfrac{\alpha}{s^2 - \alpha^2}$ if $s > |\,\alpha\,|$.

Now, the reader can easily prove trhat

(v) $L\,(\cos \alpha t) = \dfrac{s}{s^2 - \alpha^2}$ if $s >|\,\alpha\,|$

(vi) $L(\sin\alpha t) = \dfrac{\alpha}{s^2+\alpha^2}$ if $s > 0$

Proof : $L(\sin\alpha t) = \int_0^\infty e^{-st}\sin\alpha t\,dt$

$$= \lim_{t\to\infty}\left(\int_0^t e^{-st}\sin\alpha t\,dt\right)$$

$$= \lim_{t\to\infty}\left[\frac{e^{-st}}{s^2+\alpha^2}(-s\sin\alpha t - \alpha\cos\alpha t)\right]_0^t$$

$$\left[\text{using } \int e^{ax}\sin(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}(d\sin(bx+c) - b\cos(bx+c))\right]$$

$$= \lim_{t\to\infty}\frac{-1}{s^2+\alpha^2}\left(e^{-st}(s\sin\alpha t + \alpha\cos\alpha t) - e^0(s\sin 0 + \alpha\cos 0)\right)$$

$$= \lim_{t\to\infty}\frac{-1}{s^2+\alpha^2}\left(\frac{s\sin\alpha t + \alpha\cos\alpha t}{e^{st}} - \alpha\right)$$

$$= -\frac{1}{s^2+\alpha^2}(0-\alpha) \text{ if } s > 0 \qquad \left[\because \lim_{t\to\infty}\left|\frac{s\sin\alpha t + \alpha\cos\alpha t}{e^{st}}\right| \le \lim_{t\to\infty}\frac{|s+\alpha|}{e^{st}} = 0\right]$$

$$= \frac{\alpha}{x^2+\alpha^2} \text{ if } x > 0$$

$\therefore$ $L(\sin\alpha t) = \dfrac{\alpha}{s^2+\alpha^2}$ if $s > 0$.

(vii) $L(\cos\alpha t) = \dfrac{\alpha}{s^2+\alpha^2}$ if $s > 0$

Write : Do yourself

IV. Some Important Examples :

Example 1 : Find the Laplace Transform of f (t) = |t–2| + |t+2|, t ≥ 0 by first principles.

Sol. Here f (t) = |t – 2| + |t + 2|, t ≥ 0

$$= \begin{cases} -(t-2)+t+2, & 0 \le t \le 2 \\ t-2+t+2, & t > 2 \end{cases}$$

$$= \begin{cases} 4, & 0 \le t \le 2 \\ 2t, & t > 2 \end{cases}$$

$$\therefore \quad L\,(f(t)) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^2 e^{-st} f(t)\,dt + \int_2^\infty e^{-st} f(t)\,dt$$

$$= \int_0^2 e^{-st}(4)\,dt + \int_2^\infty e^{-st}(2t)\,dt$$

$$= 4\int_0^2 e^{-st}dt + 2\,\underset{t\to\infty}{\mathrm{Lim}} \int_2^t te^{-st}dt$$

$$= 4\left(\frac{e^{-st}}{-s}\right)_0^2 + 2\,\underset{t\to\infty}{\mathrm{Lim}}\left[\left(\frac{te^{-st}}{-s}\right)_2^t - \int_2^t \frac{e^{-st}}{-s}dt\right]$$

$$= -\frac{4}{s}(e^{-2s} - e^0) + 2\,\underset{t\to\infty}{\mathrm{Lim}}\left[-\frac{1}{s}(te^{-st} - e2e^{-2s}) + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)_2^t\right]$$

$$= -\frac{4}{s}(e^{-2s} - 1) + 2\,\underset{t\to\infty}{\mathrm{Lim}}\left[-\frac{1}{s}\left(\frac{t}{e^{st}} - 2e^{-2s}\right) - \frac{1}{s^2}\left(e^{-st} - e^{-2s}\right)\right]$$

$$= \frac{4\left(1-e^{-2s}\right)}{s} + 2\left(-\frac{1}{s}(0-2e^{-2s}) - \frac{1}{s^2}(0-e^{-2s})\right) \text{ if } s > 0$$

$$= \frac{4}{s} - \frac{4e^{-2s}}{s} + \frac{4e^{-2s}}{s} + \frac{2e^{-2s}}{s^2} \text{ if } s > 0$$

$$\left(\begin{array}{l} \because \lim\limits_{t \to \infty} \dfrac{t}{e^{st}} = \lim\limits_{t \to \infty} \dfrac{1}{se^{st}} = \dfrac{1}{\infty} = 0 \\ \text{and } \lim\limits_{t \to \infty} e^{-st} = 0 \text{ if } s > 0 \end{array} \right)$$

$$= \frac{4}{s} + \frac{2e^{-2s}}{s^2} \text{ if } s > 0$$

$$= \frac{2}{s}\left(2 + \frac{e^{-2s}}{s}\right) \text{ if } s > 0.$$

V. Self Check Exercise :

1. Find the Laplace transform of $f(t) = \begin{cases} e^t \text{ if } & 0 < t < 1 \\ 1 \text{ if } & t > 1 \end{cases}$ by first principles.

Suggested Readings :

1. A.R. Vasushtha & Dr. R.K. Gupta, Integral Transforms by Krishna Prakashan Media Pvt. Ltd. Meerut.

LESSON NO. 1.2 **Author : Dr. Chanchal**

LAPLACE TRANSFORMS–II

Structure :

I. Some Important Properties of Laplace Transforms

I (a) Theorem 1 (Linearity Property of Laplace Transforms)

Let f_1 (t) and f_2 (t) be any functions of t, where t ≥ 0 and their Laplace transforms exist.

Show $L(a_1 f_1(t) + a_2 f_2(t)) = a_1 L(f_1(t)) + a_2 L(f_2(t))$ for any constants a_1, a_2.

Proof : The proof is left as an exercise for the reader.

Theorem 2 (First Shifting Theorem) :

If F (s) is the Laplace transform of f (t) for t ≥ 0 and α is any number (real or complex)
Prove F (s – α) is Laplace transform of $e^{\alpha t}$ f (t)

OR

If L (f (t)) = F (s) for t ≥ 0
Prove $L(e^{\alpha t} f(t)) = F(s-\alpha)$, where α is any real or complex

Proof : We know $F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$

$$\therefore \quad L(e^{\alpha t} f(t)) = \int_0^\infty e^{-st} e^{\alpha t} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-\alpha)t} f(t)\,dt$$

$$= \int_0^\infty e^{-ut} f(t)\,dt \text{ where } s-\alpha = u$$

$$= F(u) = F(s-\alpha),$$

$\Rightarrow$ F (s – α) is the Laplace transform of $e^{\alpha t}$ f(t).

I (b) Theorem 3 (Second Shifting Theorem) :

If F (s) is Laplace transform of f(t) form $t \geq 0$ and α is any number (real or complex)

Prove that, the function $g(t) = \begin{cases} f(t-\alpha), & t > \alpha \\ 0, & t < \alpha \end{cases}$ has Laplace transform $e^{\alpha t}$ F(s)

Prove if $g(t) = \begin{cases} f(t-\alpha), & t > \alpha \\ 0, & t < \alpha \end{cases}$ then $L(g(t)) = e^{-\alpha t} F(s)$.

Proof : We know F (s) = L (f (t)) $= \int_0^\infty e^{-st} f(t)\,dt$

and given function is $g(t) = \begin{cases} f(t-\alpha), & t > \alpha \\ 0, & t < \alpha \end{cases}$

$\therefore$ $$L(g(t) = \int_0^\infty e^{-st} g(t)\,dt$$

$$= \int_0^\alpha e^{-st} g(t)\,dt + \int_\alpha^\infty e^{-st} g(t)\,dt$$

$$= \int_0^\alpha e^{-st}\, 0\,dt + \int_\alpha^\infty e^{-st} f(t-\alpha)\,dt \qquad \text{(By def. of g (t))}$$

$$= 0 + \int_\alpha^\infty e^{-st} f(t-\alpha)\,dt \text{ Put } t-\alpha = V \Rightarrow dt = dV$$

$$= \int_{V=0}^{V=\infty} e^{-s(V+\alpha)} f(V)\, dV$$

$$= e^{-\alpha s} \int_0^\infty e^{-sV} f(V)\, dV$$ (By changing variable V by t)

$$= e^{-\alpha s} \int_0^\infty e^{-st} f(t)\, dt$$

$$= e^{-\alpha s} F(s)$$

$\Rightarrow$ Laplace transform of g(t) is $e^{-\alpha s}$ F(s)

Another Form of above Theorem

If L (f(t) = F (s) for $t \geq 0$

and $\alpha \geq 0$, real, Prove L (f (t – α) h (t – α)) = $e^{\alpha s}$ F(s)

where $h(t-\alpha) = \begin{cases} 1, & t > \alpha \\ 0, & t < \alpha \end{cases}$ or $h(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$.

I (c) Theorem 4 (Change of Scale Property) :

If L (f (t)) = F(s) for $t \geq 0$

Prove for any positive constant α,

(i) $L(f(\alpha t) = \frac{1}{\alpha} F\left(\frac{s}{\alpha}\right)$ (ii) $L\left(f\left(\frac{1}{\alpha}\right)\right) = \alpha\, F(\alpha s)$

Proof : We know $F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$

(i) $\therefore L(f(\alpha\, t)) = \int_0^\infty e^{-st} (\alpha t)\, dt$ Put $\alpha t = V \Rightarrow \alpha dt = dV$

$$= \int_{V=0}^{V=\infty} e^{-s\frac{V}{\alpha}} f(V) \frac{dV}{\alpha}$$

$$= \frac{1}{\alpha} \int_0^\infty e^{-\left(\frac{s}{\alpha}\right)V} f(V)\, dV = \frac{1}{\alpha} \int_0^\infty e^{-\frac{s}{\alpha}t} f(t)\, dt$$

(Changing variable V by t)

$$= \frac{1}{\alpha} F\left(\frac{s}{\alpha}\right)$$

Now, the reader can easily prove the (ii) part.

II. Some Important Examples

Example 1 : Evaluate L {sinh³ 2r}.

