Centre for Distance and Online Education

Punjabi University, Patiala

Class : BCA II **Semester : III**

Paper : BCAB2102T **Unit-I**

(Discrete Mathematics)

Medium : English

Lesson No.

Department website : www.pbidde.org

BCA-212 : DISCRETE MATHEMATICS

Maximum Marks: 75 **Pass Percentage: 35%**
Maximum Time: 3 Hrs

INSTRUCTIONS FOR THE PAPER SETTER

The question paper will consist of three sections A, B and C. Sections A and 8 will have four questions from the respective sections of the syllabus carrying 15 marks for each question. Section C will consist of 5-10 short answer type questions carrying a total of 15 marks, which will cover the entire syllabus uniformly.

INSTRUCTIONS FOR THE CANDIDATES

Candidates are required to attempt five questions in all by selecting at least two questions each from the section A and B. Section C.

Section-A

Set Theory: Sets, Types of sets, Set operations, Principle of Inclusion – Exclusion, Cartesina product of sets, Partitions.

Logic: Propositions, Implications, Precedence of logical operators, Translating English sentences into logical expressions, Propositional equivalence.

Principle of Mathematical Induction.

Relations: Relations and diagraph, n-ary relations and their applications, Properties of relations, Representing relations, Closure of relations, Equivalence relation, Operations on relations, Partial ordering.

Section-B

Functions: Functions, One-to-one functions, Onto functions, Inverse and composition of functions, Floor function, Ceiling function.

Basic Concepts (only definition): Big-O notation, Big-Omega and Big-Theta notation.

Graphs: Introduction to graphs, Graph terminology, Representing graphs and Graph Isomorphism, Connectivity, Euler paths and circuits, Hamiltonian paths and circuits, Shortest path problems, Planar graphs.

Trees: Tress, Labelled trees, Tree traversal, Undirected trees, Spanning trees, Minimum spanning trees.

TEXT BOOK:

1. Discrete Mathematical Structures: Bernard Kolman, Robert C. Busby, Sharon C. Ross, 4th Edition, Pearson Education, Asia.

REFERENCE BOOKS:

1. Discrete Mathematics, Richard Johnsonbaugh, 5th Edition, Pearson Education, Asia.
2. Elements of Discrete Mathematics, 2nd Edition, Tata Mc-Graw Hill.
3. Discrete Mathematics, Seymon Lipschutz & Max Lans Lipson, Tata McGraw Hill.

LESSON NO. 1.1 **Author : Dr. Chanchal**
LAST UPDATED ON MAY, 2023

SET THEORY AND PMI

Structure :

1.1.1 Objectives

- To study sets, operations on sets with their illustration using Venn diagram and fundamental laws of set theory
- To study about an important relation on sets called Cartesian product of sets, which is required to understand relations
- To understand how a set can be partitioned into non-overlapping subsets
- To study inclusion exclusion principle and solve various practical problems related to it
- To study the concept Principle of Mathematical Induction.

1.1.2 Introduction to Some Basic Terms in Set Theory

Firstly, we introduce a set as :

Def. Set : A set is a well defined collection of distinct objects.

The word 'well defined' implies that we are given a rule with the help of which we can say whether a particular object belongs to the set or not. The word 'distinct' implies that repetition of objects is not allowed. Each object of the set is called an element of the set. Further, sets are generally denoted by capital letters A,B,C,..... while elements of the sets are denoted by small letters a,b,c, ...

For Example : (i) The set of days of a week.
(ii) The set of even integers.

A set can be represented in two ways :

(1) Tabular or Roster Method

(2) Set-builder or Rule Method

In roster form, we represent a set by listing all its elements within curly brackets { }, separated by commas while in the set-builder form, we do not list the elements but the set is represented by specifying the defining property.