Sol. Here $\sinh^3 2t = \left(\frac{e^{2t} - e^{-2t}}{2}\right)^3$

$$= \frac{1}{8}\left(\left(e^{2t}\right)^3 - \left(e^{-2t}\right)^3 - 3\, e^{2t}e^{-2t}\left(e^{2t} - e^{2t}\right)\right)$$

$$= \frac{1}{8}\left(e^{6t} - e^{-6t} - 3e^{2t} + 3e^{-2t}\right)$$

$$\therefore \quad L\,(\sinh^3 2t) = L\left(\frac{1}{8}\left(e^{6t} - e^{-6t} - 3e^{2t} + 3e^{-2t}\right)\right)$$

$$= \frac{1}{8} L\left(e^{6t}\right) - \frac{1}{8} L\left(e^{-6t}\right) - \frac{3}{8} L\left(e^{2t}\right) + \frac{3}{8} L\left(e^{-2t}\right)$$

$$= \frac{1}{8}\,\frac{1}{s-6} - \frac{1}{8}\,\frac{1}{s+6} - \frac{3}{8}\,\frac{1}{s-2} + \frac{3}{8}\,\frac{1}{s+2}$$

if s > 6, s > –6, s > 2, s > –2

$$= \frac{1}{8}\left(\frac{(s+6)-(s-6)}{(s-6)\,(s+6)}\right) - \frac{3}{8}\left(\frac{(s+2)-(s-2)}{(s-2)\,(s+2)}\right)$$

$$= \frac{12}{8(s^2-36)} - \frac{3}{8}\left(\frac{4}{s^2-4}\right) \text{ if } s > 6$$

$$= \frac{3}{2}\left(\frac{(s^2-4)-(s^2-36)}{(s^2-36)\,(s^2-4)}\right) \text{ if } s > 6$$

$$= \frac{3}{2}\left(\frac{32}{(s^2-36)\,(s^2-4)}\right) \text{ if } s > 6$$

$$= \frac{48}{(s^2 - 36)(s^2 - 4)} \text{ if } s > 6 \cdot$$

Example 2 : Find the Laplace transform of following functions of t, $t \geq 0$

$e^{-3/2t} \sin 6t \sin 2t$

Sol. We know L (sin 6t sin 2t) = L $\left(\frac{1}{2}(\cos 4t - \cos 8t)\right)$

$$= \frac{1}{2}(L(\cos 4t) - L(\cos 8t))$$

$$= \frac{1}{2}\left(\frac{s}{s^2 + 4^2} - \frac{s}{s^2 + 8^2}\right); s > 0$$

$$= \frac{s}{2}\left(\frac{s^2 + 64 - s^2 - 16}{(s^2 + 16)(s^2 + 64)}\right); s > 0$$

$$= \frac{s}{2}\left(\frac{48}{(s^2 + 16)(s^2 + 64)}\right)$$

$$= \frac{24s}{(s^2 + 16)(s^2 + 64)}; s > 0$$

∴ Using First Shifting Theorem

$$L(e^{-3/2\,t} \sin 6t \sin 2t) = \frac{24\left(s - \left(-\frac{3}{2}\right)\right)}{\left\{\left(s - \left(-\frac{3}{2}\right)\right)^2 + 16\right\}\left\{\left(s - \left(-\frac{3}{2}\right)\right)^2 + 64\right\}}$$

$$= \frac{24\left(s+\frac{3}{2}\right)}{\left\{\left(s+\frac{3}{2}\right)^2+16\right\}\left\{\left(s+\frac{3}{2}\right)^2+64\right\}}, s > -\frac{3}{2}.$$

Example 3 : Find the Laplace transform of $g(t) = \begin{cases} 0, & 0 < t < \frac{1}{2} \\ t+\frac{3}{2}, & t > \frac{1}{2} \end{cases}$ by Second Shifting Theorem.

Sol. Given Function is

$$g(t) = \begin{cases} 0, & 0 < t < \frac{1}{2} \\ (t+2)-\frac{1}{2}, & t > \frac{1}{2} \end{cases} \qquad \left[\begin{array}{l} \because \lim_{t\to\infty} \frac{t+\frac{3}{2}}{e^{st}} = 0 \\ \text{By L}' \text{ Hospital Rule} \end{array}\right]$$

Here $f(t) = t+2, \alpha = \frac{1}{2}$

We know L (f(t)) = L (t + 2) = L (t) + 2 L (1)

$$= \frac{1}{s^2} + \frac{2}{s} = \frac{1+2s}{s^2}$$

∴ By Second Shifting Theorem

$$L(g(t)) = e^{-\frac{1}{2}s}\left(\frac{1+2s}{s^2}\right) = \frac{2s+1}{s^2\, e^{\frac{1}{2}s}} \text{ if } s > 0.$$

Example 4 : Evaluate L (e^{10} sin 10 t cos 10 t), $t \geq 0$ by change of scale property.

Sol. Firstly, we shall evaluate

L (e^t sin t cos t), $t \geq 0$

We have L (sin t cos t) = $L\left(\frac{\sin 2t}{2}\right)$

$$= \frac{1}{2} L(\sin 2t) = \frac{1}{2}\left(\frac{2}{s^2+2^2}\right)$$

$= \frac{1}{s^2 + 4}, s > 0$

∴ By First Shifting Theorem, we have

$L(e' \sin i \cos t) = \frac{1}{(s-1)^2 + 4}; s - 1 > 0$

Further, Using Change of Scale Property

$$L(e^{10t} \sin 10t \cos 10t) = \frac{1}{10} \frac{1}{\left(\frac{s}{10} - 1\right)^2 + 4}, \frac{s}{10} > 1$$

$$= \frac{10}{(s-10)^2 + 400}, s > 10$$

$$= \frac{10}{s^2 - 20s + 500}, s > 10.$$

III. Self Check Exercise

1. Using $L(e^{\alpha t}) = \frac{1}{s - \alpha}, s > \alpha$

   Evaluate (i) L (sin h α t) (ii) L (cosh α t)
   (iii) L (sin α t) (iv) L (cos α t)

2. Find Laplace transform of $\sin \sqrt{t}, t \geq 0$.

3. Find Laplace Transform of $t^2 e^t \sin 4t$

4. Find the Laplace transform of following using second shifting Theorem

$$g(t) = \begin{cases} 0, & 0 < t < \frac{\pi}{2} \\ \sin t, & t > \frac{\pi}{2} \end{cases}$$

6. Evaluatge L (e' cosh t), t ≥ 0. Hence evaluate L (e^{5t} cosh 5t).

Suggested Readings :

1. A.R. Vasushtha & Dr. R.K. Gupta, Integral Transforms by Krishna Prakashan Media Pvt. Ltd. Meerut.

LESSON NO. 1.3 **Author : Dr. Chanchal**

LAPLACE TRANSFORMS–III

Structure :

I. Introduction

From our previous lesson, we are alreadyfamiliar with the basic concept of Laplace transforms and its properties. So, now we have theenough knowledge to understand the Laplace transforms of derivatives and integrals of functions, as discussed below.

II. Laplace Transforms of Derivatives and Integrals

For finding the Laplace transforms of derivatives, we introduce the following result:

Result 1 : Let f (t) be real function defined for $t \geq 0$

If f (t), f (t), f"(t), ... f^{n-1} (t) are continuous on $[0, \infty)$ and are of exponential order α and f"(t) is cont. or piecewise continuous on $[0, \infty)$. Then, Laplace transform of f"(t) exists and $L(f''(t)) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \ldots - sf^{n-2}(0) - f^{n-1}(0)$; $s > \alpha$ where $f''(t) = \frac{d^n f(t)}{dt^n}$.

In particular, (i.e., nth order derivative of f (t))

(i) $L(f'(t)) = s\,F(s) - f(0)$, $s > \alpha$

(ii) $L(f''(t)) = s^2 F(s) - sf(0) - f'(0)$ for $s > \alpha$

(iii) $L(f'''(t) = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$

For providing this result, we will prove it for the first derivative, as given below:

Theorem 1 : Let f(t) be real and continuous function of exponential order α on $[0, \infty)$. Also f'(t) be continuous or piecewise continuous function on $[0, \infty)$. Then prove that Laplace transform of f'(t) exists and $L(f'(t)) = s\,F(s) - f(0)$ for $s > \alpha$ if $F(s) = L(f(t))$.

Proof : Case I. When f' is continuous on $[0, \infty)$

Then $L(f'(t)) = \int_0^\infty e^{-st} f'(t)\,dt$ (by def. of Laplace transform)

$$= \lim_{t\to\infty} \int_0^t e^{-st} f'(t)\,dt$$

$$= \lim_{t\to\infty} \left(e^{-st} f(t)\right)_0^t - \int_0^t -se^{-st} f(t)\,dt$$

$$= \lim_{t\to\infty} \left(\left\{e^{-st} f(t) - f(0)\right\} + s\int_0^t e^{-st} f(t)\,dt\right)$$

$$= \lim_{t\to\infty} e^{-st} f(t) - f(0) + s\lim_{t\to\infty} \int_0^t e^{-st} f(t)\,dt$$

$= 0 - f(0) + s\,L(f(t))$

$= s\,F(s) - f(0)$, if $s > \alpha$

$$\left[\begin{array}{l} \because \text{Given } f(t) \text{ is cont. of exponential order } \alpha \\ \therefore |f(t)| < ke^{\alpha t} \text{ for some scalar } k \\ \Rightarrow |e^{-st} f(t)| = e^{-st}\,|f(t)| < e^{-st}(ke^{\alpha t}) \\ \qquad\qquad\qquad\qquad \leq \dfrac{K}{e^{(s-\alpha)t}} \to 0 \\ \qquad\qquad\qquad\qquad \text{as } t \to \infty \\ \Rightarrow \lim\limits_{t\to\infty} e^{-st} f(t) = 0 \end{array}\right.$$

Case II. When f' is piecewise cont. on $[0, \infty)$

$\Rightarrow$ f' is piecewise cont. on $[0\ t]$
where t is any positive number

$\Rightarrow$ $f'(t)$ is discontinuous at $t_1, t_2 \ldots\ldots\ldots t_n$
where $0 < t_1 < t_2 \ldots\ldots\ldots < t_n \leq t$

$$\Rightarrow \quad L(f'(t)) = \int_0^\infty e^{-st} f'(t)\,dt = \lim_{t\to\infty} \int_0^t e^{-st} f'(t)\,dt$$

$$= \lim_{t\to\infty} \left\{ \int_0^{t_1} e^{-st} f'(t)\,dt + \int_{t_1}^{t_2} e^{-st} f'(t)\,dt \ldots + \int_{t_n}^{t} e^{-st} f'(t)\,dt \right\}$$

$$= \lim_{t\to\infty} \left\{ (e^{-st} f(t))_0^{t_1} - \int_0^{t_1} -se^{-st} f(t)\,dt + (e^{-st} f(t))_{t_1}^{t_2} \right.$$

$$\left. - \int_{t_1}^{t_2} -se^{-st} f(t)\,dt + \ldots + (e^{-st} f(t))_{t_n}^{t} - \int_{t_n}^{t} -se^{-st} f(t)\,dt \right\}$$

$$= \lim_{t\to\infty} \left\{ (e^{-st_1} f(t_1) - f(0) + s\int_0^{t} \frac{1}{e^{st}} f(t)\,dt \right.$$

$$+ (e^{-st_2} f(t_2) - e^{-st_1} f(t_1)) + s\int_{t_1}^{t_2} e^{-st} f(t)\,dt + \ldots\ldots..$$

$$\left. + (e^{-st} f(t) - e^{-st_n} f(t_n) + s\int_{t_n}^{t} e^{-st} f(t)\,dt \right\}$$

$$= \lim_{t\to\infty} \left\{ e^{-st_1} f(t_1) - f(0) + e^{-st_2} f(t_2) - e^{-st_1} f(t_1) + \ldots\ldots. + e^{-st} f(t) \right.$$

$$\left. - e^{-st} f(t_n) + s\int_0^{t_1} e^{-st} f(t)\,dt + \int_{t_1}^{t_2} e^{-st} f(t)\,dt + \ldots.. \right\}$$

$$= \lim_{t\to\infty} \left(e^{-st} f(t) - f(0) + s\left(\int_0^{t} e^{-st} f(t)\,dt + \ldots \right) \right) + \int_{t_n}^{t} e^{-st} f(t)\,dt$$

[Using property of definite integrals]

$= s\,F(s) - f(0)$ if $s > \alpha$ [As explained in case (i)]

$\therefore \quad L(f'(t)) = s\,F(s) - f(0),\ s > \alpha$

Hence the result.

Now, to understand the Laplace transform of integral, we study the following theorem:

Theorem 2 : Let f (t) (for $t \geq 0$) be real and continuous function on $[0, \infty)$.