For Example : Set	Roster form	Set-builder form
(1) A set of vowels	A = {a, e, i, o, u}	A = {x : x is a vowel of english alphabet}
(2) A set of positive even integers upto 10	A = {2, 4, 6, 8, 10}	A = {x : x is a positive even integer and $x \leq 10$}

There are some basic mathematical sets, such as

N = Set of all natural numbers

W = Set of all whole numbers

I a Z = Set of all integers

Q = Set of all rational numbers

R = Set of all real numbers

Membership of a Set : If an object x is a member of the set A, we write $x \in A$, which can be read as 'x belongs to A' or A contains x. Similarly, we write $x \notin A$ to show that x is not a member of the set A.

For Example : Let A = {1, 2, 4, 6, 7}. Here $2 \in A$ but $5 \notin A$.

Finite Set : A set is said to be finite if it has finite no. of elements.

For Example : A = {2, 4, 6, 8}

Infinite Set : A set is said to be infinite if it has an infinite number of elements.

For Example : A = {1,2, 3.......} and B = {x : x is an odd integer} are infinite sets

Singleton Set : A set containing only one element is called a singleton or a unit set.

For Example : $A = \{x : x \text{ is a perfect square and } 30 \le x \le 40\} = \{6\}$

Empty, Null or Void Set : A set which contains no element, is called a null set and is denoted by ϕ (read as phi).

For Example : $A = \left\{x : x \text{ is a positive integer satisfying } x^2 = \frac{1}{4}\right\}$

Sub-Set, Super-Set : If every member of a set A is a member of a set B, then A is called sub-set of B and B is called super-set of A.

or if $x \in A \Rightarrow x \in B$, then A is a sub set of B and B is a super set of A and we write it $A \subset B$ which means A is contained in B or B contains A.

Note 1. Since every element of A belongs A

$\therefore$ $A \subset A \Rightarrow$ every set is sub set of itself.

2. The empty set ϕ is taken as a sub-set of every set.

For Example : Let $A = \{1, 2, 3, 4, 5, 6, 8, 10\}$, $B = \{2, 4, 6, 10\}$, = $C = \{1, 2, 7, 8\}$.

Now every element of B is an element of A, $\therefore B \subset A$

Again $7 \in C$, but $7 \notin A$

$\therefore$ $C \not\subset A$ i.e., C is not a sub-set of A.

Equality of Sets : Two sets A and B are said to be equal if both have the same elements. In other words, two sets A and B are equal when every element of A is an element of B and every element of B is element of A.

i.e., if $A \subset B$ and $B \subset A$, then $A = B$.

For Example : $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $B = \{x : x \text{ is a natural number and } 1 \le x \le 10\}$

Hence, A = B.

Proper Sub-set : A non-empty set A is said to be a proper subset of B if $A \subset B$ and $A \neq B$.

Note : (i) ϕ and A are called improper subsets of A.

(ii) If A has n elements, then number of subsets of A is 2^n.

Power set : The power set of a finite set is the set of all sub-sets of the given set. Power set of A is denoted by P(A).

For Example : Take $A = \{1, 2, 3\}$

$\therefore$ $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

Universal Set : If all the sets under consideration are sub-sets of a fixed set U, then U is called a universal set.

For Example : In Plane geometry, the universal set consists of points in a plane.

Comparable and Non-comparable Sets : Two sets are said to be comparable if one of the two sets is a sub-set of the other.

For Example : Let A = {2, 3, 5}, B = {2, 3, 5, 6}.

Here $A \subset B$, so A and B are comparable sets.

Order of a Finite Set : The number of different element of a finite set A is called the order of A and is denoted by O(A).

Cardinality : Number of different elements in a set is known as its cardinality.

For Example : If A = {2, 3, 6, 8}, then O(A) = 4

Equivalent Sets : Two finite sets A and B are said to be equivalent sets if the total number of elements in A is equal to the total number of elements in B.

For Example : Let A = {1, 2, 3, 4, 6}, B = {1, 2, 7, 9, 12}

$\therefore$ O(A) = 5 = O(B) $\Rightarrow$ A and B are equivalent sets, denoted as A ~ B.

1.1.3 Operations on Sets

In order to represent various operations on sets, we use a special type of diagrams, called venn diagrams defined as :

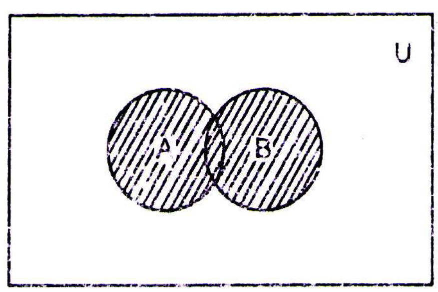
Venn Diagrams : The relations between sets can be illustrated by certain diagrams called **Venn diagrams.** In a Venn diagram, universal set U is represented by a rectangle and any sub-set of U is represented by a circle within a rectangle U.

Now, various operations of set theory are discussed below :

1.1.3.1 Union of Two Sets

If A and B be two given sets, then their union is the set consisting of all the elements of A together with all the elements in B. We should not repeat the elements. The union of two sets A and B is written as $A \cup B$.

In symbols, $A \cup B = \{x : x \in A \text{ or } x \in B\}$



A ∪ B IS SHIELDED LIKE ▨

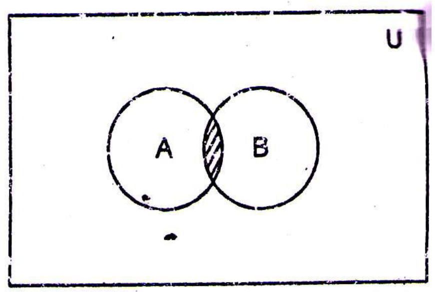
For Example : Let A = {1, 2, 3, 5, 8}, B = {2, 4, 6}

$\therefore$ $A \cup B$ = {1, 2, 3, 4, 5, 6, 8}

1.1.3.2 Intersection of two sets

The intersection of two sets A and B, denoted by $A \cap B$, is the set of elements common to A and B.

In symbols, $A \cap B = \{x : x \in A \text{ and } x \in B\}$



A ∩ B IS SHIELDED LIKE ▨

For Example : Let A = {2, 4, 6, 8, 10, 12}, B = {2, 3, 5, 7, 11}

∴ $A \cap B = \{2\}$

Disjoint Sets : If A and B are two given sets such that $A \cap B = \phi$, then the sets A and B are said to be disjoint.

For Example : Let A = {a, b, c, d}, B = {l, m, n, p},

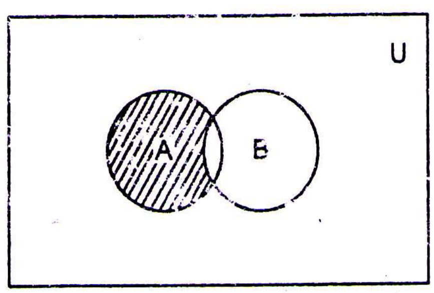
∴ $A \cap B = \phi \Rightarrow$ A and B are disjoint sets.

1.1.3.3 Difference of two sets

The difference of two sets A and B is the set of those elements of A which do not belong to B. We denote this by A – B.

In symbols, we write $A - B = \{x : x \in A \text{ and } x \notin B\}$

A – B is also sometimes written as A/B.



A – B IS SHIELDED LIKE ▨

For Example : Let A = {a, b, c, d, e}, B = {c, d, e, f, g}

Then, A – B = {a, b}

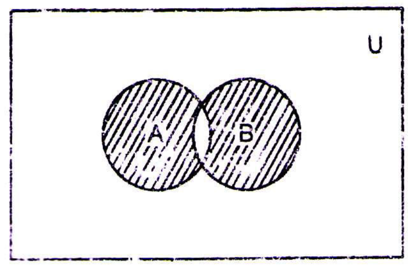
Note. $B - A \neq A - B$

Symmetric Difference of Two Sets

If A and B are any two sets, then the set $(A - B) \cup (B - A)$ is called symmetric difference of A and B and is denoted by $A \Delta B$.