If L (f (t)) = F(s), then prove Laplace transform of $\int_0^t f(z)\,dz$ exists

and $L\left[\int_0^t f(z)\,dz\right) = \frac{F(s)}{s}$ or $\frac{L\,(f(t))}{s}$

Sol. Given L (f(t) = F(s)

and let $H(t) = \int_0^t f(z)\,dz$

$\therefore$ $H(0) = \int_0^0 f(z)\,dz = 0$

Now $H'(t) = \frac{d}{dt}\left(\int_0^t f(z)\,dz\right)$

$$= \int_0^t \frac{\partial}{\partial t}(f(z))\,dz + \frac{dt}{dt}f(t) - 0$$

(Using Leibnitz Rule of differentiation under integral sign)

= 0 + f(t) – 0 = f (t)

$\Rightarrow$ H'(t) = f(t)

We know

L(H'(t)) = s L(H(t)) – H (0)

$\Rightarrow$ L (f(t) = s L (H (t) – 0 (By Theorem 1)

$\Rightarrow$ $L(H(t)) = \frac{L\,(f(t))}{s}$

$\Rightarrow$ $L\left(\int_0^t f(z)\,dz\right) = \frac{L\,(f(t))}{s}$ or $\frac{F(s)}{s}$

Now the reader can easily prove the result :

Cor : Prove $L\left(\int_0^t f(z)\,dz\right) = -\frac{F(s)}{s} + \frac{1}{s}\int_0^\infty f(z)\,dz$.

III. Multiplication and Division by 't'

On the basis of Laplace transforms of derivatives and integrals, we can easily understand the following results :

Result 2 : Let f (t) be real and piecewise continuous function of exponential order α on on $[0, \infty)$ and if L (f (t)) = F (s)

$$\text{Then, } \frac{dF(s)}{ds} = -L(tf(t))$$

$$\text{and further } \frac{d^nF(s)}{ds^n} = (-1)^n\ L\ (t^nf(t)) \text{ for } n = 1, 2.........$$

Hint : The reader can easily prove this result with the help of Principle of mathematical induction.

Result 3 : Let f (t) be real and piecewise continuous function on each interval in $[0, \infty)$ and is of exponential order α.

If L (f (t)) = F(s). Then prove $L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s)$ ds if integral is convergent.

Proof : Given L (f (t) = F(s)

$$\Rightarrow \qquad F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Taking integrals on both sides w.r.t. s from s = s to s = ∞ we get

$$\int_s^\infty F(s)\, ds = \int_{s=s}^{s=\infty}\left(\int_{t=0}^{t=\infty} e^{-st}\ f(t)\, dt\right) ds$$

$$= \int_{t=0}^{t=\infty}\left(\int_{s=s}^{s=\infty} e^{-st} ds\right) f(t)\, dt$$

($\because$ s and t are independent so order of integration can be interchanged)

$$= \int_{t=0}^{t=\infty}\left(\frac{e^{-st}}{-t}\right)_s^\infty f(t)\, dt$$

$$= \int_{t=0}^{t=\infty} -\frac{1}{t}\left(\underset{s\to\infty}{L\,im}\, e^{-st} - e^{-st}\right) f(t)\, dt$$

$$= \int_0^\infty -\frac{1}{t}(0 - e^{-st})\, f(t)\, dt$$

$$\left(\because \underset{s\to\infty}{\mathrm{Lim}}\, e^{-st} = \underset{s\to\infty}{\mathrm{Lim}}\, \frac{1}{e^{st}} = 0\right)$$

$$= \int_0^\infty e^{-st}\, \frac{f(t)}{t}\, dt$$

$$= L\left(\frac{f(t)}{t}\right) \qquad \text{(Using def. of Laplace Transform)}$$

$$\Rightarrow \qquad \int_s^\infty F(s)\, ds = L\left(\frac{f(t)}{t}\right)$$

Hence the result.

IV. Laplace Transform of Periodic Function

Theorem 3 : Let f(t) be periodic function with period T i.e. $f(t + nT) = f(t)$ for $n \in N$.

then Prove that $L(f(t)) = \int_0^T \frac{e^{-st}}{1 - e^{-sT}}\, f(t)\, dt$

Sol. We have $L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$

$$= \int_0^T e^{-st} f(t)\, dt + \int_T^{2T} e^{-st}\, f(t)\, dt + \int_{2T}^{3T} e^{-st} f(t)\, dt + ...$$

$$= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)\,T} e^{-st} f(t)\, dt \qquad \text{Put } t = y + n\,T \Rightarrow dt = dy$$

$$= \sum_{n=0}^{\infty} \int_{y=0}^{y=T} e^{-s\,(y+nT)} f\,(y + nT)\, dy$$

($\because$ f is periodic with period T so $f(y + nT) = f(y)$)

$$= \sum_{n=0}^{\infty} \int_0^T e^{-sy} e^{-snT} f(y)\, dy$$

$$= \sum_{n=0}^{\infty} e^{-snT} \int_0^T e^{-sy} f(y)\, dy$$

$$= (1 + e^{-sT} + e^{-2sT} + \ldots\ldots \text{ to } \infty)\left(\int_0^T e^{-sy} f(y)\, dy\right)$$

(Change variable y by t)

$$= \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt = \int_0^T \frac{e^{-st}}{1 - e^{-sT}} f(t)\, dt$$

Hence the result.

V. Some Important Examples

Example 1 : Evaluate L (t sin α t)

Sol. $L\ f(t) = t \sin \alpha t$

$\Rightarrow$ $f'(t) = \sin \alpha t + t(\alpha) \cos \alpha t$

and $f'(t) = \alpha \cos \alpha t + \alpha (\cos \alpha t - (\alpha \sin \alpha t)\ t)$

$= 2\alpha \cos \alpha t - \alpha^2 t \sin \alpha t$

Using $L(f''(t)) = s^2 L(f(t)) - s f(0) - f'(0)$

$\Rightarrow$ $L(2\alpha \cos \alpha t - \alpha^2(t \sin \alpha t) = s^2 L(t \sin \alpha t) - s(0) - 0$

$\Rightarrow$ $2\alpha L(\cos \alpha t) - \alpha^2 L(t \sin \alpha t) = s^2 L(t \sin \alpha t)$

$\Rightarrow$ $$2\alpha \frac{s}{s^2 + \alpha^2} = (s^2 + \alpha^2)\, L(t \sin \alpha t)$$

$\Rightarrow$ $$L(t \sin \alpha t) = \frac{2\alpha s}{(s^2 + \alpha^2)^2}$$

By Second Method

We know $L(\sin \alpha t) = \frac{\alpha}{s^2 + \alpha^2} = F(s)$

$$L(t \sin \alpha t) = (-1)\frac{dF(s)}{ds} = (-1)\frac{d}{ds}\left(\frac{\alpha}{s^2 + \alpha^2}\right)$$

$$= (-1)\,\alpha \left(\frac{(-1)\,2s}{(s^2 + \alpha^2)^2}\right) = \frac{2\alpha s}{(s^2 + \alpha^2)^2}$$

$\Rightarrow \quad L(t \sin \alpha t) = \dfrac{2\alpha s}{(s^2+\alpha^2)^2}.$

Example 2 : Show that $\int_0^\infty e^{-3t} t \cos t \, dt = \dfrac{2}{25}$

Sol. We know $L(\cos t) = \dfrac{s}{s^2+1} = F(s)$

$\therefore \quad L(t \cos t) = (-1)\dfrac{d}{ds}\left(\dfrac{s}{s^2+1}\right)$

$= (-1)\dfrac{1(s^2+1) - s(2s)}{(s^2+1)^2} = \dfrac{s^2-1}{(s^2+1)^2}$

$\Rightarrow \quad \int_0^\infty e^{-st} t \cos t \, dt = \dfrac{s^2-1}{(s^2+1)^2}$

Put s = 3

$\Rightarrow \quad \int_0^\infty e^{-3t} t \cos t \, dt = \dfrac{9-1}{(9+1)^2} = \dfrac{8}{100} = \dfrac{2}{25}.$

Example 3 : Find Laplace transform for $t \geq 0$, of $(t^2 - 3t + 2) \sin 3t$

Sol. Here $f(t) = (t^2 - 3t + 2) \sin 3t$

We know $L(\sin 3t) = \dfrac{3}{s^2+9}$

Now by Result 4.2 $L(t \sin 3t) = (-1)\dfrac{d}{ds}\left(\dfrac{3}{s^2+9}\right)$

$= (-3)\dfrac{d}{ds}(s^2+9)^{-1}$

$= (-3)(-1)(s^2+9)^{-2}(2s) = \dfrac{6s}{(s^2+9)^2}$

and $\quad L(t^2 \sin 3t) = (-1)^2 \dfrac{d^2}{ds^2}\left(\dfrac{3}{s^2+9}\right)$

$$= \frac{d}{ds}\left(\frac{-6s}{(s^2+9)^2}\right)$$

$$= -6\,\frac{1(s^2+9)^2 - 2\,(s^2+9)\,(2s)\,(s)}{(s^2+9)^4}$$

$$= -\frac{6(s^2+9)\,\{s^2+9-4s^2\}}{(s^2+9)\,(s^2+9)^3}$$

$$= \frac{-6\,(-3s^2+9)}{(s^2+9)^3} = \frac{18\,(s^2-3)}{(s^2+9)^3}$$

$\therefore$ L $((t^2 - 3t + 2) \sin 3t)$ = L $(t^2 \sin 3t)$ – 3L $(t \sin 3t)$ + 2L $(\sin 3t)$

$$= \frac{18(s^2-3)}{(s^2+9)^3} - 3\,\frac{6s}{(s^2+9)^2} + 2\,\frac{3}{(s^2+9)}$$

$$= \frac{6\,\{3s^2-9-3s\,(s^2+9)+(s^2+9)^2\}}{(s^2+9)^3}$$

$$= \frac{6(s^4-3s^3+21s^2-27s+72)}{(s^2+9)^3}.$$

Example 4 : Find Laplace Transform of

$$\frac{e^{-\alpha t}\sin \beta t}{t}$$

Sol. We know $L\,(\sin \beta t) = \dfrac{\beta}{s^2+\beta^2}$

By first shifting Theorem.

$$L\,(e^{-\alpha t}\sin \beta t) = \frac{\beta}{(s+\alpha)^2+\beta^2} = F(s)$$

$\therefore$ By Result 4.3

$$L\left(\frac{e^{-\alpha t}\sin \beta t}{t}\right) = \int_s^\infty \frac{\beta}{(s+\alpha)^2+\beta^2} = F(s)$$

$$= \underset{u\to\infty}{L\,im}\int_s^u \frac{\beta}{(s+\alpha)^2+\beta^2}\,ds$$

$$= \underset{u\to\infty}{L\,im}\ \beta\,.\,\frac{1}{\beta}\tan^{-1}\left(\frac{s+\alpha}{\beta}\right)\Bigg]_s^u$$

$$= \underset{u\to\infty}{L\,im}\left(\tan^{-1}\frac{u+\alpha}{\beta}-\tan^{-1}\frac{s+\alpha}{\beta}\right)$$

$$= \underset{u\to\infty}{L\,im}\tan^{-1}\frac{u+\alpha}{\beta}-\tan^{-1}\frac{s+\alpha}{\beta}$$

$$= \tan^{-1}\infty-\tan^{-1}\frac{s+\alpha}{\beta}=\frac{\pi}{2}-\tan^{-1}\frac{s+\alpha}{\beta}$$

$$= \cot^{-1}\frac{s+\alpha}{\beta}.$$

Example 5 : Evaluate $\int_0^\infty t^3e^{-1}\sin t\,dt$.