In symbols, we write

$A \Delta B = \{x : (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)\}$



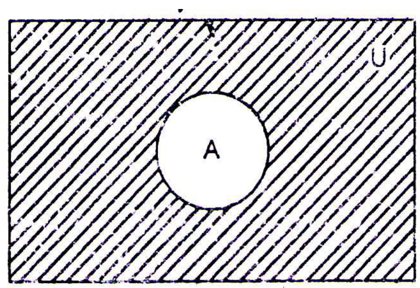
A Δ B IS SHIELDED LIKE

For Example : Let A = {1, 2, 4}, B = {1, 2, 3, 4, 6}

$\therefore$ $A \Delta B = (A/B) \cup (B/A) = (A - B) \cup (B - A) = \{4\} \cup \{3, 5, 6\} = \{3, 4, 5, 6\}$.

1.1.3.4 Complement of a Set

Let A be a subset of universal set U. Then the complement of A is the set of all those elements of U which do not belong to A and we denote complement of A by A^c or A'. We can write $A^c = \{x : x \in U, x \notin A\}$



A^C IS SHIELDED LIKE

For Example : If U = {2, 4, 6, 8,10}, A = {4, 8} then A^c = {2, 6, 10}

Note. $U^c = \phi$ and $\phi^c = U$, $(A^c)^c = A$.

1.1.4 Some Fundamental Laws of Algebra of Sets

I. Idempotent Laws

Statement : If A is any set, then (i) $A \cup A = A$ (ii) $A \cap A = A$

Proof : (i) L.H.S. $= A \cup A$

$= \{x : x \in A \cup A\} = \{ x : x \in A \text{ or } x \in A\}$

$= \{x : x \in A\} = A$

= R.H.S.

(ii) Do Yourself.

II. Identity Laws

Statement. If A is any set, then (i) $A \cup \phi = A$ (ii) $A \cap U = A$

Proof : (i) L.H.S. $= A \cup \phi = \{x : x \in A \cup \phi\}$

$= \{x : x \in A \text{ or } x \in \phi\} = \{x : x \in A\}$

$= A =$ R.H.S.

(ii) Do Yourself.

III. Commutative Laws

Statement. If A and B are any two sets, then (i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$

Proof : Do Yourself.

IV. Associative Laws

Statement. If A, B and C are any three sets, then

(i) $A \cup (B \cup C) = (A \cup B) \cup C$ (ii) $A \cap (B \cap C) = (A \cap B) \cap C$

Proof : Do Yourself.

V. Distributive Laws

Statement. If A, B, C are any three sets, then

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof : L.H.S. $= A \cup (B \cap C)$

$= \{x : x \in A \cup (B \cap C)\}$

$= \{x : x \in A \text{ or } x \in (B \cap C)\}$

$= \{x : x \in A \text{ or } (x \in B \text{ and } x \in C)\}$

$= \{x : (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)\}$

$= \{x : x \in (A \cup B) \text{ and } x \in (A \cup C)\}$

$= \{x : x \in (A \cup B) \cap (A \cup C)\}$

$= \{(A \cup B) \cap (A \cup C)\}$

$=$ R.H.S.

$\therefore$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Note. We can also prove above result by showing that

$A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$ and $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$

(ii) Do Yourself.

VI. De Morgan's Laws

Statement. If A and B are two sub-sets of U, then

(i) $(A \cup B)^c = A^C \cap B^c$ (ii) $(A \cap B)^c = A^c \cup B^c$

Proof : (i) L.H.S. $= (A \cup B)^c = \{x : x \in (A \cup B)^c\}$

$= \{x : x \notin (A \cup B)\}$

$= \{x : x \notin A \text{ and } x \notin B\}$

$= \{x : x \in A^c \text{ and } x \in B^c\}$

$= \{x : x \in (A^c \cap B^c)\}$

$= A^c \cap B^c = \text{R.H.S.}$

$\therefore \quad (A \cup B)^c = A^c \cap B^c$

(ii) Do Yourself.

1.1.5 Some Important Examples

Example 1 : Let U be the set of integers and let A = {x : x is divisible by 3}, let B = {x : x is divisible by 2}. Let C = {x : x is divisible by 5} Find the elements in each of the following set :

(a) $A \cap B$ (b) $A \cup C$ (c) $A \cap (B \cup C)$ (d) $(A \cap B) \cup C$

(e) $A^c \cap B^c$ (f) $(A \cap B)^c$ (g) B/A (h) A/B (i) A/(B/C)

Sol. : $U = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots\ldots\ldots\}$

$A = \{x : x \text{ is divisible by } 3\} = \{x : x = 3n, n \in I\} = N_3$

$B = \{x : x \text{ is divisible by } 2\} = \{x : x = 2n, n \in I\} = N_2$

$C = \{x : x \text{ is divisible by } 5\} = \{x : x = 5n, n \in I\} = N_5$

(a) $A \cap B = N_3 \cap N_2 = N_6 \quad \because \text{l.c.m. } \{2, 3\} = 6$

$= \{\ldots\ldots\ldots -12, -6, 0, 6, 12, \ldots\ldots\}$

(b) $A \cup C = N_3 \cup N_5 = \{\ldots\ldots\ldots -9, -6, -5, -3, 0, 3, 5, 6, \ldots\ldots\}$

(c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C) = (N_3 \cap N_2) \cup (N_3 \cap N_5) = N_6 \cup N_{15}$

$= \{\ldots\ldots -15, -12, -6, 0, 6, 12, 15\ldots\ldots\ldots\}$

(d) $(A \cap B) \cup C = \{N_3 \cap N_2\} \cup N_5 = N_6 \cup N_5$

$= \{\ldots\ldots\ldots -12 - 10, 6, -5, 0, 5, 6, 10, 12, \ldots\ldots\ldots\}$

(e) $A^c \cap B^c = (A \cup B)^c = (N_3 \cup N_2)^c$

$= \{\ldots\ldots\ldots -11, -7, -5, -1, 1, 5, 7, 11, \ldots\ldots\ldots\}$

(f) $(A \cap B)^c = (N_3 \cap N_2)^c = (N_6)^c$

$= \{-7, -5, -3, -2, -1, 1, 2, 3, 4, 5, \ldots\ldots\ldots\}$

(g) $A/B = N_3/N_2 = N_3 - N_2$

$= \{\ldots\ldots - 10, -8, -4, -2, 2, 4, 8, 10 \ldots\ldots\}$

(h) $B/A = N_2/N_3 = N_2 - N_3$

$= \{\ldots\ldots\ldots -15, -9, -3, 3, 9, 15, \ldots\ldots\}$

(f) $A/(B/C) = (A/B) \cup (A \cap C)$

$= (N_3/N_2) \cup (N_3 \cap N_5) = (N_3/N_2) \cup N_{15}$

$= \{\ldots\ldots\ldots, -15, -9, -3, 3, 9, 15, \ldots\ldots\ldots\} \cup \{\ldots\ldots\ldots 30, 15, 0, 15, 30 \ldots\ldots\}$

$= \{\ldots\ldots\ldots, -15, -9, -3, 3, 9, 15, \ldots\ldots\ldots).$

Example 2 : Prove that $A \cup (B/A) = A \cup B$.

Sol. : L.H.S. $= A \cup (B/A) = A \cup (B - A)$

$= A \cup (B \cap A^c)$ [$A - B = A \cap B^c$]

$= (A \cup B) \cap (A \cup A^c)$ [Distributive Law]

$= (A \cup B) \cap X$ [$A \cup A^c = X$]

$= A \cup B =$ R.H.S.

Example 3 : Let $A = \{1, 2, 4\}$, $B = (4, 5, 6\}$,

Find $A \cup B$, $A \cap B$ and $A - B$.

Sol. $A = \{1, 2, 4\}$ and $B = \{4, 5, 6\}$

(i) $A \cup B = \{1, 2, 4\} \cup \{4, 5, 6\} = \{1, 2, 4, 5, 6\}$

(ii) $A \cap B = \{1, 2, 4\} \cap \{4, 5, 6\} = \{4\}$

(iii) $A - B = \{1, 2, 4\} - (4, 5, 6\} = \{1, 2\}$.