Sol. $\int_0^\infty t^3e^{-1}\sin t\,dt=\int_0^\infty e^{-t}(t^3\sin t)\,dt$, where $s=1$

$= L\ (t^3 \sin\ t)$ (By def. of Laplace Transform)

$$= (-1)^3\frac{d^3}{ds^3}(L(\sin t))$$

$$= -\frac{d^3}{ds^3}\left(\frac{1}{s^2+1}\right)=-\frac{d^2}{ds^2}\left(\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right)$$

$$= -\frac{d^2}{ds^2}\left(-\frac{2s}{(s^2+1)^2}\right)$$

$$= 2\frac{d}{ds}\left(\frac{d}{ds}\left(\frac{s}{(s^2+1)^2}\right)\right)$$

$$= 2\frac{d}{ds}\left(\frac{(s^2+1)^2 - s\,(2(s^2+1))\,(2s)}{(s^2+1)^4}\right)$$

$$= 2\frac{d}{ds}\left(\frac{s^2+1-4s^2}{(s^2+1)^3}\right) = 2\frac{d}{ds}\left(\frac{1-3s^2}{(s^2+1)^3}\right)$$

$$= 2\,\frac{(-6s)\,(s^2+1)^3 - 3(s^2+1)^2(2s)\,(1-3s^2)}{(s^2+1)^6}$$

$$= \frac{-2\,(6s)\,(s^2+1)^2\ \{(s^2+1)+1-3s^2\}}{(s^2+1)^6}$$

$$= \frac{-12s\,(2-2s^2)}{(s^2+1)^4} = \frac{24s\,(s^2-1)}{(s^2+1)^4}$$

Put s = 1 (From above)

$$\Rightarrow \quad \int_0^\infty t^3 e^{-1} \sin t\, dt = \frac{24(1)\,(1-1)}{(1+1)^4} = 0.$$

VI. Self Check Exercise

1. Show that $\int_0^\infty e^{-3t} t \sin t\, dt = \frac{3}{50}$.

2. Evaluate L $(\sin^2 \alpha t \cos \alpha t)$ for $t \geq 0$.

3. Find L (f(t)) where $f(t) = \begin{cases} \cos t, & 0 < 1 < \pi \\ 0, & \pi < 1 < 2\pi \end{cases}$ if f(t) is periodic with period 2π.

4. Evaluate $L\left(\frac{\cos 4t - \cos 5t}{t}\right)$ for $t > 0$.

5. Show that $\int_0^\infty te^{-2t} \cos t\, dt = \frac{3}{25}$.

Suggested Readings :

1. A.R. Vasushtha & Dr. R.K. Gupta, Integral Transforms by Krishna Prakashan Media Pvt. Ltd. Meerut.

LESSON NO. 1.4 **Author : Dr. Chanchal**

INVERSE LAPLACE TRANSFORMS

Structure :

I. Introduction
II. Properties of Inverse Laplace Transforms
III. Inverse Laplace Transforms of Derivatives and Integrals
IV. Convolution Theorem
V. Some Important Examples
VI. Self Check Exercise
VII. Suggested Readings

I. Introduction

If $L(f(t)) = F(s)$ for $t \geq 0$ i.e. Laplace transform of f(t) exists then the function f(t) is called the **Inverse Laplace Transform** of F(s) and is written as $L^{-1}(F(s)) = f(t)$.
i.e. $L^{-1}(F(s))$ is the function whose Laplace Transform is F(s).

For Example : $L(\sin 5t) = \dfrac{5}{s^2 + 25}$

$$\Rightarrow \quad L^{-1}\left(\frac{5}{s^2+25}\right) = \sin 5t$$

Further, we have an important result concerning the existence and uniqueness of inverse Laplace transforms :

Result 1 : If f(t) is piecewise continuous function on each interval $[0, \alpha]$ and is of exponential order for $t > \alpha$.

Also if $L(f(t)) = F(s)$; then prove that the **Inverse Laplace Transform** f (t) is unique.

i.e. If $L^{-1}(F(s)) = f_1(t)$ and $L^{-1}F(s)) = f_2(t)$
Then $f_1(t) = f_2(t)$ for all $t \geq 0$.

II. Properties of Inverse Laplace Transforms

Theorem 1 (Linearity Property) :

If L(g(t)) = G(s) and L(h(t) = H(s) for any two functions g(t), h(t) ($t \geq 0$) and α, β are any constants then prove

$L^{-1}\{\alpha G(s) + \beta H(s)\} = \alpha L^{-1}(G(s)) + \beta L^{-1}(H(s))$

Proof : Given L(g(t)) = G9s) and L(h(t)) = H(s) for $t \geq 0$

$\therefore \quad \alpha G(s) + \beta H(s) = \alpha L(g(t)) + \beta L(h(t))$

$= L(\alpha g(t) + \beta h(t)$

(Using Theorem 3 i.e. Linear property of Laplace Transform)

By def. of Inverse Laplace Transform, we get

$L^{-1}(\alpha G(s) + \beta H(s)) = \alpha g(t) + \beta h(t) = \alpha L^{-1}(G(s)) + \beta L^{-1}(H(s))$

($\because$ Given implies that $g(t) = L^{-1}(G(s))$ and $h(t) = L^{-1}(H(s))$)

Hence the result.

Theorem 2 : (First Shifting Theorem of Inverse Laplace Transform)

if $L^{-1}(F(s)) = f(t)$ for $t \geq 0$

Prove $L^{-1}(F(s-\alpha) = e^{\alpha t} f(t) = e^{\alpha t} L^{-1}(F(s))$

Proof : Given $L^{-1}(F(s)) = f(t)$

$$\Rightarrow \quad F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$$

$$F(s-\alpha) = \int_0^\infty e^{-(s-\alpha)t} f(t)\, dt$$

(By def. of Laplace Transform)

$$= \int_0^\infty e^{-st}\{e^{\alpha t} f(t)\}\, dt$$

$$= L(e^{\alpha t} f(t))$$

so by def. of Inverse Laplace Transform we get

$$L^{-1}(F(s-\alpha)) = e^{\alpha t} f(t)$$

$$= e^{\alpha t} L^{-1}(F(s))$$

Hence the result

Note : Changing α to $-\alpha$ in above result.

we get $L^{-1}(F(s+\alpha)) = e^{-\alpha t} L^{-1}(F(s))$.

Theorem 3 : (Second Shifting Theorem of Inverse Laplace Transform)

If $L^{-1}(F(s)) = f(t)$ for $t \geq 0$

then prove $L^{-1}(e^{-\alpha s} F(s)) = g(t)$ where $g(t) = \begin{cases} f(t-\alpha), & t > \alpha \\ 0, & t < \alpha \end{cases}$.

Proof : We have $L^{-1}(F(s)) = f(t)$

$$\Rightarrow \quad F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$$

and given function is $g(t) = \begin{cases} f(t-\alpha), & t > \alpha \\ 0, & t < \alpha \end{cases}$

$$\therefore \quad L(g(t)) = \int_0^\infty e^{-st} g(t)\, dt$$

$$= \int_0^\alpha e^{-st} g(t)\, dt + \int_\alpha^\infty e^{-st} g(t)\, dt$$

$$= \int_0^\alpha e^{-st} (0)\, dt + \int_\alpha^\infty e^{-st} f(t-\alpha)\, dt$$

(By def. of g(t))

$$= 0 + \int_\alpha^\infty e^{-st} f(t-\alpha)\, dt$$

$$= \int_{V=0}^{V=\infty} e^{-s(V+\alpha)} f(V)\, dV \qquad [\text{Put } t - \alpha = V \Rightarrow dt = dV]$$

$$= e^{-\alpha s} \int_0^\infty e^{-sV} f(V)\, dt$$

$$= e^{-\alpha s} \int_0^\infty e^{-st} f(t)\, dt \qquad (\text{By changing variable V by t})$$

$$= e^{-\alpha s} F(s)$$

$$\Rightarrow \quad g(t) = L^{-1}(e^{-\alpha s} F(s))$$

or $$L^{-1}(e^{-\alpha s}F(s)) = g(t) = \begin{cases} f(t-\alpha), & f > \alpha \\ 0, & t < \alpha \end{cases}$$

Another Form of above Theorem

If $L^{-1}(F(s)) = f(t)$ for $t \geq 0$ and $\alpha \geq 0$ real.

Prove $L^{-1}(e^{-\alpha s}F(s)) = f(t-\alpha)\, h\,(t-\alpha)$

where $h(t-\alpha) = \begin{cases} 1, & t > \alpha \\ 0, & t < \alpha \end{cases}$ or $h(t) = \begin{cases} t, & t > 0 \\ 0, & t < 0 \end{cases}$

is known as Unit Step Function or Heavyside's unit step function.

Theorem 4 : (Change of Scale Property)

If $L^{-1}(F(s)) = f(t)$ for $t \geq 0$

Prove (i) $L^{-1}\left(F\left(\frac{s}{\alpha}\right)\right) = \alpha f\,(\alpha t)$

(ii) $L^{-1}(F(\alpha\, s)) = \frac{1}{\alpha} f\left(\frac{t}{\alpha}\right)$ where $\alpha > 0$.

Proof : We have $L^{-1}(F(s)) = f(t)$ for $t \geq 0$

$$\Rightarrow \quad F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$$

(i) $\therefore$ $$L\,(f(\alpha t)) = \int_0^\infty e^{-st} f(\alpha t)\, dt$$ Put $\alpha t = V \Rightarrow \alpha\, dt = dV$

$$= \int_{V=0}^{V=\infty} e^{-s\frac{V}{\alpha}} f(V) \frac{dV}{\alpha}$$

$$= \frac{1}{\alpha}\int_0^\infty e^{-\left(\frac{s}{\alpha}\right)V} dV$$

$$= \frac{1}{\alpha}\int_0^\infty e^{-\frac{s}{\alpha}t}\, f(t)\, dt$$

$$= \frac{1}{\alpha} F\left(\frac{s}{\alpha}\right) \qquad \text{(Change the variable V by t)}$$

$$\Rightarrow \quad f(\alpha\, t) = L^{-1}\left(\frac{1}{\alpha} F\left(\frac{s}{\alpha}\right)\right)$$

$$= \frac{1}{\alpha} L^{-1}\left(F\left(\frac{s}{\alpha}\right)\right)$$

$$\Rightarrow \quad L^{-1}\left(F\left(\frac{s}{\alpha}\right)\right) = \alpha f(\alpha t).$$

Now, the reader can easily prove the (ii) part.

III. Inverse Laplace Transforms of Derivatives and Integrals

For the concerned topic, we have the following results :

Result 2 : Inverse Laplace Transform of Derivatives

(i) If $L^{-1}(F(s)) = f(t)$ for $t \geq 0$

Prove $L^{-1}(F'(s)) = -t\, f(t)$.

(ii) Generalisation : If $L^{-1}(F(s)) = f(t)$, for $t \geq 0$

Prove $L^{-1}(F^n(s)) = (-1)^n t^n f(t)$.

Result 3 : (i) If $L^{-1}(F(s)) = f(t)$ for $t \geq 0$ and $f(0) = 0$.

Then prove $L^{-1}(s\, F(s)) = f'(t)$

(ii) Generalisation : If $L^{-1}(F(s)) = f(t)$ for $t \geq 0$

and $f(0) = f'(0) = f''(0) = \ldots\ldots. = f^{n-1}(0) = 0$.

Then prove $L^{-1}(s^n F(s)) = f^n(t) = \dfrac{d^n f(t)}{dt^n}$.

Result 4 : Inverse Laplace Transform of Integrals

(i) If $L^{-1}(F(s)) = f(t)$ for $t \geq 0$

Then prove $L^{-1}\left(\int_s^\infty F(s)\, ds\right) = \dfrac{f(t)}{t}$

(ii) If $L^{-1}(F(s)) = f(t)$ for $t \geq 0$

Then prove $L^{-1}\left(\dfrac{F(s)}{s}\right) = \int_0^t f(t)\, dt$.

IV. Convolution Theorem

To find Inverse Laplace Transform of the product of two functions, whose inverse Laplace Transforms are known or can be easily evaluated, the **Convolution Theorem** will help us, which is stated as

Theorem 5 : If $L^{-1}(F(s)) = f(t)$ and $L^{-1}(G(s)) = g(t)$ then prove

$$L^{-1}(F(s)\,G(s)) = \int_0^t f(z)\,g(t-z)\,dz \text{ for } t \geq 0$$

Note : The integral on R.H.S is known as convolution of f and g and denoted as f = g

Proof : To prove $L^{-1}(F(s)\,G(s)) = \int_0^t f(z)\,g(t-z)\,dz = f * g$

We have to show $L(f * g) = F(s)\,G(s)$.
By def. of Laplace transform,

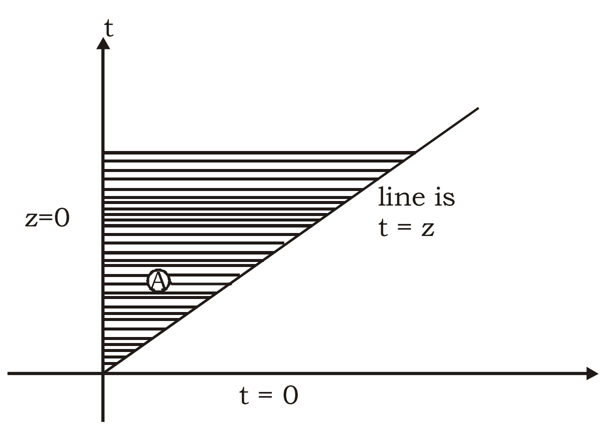
$$L(f * g) = \int_0^\infty e^{-st}(f * g)(t)\,dt$$

$$= \int_0^\infty e^{-st}\left(\int_0^t f(z)\,g(t-z)\,dz\right)dt$$

$$= \int_0^\infty \int_0^1 e^{-st} f(z)\,g(t-z)\,dz\,dt$$

$$= \iint_A e^{-st} f(z)\,g(t-z)\,dz\,dt$$

where A is the shaded region



Put z = q and t – z = r

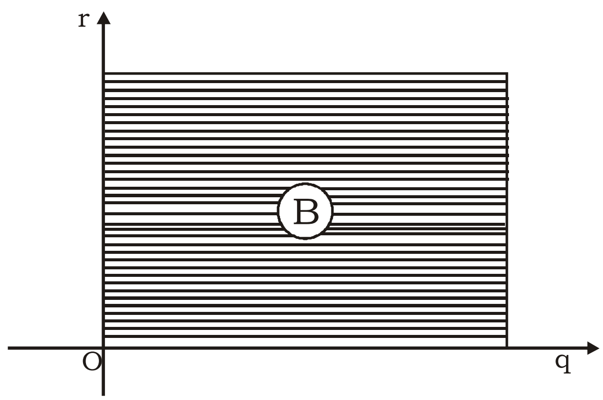
$\Rightarrow$ z = q and t = q + r

$$\therefore \quad \frac{\partial(t,z)}{\partial(q,r)} = \begin{vmatrix} \frac{\partial z}{\partial q} & \frac{\partial z}{\partial r} \\ \frac{\partial t}{\partial q} & \frac{\partial t}{\partial r} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

Hence dz dt = (1) dq dr = dq dr

So, the new variables transform region A into region

B = {(q, r) | q ≥ 0, r ≥ 0}



$$\therefore \quad L(f * g) = \iint_B e^{-s(q+r)} f(q)\, g(r)\, dq\, dr$$

$$= \int_0^\infty \int_0^\infty e^{-sq} e^{-sr} f(q)\, g(r)\, dq\, dr$$

$$= \left(\int_0^\infty e^{-sq} f(q)\, dq \right) \left(\int_0^\infty e^{-sr} g(r)\, dr \right)$$

Change variable q, r by t (By Property of double integrals)

$$= \left(\int_0^\infty e^{-st} f(t)\, dt \right) \left(\int_0^\infty e^{-st} g(t)\, dt \right)$$

= L (f(t) L (g(t)) = F (s) G (s).

($\because$ L^{-1} (F(s)) = f(t) and L^{-1} (G (s)) = g (t))

Hence the Result.

Remarks : 1. f*g = g*f $\left(L^{-1}(F(s)\,G(s) = \int_0^t f(z)g(t-z)\,dz = \int_0^t g(z)f(t-z)\,dz\right)$

2. f* (g+h) = f*g + f*h

V. Some Important Examples

Example 1 : Show that

(i) $L^{-1}\left(\dfrac{s}{(s^2+\alpha^2)^2}\right) = \dfrac{1}{2\alpha}\,t\sin\alpha\,t$

(ii) $L^{-1}\left(\dfrac{s^2-\alpha^2}{(s^2+\alpha^2)^2}\right) = t\cos\alpha t$

Sol. We know $L\,(e^{i\alpha t}) = \dfrac{1}{s-i\,\alpha}$

$\Rightarrow$ $L\,(te^{i\alpha t}) = (-1)\dfrac{d}{ds}\left(\dfrac{1}{s-i\,\alpha}\right)$

$= \dfrac{1}{(s-i\,\alpha)^2}$

$\Rightarrow$ $L\,(t\,(\cos\alpha t + i\sin\alpha t) = \dfrac{(s+i\alpha)^2}{((s-i\alpha)\,(s+i\alpha))^2}$

$\Rightarrow$ $L\,(t\cos\alpha t) + iL\,(t\sin\alpha t) = \dfrac{(s^2-\alpha^2)+(2i\,\alpha)s}{(s^2+\alpha^2)^2}$

weget $L\,(t\cos\alpha t) = \dfrac{s^2-\alpha^2}{(s^2+\alpha^2)^2}$ and $L(t\sin\alpha\,t) = \dfrac{2\alpha s}{(s^2+\alpha^2)^2}$

$\therefore$ By def. of Inverse Laplace Transform

we get $t\cos\alpha t = L^{-1}\left(\dfrac{s^2-\alpha^2}{(s^2+\alpha^2)^2}\right)$

and $$t \sin \alpha t = L^{-1}\left(\frac{2\alpha s}{(s^2+\alpha^2)^2}\right) = 2\,\alpha\, L^{-1}\left(\frac{s}{(s^2+\alpha^2)^2}\right)$$

$$\Rightarrow \quad L^{-1}\left(\frac{s^2-\alpha^2}{(s^2+\alpha^2)^2}\right) = t\cos\alpha t$$

$$\Rightarrow \quad L^{-1}\left(\frac{s}{(s^2+\alpha^2)^2}\right) = \frac{t\sin\alpha t}{2\alpha}.$$

Example 2 : Find Inverse Laplace Transform of following functions

(i) $\dfrac{s^2}{(s-\alpha)^3}$ (ii) $L^{-1}\left(\dfrac{8s+3}{s^2e^{8s}}\right)$

Sol. (i) $$L^{-1}\frac{s^2}{(s-\alpha)^3} = L^{-1}\left(\frac{((s-\alpha)+\alpha)^2}{(s-\alpha)^3}\right)$$

$$= e^{\alpha t}L^{-1}\left(\frac{(s+\alpha)^2}{s^3}\right)$$

$$= e^{\alpha t}\,L^{-1}\left(\frac{s^2+2\alpha s+\alpha^2}{s^3}\right)$$

$$= e^{\alpha t}\,L^{-1}\left(\frac{1}{s}+2\alpha\,\frac{1}{s^2}+\alpha^2\,\frac{1}{s^3}\right)$$

$$= e^{\alpha t}\left(L^{-1}\left(\frac{1}{s}\right)+2\,\alpha L^{-1}\left(\frac{1}{s^2}\right)+\alpha^2 L^{-1}\left(\frac{1}{s^3}\right)\right)$$

$$= e^{\alpha t}\left(1+2\alpha t+\alpha^2\,\frac{t^2}{2}\right)$$

(ii) Here $$L^{-1}\left(\frac{8s+3}{s^2}\right) = L^{-1}\left(\frac{8}{s}+\frac{3}{s^2}\right)$$

$$= 8L^{-1}\left(\frac{1}{s}\right)+3L^{-1}\left(\frac{1}{s^2}\right)$$

= 8 (1) + 3t = 3t + 8 = f(t), say

∴ Using Second shifting Theorem.

Example 3 : Evaluate : $L^{-1}\left(\dfrac{3s}{9s^2+27}\right)$

Sol. Take $F(s) = \dfrac{s}{s^2+27}$

$\Rightarrow$ $L^{-1}\left(\dfrac{s}{s^2+27}\right) = \cos(3\sqrt{3}\,t) = f(t)$ (say)

∴ By change of scale Property

$$L^{-1}\left(\frac{3s}{9s^2+27}\right) = L^{-1}\left(\frac{3s}{(3s)^2+27}\right)$$

$$= L^{-1}(F(3s)) = \frac{1}{3} f\left(\frac{t}{3}\right) = \frac{1}{3}\cos\left(3\sqrt{3}\,\frac{t}{3}\right)$$

$$= \frac{1}{3}\cos\left(\sqrt{3}\,t\right).$$

Example 4 : Evaluate : (i) $L^{-1}\left(4\tan^{-1}\dfrac{2}{s}\right)$ (ii) $L^{-1}\left(\dfrac{s}{(s^2+\alpha^2)^2}\right)$

Sol. (i) Let $F(s) = 4\tan^{-1}\dfrac{2}{s}$

$$\Rightarrow \quad F'(s) = \frac{4\dfrac{d}{ds}\left(\dfrac{2}{s}\right)}{1+\left(\dfrac{2}{s}\right)^2} = \frac{4s^2\left(-\dfrac{2}{s^2}\right)}{s^2+4}$$

$$= \frac{-8}{s^2+4}$$

$$\therefore \quad L^{-1}(F'(s)) = L^{-1}\left(\frac{-8}{s^2+4}\right)$$

$$= (-4)\, L^{-1}\left(\frac{2}{s^2 + 2^2}\right) = -4 \sin 2t.$$

$\Rightarrow$ $-tf(t) = -4 \sin 2t$ (Using Result 2)

$\Rightarrow$ $f(t) = \dfrac{4 \sin 2t}{t}$

$\Rightarrow$ $L^{-1}(F(s)) = \dfrac{4 \sin 2t}{t}$.