Example 4 : Prove that $A \cup B = A \cap B$ iff $A = B$

Sol. : (i) Assume that $A \cup B = A \cap B$ ... (1)

Let x be any element of A

$\therefore$ $x \in A \Rightarrow x \in A \cup B \Rightarrow x \in A \cap B$ [$\because$ of (1)]

$\Rightarrow x \in B$

$\therefore$ $x \in A \Rightarrow x \in B$

$\therefore$ $A \subset B$

Similarly, $B \subset A$ ... (2)

From (2) and (3), $A = B$. ... (3)

$\therefore$ $A \cup B = A \cap B \Rightarrow A = B$

(ii) Assume that $A = B$

$\therefore$ $A \cup B = A \cup A = A$

$A \cap B = A \cap A = A$

$\therefore$ $A \cup B = A \cap B$

$\therefore$ $A = B \Rightarrow A \cup B = A \cap B$.

Example 5 : For any two sets A and B, prove that $A \cap B = \phi \Rightarrow A \subset B^c$.

Sol. : We are given that $A \cap B = \phi$ ... (1)

Let x be any element of A

$\therefore$ $x \in A \Rightarrow x \notin B$ [$\because$ of (1)]

$\Rightarrow$ $x \in B^c$

$\therefore$ $x \in A \Rightarrow x \in B^c$

But x is any element of A

∴ every element of A is an element of B^c

∴ $A \subset B^c$.

Example 6 : Prove that $A^c/B^c = B/A$

Sol. : L.H.S. = A^c/B^c

$= A^c - B^c = \{x : x \in (A^c - B^c)\} = \{x : x \in A^c \text{ and } x \notin B^c\}$

$= \{x : x \notin A \text{ and } x \in B\} = \{x : x \in B \text{ and } x \notin A\}$

$= \{ x : x \in (B - A)\} = B - A = B/A$

= R.H.S.

∴ $A^c/B^c = B/A$.

Example 7 : Show that $A \cap (B - C) = (A \cap B) - (A \cap C)$

Sol. : R.H.S. = $(A \cap B) - (A \cap C)$

$= (A \cap B) \cap (A \cap C)^c = (A \cap B) \cap (A^c \cup C^c)$

$= [(A \cap B) \cap A^c] \cup [A \cap B) \cap C^c]$

$= [(A \cap A^c) \cap B] \cup [A \cap (B \cap C^c)] = (\phi \cap B) \cup [A \cap (B - C)]$

$= \phi \cup [A \cap (B - C)] = A \cap (B - C)$ = L.H.S.

1.1.6 Cartesian Product of Sets

Ordered-Pair : By an ordered pair of elements, we mean a pair (a, b) such that $a \in A$ and $b \in B$. The ordered pairs (a, b), (b, a) are different unless a = b. Also} (a, b) = (c, d) iff a = c, b = d.

Cartesian Product of Two Sets : The set of all ordered pairs (a, b) of element $a \in A$, $b \in B$ is called the cartesian product of the sets A and B and is denoted by $A \times B$.

In symbols, $A \times B = \{(a, b) : a \in A, b \in B\}$

Note 1. $A \times B$ and $B \times A$ are different sets if $A \neq B$.

2. $A \times B = \phi$ when one or both of A, B are empty.

Art 1.1 : Prove that

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Proof : (i) L.H.S. = $A \times (B \cup C)$

$= \{(x, y) : x \in A \text{ and } y \in (B \cup C)\}$

$= \{(x, y) : x \in A \ (y \in B \text{ or } y \in C)\}$

$= \{(x, y) : (x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C)\}$

$= \{(x, y) : (x, y) \in (A \times B) \text{ or } (x, y) \in (A \times C)\}$

$= \{(x, y) : (x, y) \in (A \times B) \cup (A \times C)\}$

$= (A \times B) \cup (A \times C)$ = R.H.S.