(ii) Firstly find a function whose derivative is $\dfrac{s}{(s^2 + \alpha^2)^2}$

Here $\displaystyle\int \frac{s}{(s^2 + \alpha^2)^2} = \frac{1}{2}\int (s^2 - \alpha^2)^{-2}(2\,s)\,ds$

$$= \frac{1}{2}\,\frac{(s^2 + \alpha^2)^{-2+1}}{-2+1}$$

$$= -\frac{1}{2(s^2 + \alpha^2)} = F(s) \qquad \text{(Say)}$$

$\therefore$ $\dfrac{d}{ds}\left(-\dfrac{1}{2\,(s^2 + \alpha^2)}\right) = \dfrac{s}{(s^2 + \alpha^2)^2}$

Now $L^{-1}(F(s)) = L^{-1}\left(-\dfrac{1}{2\,(s^2 + \alpha^2)}\right)$

$$= -\frac{1}{2\alpha}\, L^{-1}\left(\frac{\alpha}{s^2 + \alpha^2}\right)$$

$$= -\frac{1}{2\alpha} \sin \alpha t$$

$= f(t)$ (say)

Using $L^{-1}(F'(s)) = -t\, f(t)$ (By Result 2)

we get $L^{-1}\left(\dfrac{s}{(s^2 + \alpha^2)^2}\right) = -t\left(-\dfrac{1}{2\alpha} \sin \alpha t\right)$

$$= \frac{1}{2\alpha} t \sin \alpha t.$$

Example 5 : Evaluate : $L^{-1}\left(2\log \frac{s^2+\beta^2}{s^2+\alpha^2}\right)$

Sol. Let $F(s) = 2 \log \frac{s^2+\beta^2}{s^2+\alpha^2}$

$= 2\ (\log (s^2 + \beta^2)) - \log (s^2 + \alpha^2))$

$\Rightarrow \quad F'(s) = \frac{2}{s^2+\beta^2}(2s) - 2 \cdot \frac{1}{s^2+\alpha^2}(2s)$

$\Rightarrow \quad F'(s) = 4\left(\frac{s}{s^2+\beta^2} - \frac{s}{s^2+\alpha^2}\right)$

$\Rightarrow \quad L^{-1}(F'(s)) = 4\left(L^{-1}\left(\frac{s}{s^2+\beta^2}\right) - L^{-1}\left(\frac{s}{s^2+\alpha^2}\right)\right)$

$= 4\ (\cos \beta t - \cos \alpha t)$

$\Rightarrow \quad -t\ f(t) = 4\ (\cos \beta t - \cos \alpha t)$ (Using Result 2)

$\Rightarrow \quad f(t) = \frac{4\ (\cos \alpha t - \cos \beta t)}{t}.$

Example 6 : Find inverse Laplace transform of $\frac{1}{s^3(s+1)}$

Sol. We know $L^{-1}\left(\frac{1}{s+1}\right) = e^{-t}$

$\therefore$ By using property of Inverse Laplace Transform repeatedly Result 4

We have $L^{-1}\left(\frac{1}{s^3(s+1)}\right) = \int_0^t\left(\int_0^t\left(\int_0^t e^{-t}dt\right)dt\right)dt$

$$= \int_0^t\left(\int_0^t\left(\frac{e^{-t}}{-1}\right)_0^t dt\right)dt = \int_0^t\left(\int_0^t -(e^{-t}-1)\,dt\right)dt$$

$$= \int_0^t \left(e^{-1} + t\right)_0^t dt = \int_0^t \left\{(e^{-t} + t) - (e^0 + 0)\right\} dt$$

$$= \int_0^t (e^{-t} + t - 1)\, dt = \left(\frac{e^{-t}}{-1} + \frac{t^2}{2} - t\right)_0^t$$

$$= \left(-e^{-t} + \frac{t^2}{2} - t\right) - (-e^0 + 0 - 0) = -e^{-t} + \frac{t^2}{2} - t + 1 \cdot$$

Example 7 : Find inverse Laplace Transform of following functions $\frac{1}{(s+\alpha)(s+\beta)}$

Sol. Let $\frac{1}{s+\alpha} = F(s)$ and $\frac{1}{s+\beta} = G(s)$

So that given function = F (s) G (s)

We know $L^{-1}(F(s)) = L^{-1}\left(\frac{1}{s+\alpha}\right) = e^{-\alpha t} = f(t)$ say

and $L^{-1}(G(s)) = L^{-1}\left(\frac{1}{s+\beta}\right) = e^{-\beta t} = g(t)$ say

Now using convolution Theorem

$$L^{-1}\left(\frac{1}{(s+\alpha)(s+\beta)}\right) = L^{-1}\,(F(s)\,G(s))$$

$$= \int_0^t f(z)\, g(t-z)\, dz = \int_0^t e^{-\alpha z} e^{-\beta(t-z)} dz$$

$$= \int_0^t e^{-\beta t + (\beta-\alpha) z}\, dz = e^{-\beta t} \int_0^t e^{(\beta-\alpha) z}\, dz$$

$$= e^{-\beta t} \left(\frac{e^{(\beta-\alpha) z}}{\beta - \alpha}\right)_0^t$$

$$= \frac{e^{-\beta}(e^{(\beta-\alpha)} - e^{0})}{\beta - \alpha}$$

$$= \frac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha}.$$

Example 8 : Apply convolution Theorem to show that

$$\int_0^t z e^{-t-8z} dz = \frac{e^{-t}}{64}(1 - (1 + 8t)\, e^{-8t})$$

Sol. Given integral can be written as

$$\int_0^t e^{-9z-t+z} z\, dz = \int_0^t e^{-9z} z e^{(-t+z)} dz = \int_0^t f(z)\, g(t-z)\, dz$$

$= L^{-1}(F(s)\ G(s))$ where $f(z) = ze^{-9z}$
$g(z) = e^{-z}$

$$= L^{-1}\left(\frac{1}{(s+9)^2}\frac{1}{s+1}\right)$$ For this step reason is

$$= L^{-1}\left(\frac{1}{(s+1)(s+9)^2}\right)$$

$$= e^{-t} L^{-1}\left(\frac{1}{s(s+8)^2}\right)$$

$$\left[\because L^{-1}\left(\frac{1}{(s+8)^2}\right) = e^{-8t} L^{-1}\left(\frac{1}{s^2}\right) = e^{-8t}\, t\right.$$

$$= e^{-t}\left(-\frac{1}{8}(te^{-8t} - 0) + \frac{1}{8}\left(\frac{e^{-8t}}{-8}\right)_0^t\right)$$

$$= e^{-t}\left(-\frac{1}{8}te^{-8t} - \frac{1}{6}\left(e^{-8t} - 1\right)\right)$$

$$= \frac{e^{-t}}{64}(-(8t+1)e^{-8t}+1) = \frac{e^{-t}}{64}(1-(1+8t)e^{-8t})$$

Hence the result.

VI. Self Check Exercise

1. Evaluate (i) $L^{-1}\left(\frac{1}{s}\sin\frac{1}{s}\right)$ (ii) $L^{-1}\left(\frac{2s}{s^4+s^2+1}\right)$

2. Evaluate (i) $L^{-1}\left\{\frac{1}{(s-1)^5\,(s+2)}\right\}$ (ii) $L^{-1}\left\{\frac{30}{\left(\frac{s}{50}\right)^2-50}\right\}$

3. Evaluate : (i) $L^{-1}\left(\frac{s^2-\pi^2}{(s^2+\pi^2)^2}\right)$ (ii) $\frac{13}{s\,(s^2+169)}$ (iii) $L^{-1}\left(\frac{1}{s\,(\alpha^2 s^2+\beta^2)}\right)$

4. Prove 2 * 2 * 2 * 2 * .... * 2 (k times) = $\frac{2^k t^{k-1}}{\lfloor k-1}$

5. Use Convolution Theorem to show that

$$\beta\,(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}\ (m, n > 0)$$

(i.e. relation between Beta and Gamma Function)

Hence evaluate

$$\int_0^t \sinh z \cosh(t-z)\,dz = \frac{1}{2}\,t \sin ht$$

Suggested Readings :

1. A.R. Vasushtha & Dr. R.K. Gupta, Integral Transforms by Krishna Prakashan Media Pvt. Ltd. Meerut.

LESSON NO. 2.1 **Author : Dr. Chanchal**

SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS USING LAPLACE TRANSFORMS

Structure :

2.1.1 Introduction

From our previous study, we are already familiar with the general methods of finding the solutions of ordinary differential equations. In this lesson, we introduce the methods to solve linear differential equations (with constent and variable coefficients) using Laplace transforms.

2.1.2 Method to Solve Linear Differential Equation with Constant Coefficients

As we know $L(f'(t)) = s\,F(s) - f(0)$

$L(f''(t)) = s^2 F(s) - f(0) - f'(0)$

....

....

$L(f^n(t)) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \ldots\ldots\ldots\ldots f^{n-1}(0)$ where $L(f(t) = F(s)$

If function is taken as $y = f(t)$, then we denote $\bar{y}$ for $F(s)$ i.e., $F(s) = \bar{y}$

$$\therefore \quad L\left(\frac{dy}{dt}\right) = s\,\bar{y} - y(0)$$

$$L\left(\frac{d^2y}{dt^2}\right) = s^2\,\bar{y} - s\,(y(0) - y'(0) \text{ and so on}$$

Let the initial value problem is given by

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + y = h(t),\ y(0) = \alpha,\ y'(0) = \beta$$

where a, b are constants and h (t) is any function of t

Step (i) Take Laplace transform on both sides, we get

$$a\,L\left(\frac{d^2y}{dt^2}\right) + bL\left(\frac{dy}{dt}\right) + L\,(y) = L\,(h\,(t))$$

$\Rightarrow$ $$a\,(s^2\bar{y} - sy(0) - y'(0)) + b\,(s\,\bar{y} - y(0)) + \bar{y} = H(s)$$

which is known as **subsidiary Equation,** where H (s) = L (h (t)) and L (y (t)) = $\bar{y}$

Step (ii) Put given values of y (0) and y' (0), we have

$(as^2 + bs + 1)\ \bar{y} - a\,\alpha\,s - \beta - b\,\alpha = H\,(s)$

$\Rightarrow$ $$\bar{y} = \frac{(as + b)\,\alpha + \beta + H(s)}{as^2 + bs + 1} \quad \text{(say)}$$

Step (iii) Now take inverse transform on both sides

$$L^{-1}(\bar{y}) = L^{-1}(G(s))$$

$\Rightarrow$ y (t) = L^{-1} G (s)) which gives the required solution.

Example 1 : Solve $\frac{d^2y}{dt^2} + y = 6$ sin 2t given is y (0) = 2, y' (0) = 1.

Sol. The given differential equation is

$$\frac{d^2y}{dt^2} + y = 6\sin 2t \quad \dots (i)$$

Taking Laplace transform on both sides of (i) we get

$$s^2\bar{y} - sy(0) - y'(0) + \bar{y} = 6L\,(\sin 2t) \text{ where } \bar{y} = L\,(y) = 6\left(\frac{2}{s^2 + 4}\right) \quad \dots (ii)$$

But given is y (0) = 2, y' (0) = 1

$\therefore$ Equation (ii) becomes

$$s^2\bar{y} - 2s - 1 + \bar{y} = \frac{12}{s^2 + 4}$$

$\Rightarrow$ $$\bar{y}(s^2 + 1) = 2\,s + 1 + \frac{12}{s^2 + 4}$$

$\Rightarrow \quad \bar{y} = \frac{2s+1}{s^2+1} + \frac{12}{(s^2+1)(s^2+4)}$

$\Rightarrow \quad \bar{y} = \frac{2s}{s^2+1} + \frac{1}{s^2+1} + 12\left(\frac{1}{3(s^2+1)} - \frac{1}{3(s^2+4)}\right)$ (By Partial fractions)

$= \frac{2s}{s^2+1} + \frac{5}{s^2+1} - \frac{4}{s^2+4}$

Taking inverse Laplace transform on both sides, we get

$$L^{-1}(\bar{y}) = 2\,L^{-1}\left(\frac{s}{s^2+1}\right) + 5\,L^{-1}\left(\frac{1}{s^2+1}\right) - 4L^{-1}\left(\frac{1}{s^2+4}\right)$$

$\Rightarrow \quad y = 2\cos t + 5\sin t - \frac{4}{2}\sin 2t.$

Example 2 : $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 5y = \frac{1}{2}t^2$; given is y (0) = y'(0) = 0.

Sol. The given differential equation is

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} * 5y = \frac{1}{2}t^2 \qquad \ldots \text{(i)}$$

Taking Laplace transform on both sides of (i) we get

$$s^2\bar{y} - sy(0) - y'(0) - 4(s\,\bar{y} - y(0)) + 5\bar{y} = \frac{1}{2}\left(\frac{2}{s^3}\right)$$

But given is y (0) = y' (0) = 0

$\therefore \quad s^2\bar{y} - 0 - 0 - 4s\bar{y} + 0 + 5\bar{y} = \frac{1}{s^3} \Rightarrow (s^2 - 4s + 5)\,\bar{y} = \frac{1}{s^3}$

$$\Rightarrow \quad \bar{y} = \frac{1}{s^3(s^2 - 4s + 5)} \qquad \ldots \text{(ii)}$$

Let $\frac{1}{s^3(s^2-4s+5)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D_s + E}{s^2 - 4s + 5}$

Multiply by $s^3(s^2 - 4s + 5)$, we get

$$1 = As^2(s^2 - 4s + 5) + B\,s(s^2 - 4s + 5) + c\,(s^2 - 4s + 5) + (D\,s + E)\,s^3 \qquad \ldots \text{(ii)}$$

Put s = 0

$\therefore \qquad 1 = c\,(0 - 0 + 5) \qquad \Rightarrow \boxed{C = \frac{1}{5}}$

Equating coefficient of s^4, s^3, s^2, s on both sides of (ii) we get

$0 = A + D$, $0 = -4A + B + E$, $0 = 5A - 4B + C$ and $0 = 5B - 4c$

Solving $\boxed{B = \frac{4}{25}}$, $\boxed{A = \frac{11}{125}}$, $\boxed{D = -\frac{11}{125}}$ and $\boxed{E = \frac{24}{125}}$

$\therefore$ (ii) becomes

$$\bar{y} = \frac{11}{125}\left(\frac{1}{s}\right) + \frac{4}{25}\left(\frac{1}{s^2}\right) + \frac{1}{5}\left(\frac{1}{s^3}\right) + \frac{\frac{-11s}{125} + \frac{24}{125}}{s^2 - 4s + 5}$$

$$= \frac{11}{125}\left(\frac{1}{s}\right) + \frac{4}{25}\left(\frac{1}{s^2}\right) + \frac{1}{5}\left(\frac{1}{s^3}\right) - \frac{11}{125}\left(\frac{s}{(s-2)^2 + 1}\right) + \frac{24}{125}\left(\frac{1}{(s-2)^2 + 5}\right)$$

Taking inverse Laplace transform on both sides

$$L^{-1}(\bar{y}) = \frac{11}{125}L^{-1}\left(\frac{1}{s}\right) + \frac{4}{25}L^{-1}\left(\frac{1}{s^2}\right) + \frac{1}{5}L^{-1}\left(\frac{1}{s^3}\right) - \frac{11}{125}L^{-1}\left(\frac{s}{(s-2)^2 + 1}\right)$$

$$+ \frac{24}{125}L^{-1}\left(\frac{1}{(s-2)^2 + 1}\right)$$

$$\Rightarrow \qquad y = \frac{11}{125}(1) + \frac{4t}{25} + \frac{1}{5}\left(\frac{t^2}{2}\right) - \frac{11}{125}e^{2t}L^{-1}\left(\frac{s+2}{s^2+1}\right) + \frac{24}{125}e^{2t}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$\Rightarrow \qquad y = \frac{11}{125} + \frac{4t}{25} + \frac{t^2}{10} - \frac{11}{125}e2t(\cos t + 2\sin t) + \frac{24}{125}e^{2t}\sin t$$

$$= \frac{1}{125}\left(11 + 20t + \frac{25}{2}t^2 + e^{2t}(-11\cos t - 22\sin t + 24\sin t)\right)$$

$$y = \frac{1}{125}\left(11 + 20t + \frac{25}{2}t^2 + e^{2t}(2\sin t - 11\cos t)\right)$$

which is required solution of (i).

2.1.3 Method to Solve Linear Differential Equation with Variable Coefficients

Now, we explain the method with the help of following example :

Example 3 : Solve $\frac{d^2y}{dt^2}+t\frac{dy}{dy}-y=0$ when y(0) = 0, y'(0) = 2.

Sol. The given differential equation is

$$\frac{d^2y}{dt^2}+t\frac{dy}{dy}-y=0 \qquad \dots (i)$$

Taking Laplace transform on both sides we get

$$-\frac{d}{ds}(s^2\bar{y}-s\,y(0)-y'(0))-\frac{d}{ds}(s\bar{y}-y'(0))-\bar{y}=0$$

But given is y(0) = 0, y'(0) = 2

$$\therefore \qquad -\frac{d}{ds}(s^2\bar{y}-0-2)-\frac{d}{ds}(s\,\bar{y}-2)-\bar{y}=0$$

$$\Rightarrow \qquad -\left(2s\bar{y}+s^2\frac{d\bar{y}}{ds}\right)-\left(\bar{y}+s\frac{d\bar{y}}{ds}\right)-\bar{y}=0$$

$$\Rightarrow \qquad -(s^2+s)\frac{d\bar{y}}{ds}=+(2s+2)\,\bar{y} \Rightarrow -s\,(s+1)\frac{d\bar{y}}{ds}=2\,(s+1)\,\bar{y}$$

$$\Rightarrow \qquad -s\frac{d\bar{y}}{ds}=2\bar{y}$$

Separating variables, we get

$$\frac{d\bar{y}}{y}=-2\frac{ds}{s}$$

Integrating

$$\log\bar{y}=-2\log s+\log c=\log s^{-2}+\log c=\log\frac{c}{s^2}$$

$$\Rightarrow \qquad \bar{y}=\frac{c}{s^2}$$

Taking inverse Laplace transform, so

$$L^{-1}(\bar{y})=cL^{-1}\left(\frac{1}{s^2}\right)\Rightarrow y=ct$$

But given is y = 0 when t = 0 $\therefore$ 0 = c (0) = 0

which is time for every arbitrary constant c.

$\therefore$ solution is y = ct where c is any scalar.

2.1.4 Method to Solve Simultaneous Ordinary Differential Equations

The method is explained through the following example :

Example 4 : Solve $\frac{dx}{dt} = ax + by, \frac{dy}{dt} = bx + ay$

subject to conditions x(0) = a, y(0) = b.

Sol. Given differential equations are

$$\frac{dx}{dt} = ax + by \qquad \ldots \text{(i)}$$

and $$\frac{dy}{dt} = bx + ay \qquad \ldots \text{(ii)}$$

Taking Laplace transform on both sides, we get

$$s\bar{x} - x(0) = a\bar{x} + b\bar{y} \text{ and } s\bar{y} - y(0) = b\bar{x} + a\bar{y}$$

But given is x(0) = a, y (0) = b

$\therefore$ Eqs. becomes

$$s\bar{x} - a = a\bar{x} + b\bar{y}$$

and $$s\bar{y} - b = b\bar{x} + a\bar{y}$$

$\Rightarrow$ $$(s - a)\,\bar{x} - b\,\bar{y} = a \qquad \ldots \text{(iii)}$$

and $$b\bar{x} - (s - a)\,\bar{y} = -b \qquad \ldots \text{(iv)}$$

Apply (s–a) (iii) – b (iv), we get

$$(s - a)^2\bar{x} = b\,(s - a)\,\bar{y} - b^2\bar{x} + b\,(s - a)\,\bar{y} = a\,(s - a) + b^2$$

$\Rightarrow$ $$((s - a)^2 - b^2)\,\bar{x} = a\,(s - a) + b^2$$

$\Rightarrow$ $$\bar{x} = \frac{a\,(s - a) + b^2}{(s - a)^2 - b^2}$$

Taking inverse Laplace transform on the sides,

$$L^{-1}(\bar{x}) = L^{-1}\left(\frac{a(s - a) + b^2}{(s - a)^2 - b^2}\right) = e^{at}L^{-1}\left(\frac{as + b^2}{s^2 - b^2}\right)$$

$$\Rightarrow \quad x = e^{at}\left(a\,L^{-1}\left(\frac{s}{s^2 - b^2}\right) + b\,L^{-1}\left(\frac{b}{s^2 - b^2}\right)\right)$$

$x = e^{at}$ (a cosh b t + b sinh b t)

And apply b (iii) – (s – a) (iv), we get

$$b(s-a)\,\bar{x} - b^2\bar{y} - b\,(s-a)\,\bar{x} + (s-a)^2\,\bar{y} = ab + b\,(s-a)$$

$$\Rightarrow \quad ((s-a)^2 - b^2)\,\bar{y} = b(s-a) + ab$$

$$\Rightarrow \quad \bar{y} = \frac{b\,(s-a) + ab}{(s-a)^2 - b^2}$$

Taking inverse Laplace transform on both sides, we get

$$L^{-1}(\bar{y}) = L^{-1}\left(\frac{b\,(s-a) + ab}{(s-a)^2 - b^2}\right) \Rightarrow y = e^{at}L^{-1}\left(\frac{bs + ab}{s^2 - b^2}\right)$$

$$\Rightarrow \quad y = e^{at}\left(b\,L^{-1}\left(\frac{s}{s^2 - b^2}\right) + a\,L^{-1}\left(\frac{b}{s^2 - b^2}\right)\right)$$

$\Rightarrow$ $y = e^{at}$ (b cos bt + a sinh b t)

$\therefore$ $x = e^{at}$ (a cosh bt + b sinh b t)

$y = e^{at}$ (b cosh bt + a sinh b t)

gives solution.

2.1.5 Self Check Exercise

1. Solve $\frac{d^2y}{dt^2} + y = g(t),\ y(0) = y'(0) = 0.$ Hence solve $\frac{d^2y}{dt^2} + y = e^t$.

2. Solve $(D^4 + 2D^2 + 1)\,x = 0$ given x = 0, Dx = 1, $D^2x = 2$, $D^3x = -3$ at t = 0.

3. Solve $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 3x =$ sin t, given is x(0) = x^t (0) = 0.

4. Solve $t\frac{d^2y}{dt^2} + (2t+3)\frac{dy}{dt} + (t+3)\,y = e^{-t}$.

5. Solve $\frac{dx}{dt} = 3y - 2x, \frac{dy}{dt} + y - 4x = 0$ given x (0) = 4, y(0) = 3.

6. Solve $\frac{d^2y}{dt^2} - x = e^{-t}, \frac{dy}{dt} + \frac{dy}{dt} = t;$ when x(0) = y(0) = y'(0) = 0.

LESSON NO. 2.2 **Author : Dr. Chanchal**

SOLUTIONS OF SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

Structure :

2.2.1 Introduction

In this lesson, we learn the mathodology to solve the second order partial differential equations particularly heat, wave and Laplace equations, with the help of Laplace transforms. Before explaining the methods, we introduce some results concerning Laplace transforms :

Results of Laplace Transforms : Let a function u (x, t) is defined for $x \in [a\ b]$ where $t \geq 0$ and it is function of some exponential order as t tends to infinity and is sectionally continuous on eachfinite interval,

then, $$L\left(\frac{\partial u}{\partial t}\right) = s\bar{u}(x,s) - u\,(x,0)$$

and $$L\left(\frac{\partial^2 u}{\partial t^2}\right) = s^2\bar{u}(x,s) - su\,(x,0) - u_t\,(x,0)$$

and $$L\left(\frac{\partial u}{\partial x}\right) = \frac{d\bar{u}}{dx},\ L\left(\frac{\partial^2 u}{\partial x^2}\right) = \frac{d^2\bar{u}}{dx^2}$$

where $$\bar{u}\,(x,s) = L\,(u(x,t)) = \int_0^\infty e^{-st}u(x,\,t)\,dt$$

and $\frac{\partial u}{\partial t} = u_t\ (x, t), \frac{\partial u}{\partial x} = u_x\ (x, t).$

Results : (i) $L^{-1}\left(\frac{e^{-\alpha\sqrt{s}}}{\sqrt{s}}\right) = \frac{1}{\sqrt{\pi\, t}} e^{-\frac{\alpha^2}{4t}}$

(ii) $$L^{-1}\left(\frac{e^{-\alpha\sqrt{s}}}{s}\right) = \text{erfc}\left(\frac{\alpha}{2\sqrt{t}}\right) = 1 - \text{erf}\left(\frac{\alpha}{2\sqrt{t}}\right) = \frac{2}{\sqrt{\pi}} \int_{\frac{\alpha}{2\sqrt{t}}}^{\infty} e^{-x^2} dx$$

(iii) $$L^{-1}\left(e^{-\alpha\sqrt{s}}\right) = \frac{\alpha}{2\sqrt{\pi\, t^3}} e^{\frac{\alpha^2}{4t}}.$$

Results of Fourier Transforms :

(i) $$F_s\left(\frac{\partial^2 V}{\partial x^2}\right) = s\ V\ (0, t) - s^2 \bar{V}_s (s, t)$$

where $\bar{V}_s\ (s, t)$ is Fourier sine transform of V (x, t) w.r.t.x

(ii) $$F_c\left(\frac{\partial^2 V}{\partial x^2}\right) = -\left(\frac{\partial V}{\partial x}\right)_{x=0} - s^2 \bar{V}_c\ (s, t)$$

where $\bar{V}_c (s, t)$ is Fourier cosine transform of V (x, t) w.r.t.x

(iii) $$F\left(\frac{\partial^2 V}{\partial x^2}\right) = -s^2 F(V)$$

where F (V) is Fourier transform of V w.r.t. x

Note : If V (x, t) at x = 0 is given, take Fourier sine transform and if V_x (x, t) at x = 0 is given, take Fourier cosine transform.

Now, we explain all the methods through suitable examples.

2.1.2 Solution of Second Order PDE by Laplace Transforms

Example 1 : Solve $\frac{\partial^2 U}{\partial x^2} = \frac{1}{9}\frac{\partial U}{\partial t}$ where U (0, t) = U (2, t) = 0 and U(x, 0) = 10

$\sin 2\pi x - 20 \sin 5\pi x$.

Sol. The given P.D.E. is

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{9}\frac{\partial U}{\partial t} \qquad \ldots \text{(i)}$$

Taking Laplace transform on both sides,

We get, $\dfrac{d^2\bar{U}}{dx^2} = \dfrac{1}{9}\left(s\bar{U}(x,s) - U(x,0)\right)$

$$\Rightarrow \quad \frac{d^2\bar{U}}{dx^2} - \frac{1}{9}s\,\bar{U}(x,s) = -\frac{1}{9}(10\sin 2\pi x - 20\sin 5\pi x)$$

$$\Rightarrow \quad \left(D^2 - \frac{1}{9}s\right)\bar{U} = \frac{1}{9}(20\sin 5\pi x - 10\sin 2\pi x) \qquad \ldots \text{(ii)}$$

To Solve (ii),

A.E. is $D^2 - \dfrac{s}{9} = 0 \Rightarrow D = \pm\dfrac{\sqrt{s}}{3}$

$\therefore$ C.F. is $\bar{U}_c = Ae^{\frac{\sqrt{s}}{3}} + Be^{\frac{-\sqrt{s}}{3}x}$

where A and B are arbitrary constants.

and P.I. is $\bar{U}_P = \dfrac{1}{D^2 - \frac{s}{9}}\left(\dfrac{20}{9}\sin 5\pi x - \dfrac{10}{9}\sin 2\pi x\right)$

$$\Rightarrow \quad \bar{U}_p = \frac{20}{9}\left(\frac{1}{-25\pi^2 - \frac{s}{9}}\right)\sin 5\pi x - \frac{10}{9}\left(\frac{1}{-4\pi^2 - \frac{s}{9}}\right)\sin 2\pi x$$

$$= -\frac{20\sin 5\pi x}{225\pi^2 + s} + \frac{10\sin 2\pi x}{36\pi^2 + s}$$

Hence general sol of (ii) is

$$\bar{U} + \bar{U}_c + \bar{U}_p = Ae^{\frac{\sqrt{s}\,x}{3}} + Be^{\frac{\sqrt{s}\,x}{3}} - \frac{20\sin 5\pi x}{225\pi^2 + s} + \frac{10\sin 2\pi x}{36\pi^2 + s} \qquad \ldots \text{(iii)}$$

We are given U (0, t) = U (2, t) = 0

Taking Laplace transform, we get

$\bar{U}(0, s) = \bar{U}(2, s) = 0$

Now (ii) $\Rightarrow 0 = A + B$ (Putting x = 0)

and $0 = Ae^{\frac{2\sqrt{s}}{3}} + Be^{\frac{-2\sqrt{s}}{3}}$ (Putting x = 2) $(\because \sin\pi = \sin 10\pi = 0)$

Solving A = B = 0

$\therefore$ (ii) becomes $\bar{U} = \dfrac{-20\sin 5\pi x}{225\pi^2 + s} + \dfrac{10\sin 2\pi x}{36\pi^2 + s}$

Taking inverse Laplace transform on both sides,

We get U (x, t) = $L^{-1}\left(\bar{U}\right)$

$$-20\sin 5\pi\, xL^{-1}\left(\frac{1}{s + 225\pi^2}\right) + 10\sin 2\pi x$$

$$= -20\, e^{-225\pi^2 t}\sin 5\pi x + 10^{-36\pi^2 t}\sin 2\pi x\,.$$

2.2.3 Solution of Heat Conduction Problems by Laplace Transforms

Example 2 : A semi-infinite solid x > 0 have initial temperature zero. At time t = 0, a constant temperature $V_0 > 0$ is applied and maintained at the face x = 0. Find the temperature at any point of solid and at any time t > 0.

Sol. Let V (x, t) be temperature in the solid at any point x and at any time t. The equation which governs the flow of heat in solid is given by

$$\frac{\partial V}{\partial t} - \frac{\lambda\, .\, \partial^2 V}{\partial x^2};\ x > 0,\ t > 0,\ \lambda > 0 \qquad \text{… (i)}$$

Subject to boundary condition

$V(0, t) = V_0$

and initial condition V (x, 0) = 0

and V is finite $\forall$ x, t.

To solve (i), Take Laplace transform on both sides of (i), we get

$$s\bar{V}(x, s) - V(x,0) = \frac{\lambda\, d^2\bar{V}}{dx^2} \text{ where } \bar{V}(x,s) = L\,(V(x,t))$$

$$\Rightarrow \qquad s\,\bar{V} - 0 = \frac{\lambda\, d^2\bar{V}}{dx^2}$$

$\Rightarrow \quad \left(D^2 - \frac{s}{\lambda}\right)\bar{V} = 0 \qquad$ where $D = \frac{d}{dx}$

Its solution is given by

$$\bar{V} = Ae^{\sqrt{\frac{s}{\lambda}}\,x} + Be^{-\sqrt{\frac{s}{\lambda}}\,s} \qquad \dots \text{(ii)}$$

where A, B are the arbitrary constants.

As V is finite $\forall$ x so that $\bar{V}$ (x, s) is also finite as $x \to \infty$.

$\therefore$ A must be zero i.e., A = 0

so that $\bar{V} = Be^{-\sqrt{\frac{s}{\lambda}}\,s}$ ... (iii)

Given V (0, t) = V_0

Taking Laplace transforms

$$\bar{V}(0, s) = V_0\, L(1) = \frac{V_0}{s}$$

Put x = 0 in (iii), we get $\frac{V_0}{s} = B$

Hence $\bar{V} = \frac{V_0}{s} e^{-\sqrt{\frac{s}{\lambda}}\,x}$

Taking inverse Laplace transform, we get

$$V(x, t) = L^{-1}(\bar{V}) = V_0\, L^{-1}\left(\frac{1}{s} e^{-\sqrt{\frac{s}{\lambda}}\,x}\right)$$

$$= V_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{\lambda}\sqrt{t}}\right) = V_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{\lambda t}}\right)$$

$$= V_0\left(1 - \operatorname{erf}\left(\frac{x}{2\sqrt{\lambda t}}\right)\right).$$

2.2.4 Solution of Wave Equation by Laplace Transform

Example 3 : Find the solution of one-dimensional wave equation i.e., $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}; x, t > 0$

where $y(x, 0) = 0$, $y_t(x, 0) = 0$

$y(0, t) = f(t)$ and $y(x, t) \to 0$ as $x \to \infty$.

Sol. Given P.D. E i.e., wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \ldots (i)$$

where boundary conditions are $y(0, t) = f(t)$ and $y(x, t) \to 0$ as $x \to \infty$.
and initial conditions are $y(x, 0) = 0$, $y_t(x, 0) = 0$

To solve (i), Take Laplace transform on both sides of (i), we get

$$s^2 \bar{y}(x, s) - s y(x, 0) - y_t(x, 0) = c^2 \frac{d^2 \bar{y}}{dx^2}$$

$$\Rightarrow \quad s^2 \bar{y}(x, s) - s(0) - 0 = c^2 \frac{d^2 \bar{y}}{dx^2}$$

$$\Rightarrow \quad \frac{d^2 \bar{y}}{dx^2} - \frac{s^2}{c^2} \bar{y}(x, s) = 0$$

$$\Rightarrow \quad \left(D^2 - \frac{s^2}{c^2}\right) \bar{y} = 0$$

Its solution is $\bar{y} = Ae^{\frac{s}{c}x} + Be^{\frac{-s}{c}x}$ … (ii)

where A, B are arbitrary constants.

We have $y(0, t) = f(t)$

Take Laplace transform, we get

$$\bar{y}(0, s) = L(f(t)) = \phi(s) \text{ (say)}$$

And $y(x, t) \to 0$ as $x \to \infty \Rightarrow \bar{y}(x, s) \to 0$ as $x \to \infty$

$\because \quad \bar{y} \to 0$ as $x \to \infty$

so let A = 0 in (ii), which gives $\bar{y}(x, s) = Be^{\frac{-s}{c}x}$

Put x = 0, so that $\bar{y}(0, s) = B \Rightarrow \phi(s) = B$

$$\therefore \quad \bar{y}(x, s) = \phi(s)\, e^{\frac{-s}{c}x}$$

Take inverse Laplace transform, we get

$$y\ (x, t) = L^{-1}\left(\phi(s)\, e^{\frac{-s}{c}x}\right)$$

$$= f\left(t - \frac{x}{c}\right) h\left(t - \frac{x}{c}\right) (\because L^{-1}(\phi(s)) = f(t) \text{ and use second shifting theorem})$$

where $h\left(t - \frac{x}{c}\right)$ is Heaviside's unit step function.

2.2.5 Self Check Exercise

1. Solve $\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}; 0 < x < 1, t > 0$ where, y (x, 0) = 0, y_t (x, 0) = sin 3πx and y (, t) = y (1, t) = 0.
2. A slab has ends x = 0 and x = *l* which are kept at temperature zero. It has initial temperature as sin $\left(\frac{\pi x}{l}\right)$. Find the temperature V (x, t) at any point and at any time t.
3. A string is stretched between two points (0 0) and (*l*, 0). If it is displaced along the curve y = K sin $\left(\frac{\pi x}{l}\right)$ and released from rest in that position at time t = 0. Find displacement y (x, t) at any time t > 0 and at any point x, 0 < x < *l*.
4. Find solution of $\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}$

   Subject to V (0, t) = 1, V (π, t) = 3 and V (x, 0) = 1 for x ∈ (0π), t > 0
   Also interpret the above problem physically.