∴ $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) L.H.S. $= A \times (B \cap C)$

$= \{(x, y) : x \in A \text{ and } y \in (B \cap C)\}$

$= \{(x, y) : x \in A \text{ and } (y \in B \text{ and } y \in C)\}$

$= \{(x, y) : (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C)\}$

$= \{(x, y) : (x, y) \in (A \times B) \text{ and } (x, y) \in (A \times C)\}$

$= \{(x, y) : (x, y) \in (A \times B) \cap (A \times C)\}$

$= (A \times B) \cap (A \times C)$ = R.H.S

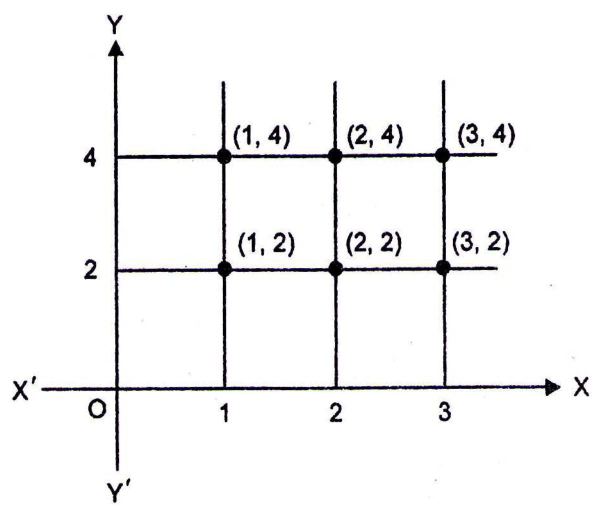
∴ $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Example 8 : Let A = {1, 2, 3}, B = {2, 4}. Find A × B and show it graphically.

Sol. Here A = {1, 2, 3}, B = {2, 4}

A × B = {1, 2, 3} × {2, 4} = {(1, 2), (1, 4), (2, 2), (2,4), (3, 2), (3, 4)}.

Now to represent (1, 2), we draw a vertical line through 1 and a horizontal line through 2. These two lines meet in the point which represents (1, 2). Similarly we can represent the other points in A × B and get the graphical representation of A × B.



Example 9 : A, B, C are any three sets, then prove that

$(A \cap B) \times C = (A \times C) \cap (B \times C)$

Sol. : L.H.S. $= (A \cap B) \times C$

$= \{(x, y) : x \in (A \cap B) \text{ and } y \in C\}$

$= \{(x, y) : (x \in A \text{ and } x \in B) \text{ and } y \in C\}$

$= \{(x, y) : (x \in A \text{ and } y \in C) \text{ and } (x \in B \text{ and } y \in C)\}$

$= \{(x, y) : (x, y) \in (A \times C) \text{ and } (x, y) \in (B \times C)\}$

$= \{(x, y) : (x, y) \in (A \times C) \cap (B \times C)\}$

$= (A \times C) \cap (B \times C) = \text{R.H.S.}$

1.1.7 Partition of Sets

A partition of a non-empty set A is a collection $P = \{A_1, A_2, A_3, \ldots\ldots\ldots\ldots\}$ of subsets of A if and only if

(i) $A = A_1 \cup A_2 \cup A_3 \cup \ldots\ldots\ldots\ldots$

and (ii) $A_i \cap A_j = \phi$ for $i \neq j$

$A_1, A_2, A_3, \ldots\ldots$ are called cells or blocks of the partition P.

For Example : (i) Let A = {a, b, c} be any set.

Then, $P_1 = \{\{a\}, \{b\}, \{c\}\}$, $P_2 = \{\{a\}, \{b, c\}\}$, $P_3 = \{\{b\}, \{a, c\}\}$,

$P_4 = \{\{c\}, \{a, b\}\}$, $P_5 = \{\{a, b, c\}\}$ are partitions of the set A.

(ii) Let **Z** = set of integers. Then the collection

$P = \{\{n\} : n \in \mathbf{Z}\}$ is a partition of **Z**.

Minimum Set or Minset or Minterm :

Let A be any non-empty set and $B_1, B_2, \ldots\ldots\ldots\ldots, B_n$ be any subsets of A. Then the minimum set generated by the collection $\{B_1, B_2, \ldots\ldots, B_n\}$ is a set of the type $D_1 \cap D_2 \cap \ldots\ldots\ldots\ldots \cap D_n$, where each $D_1, D_2, \ldots\ldots\ldots\ldots, D_n$ is B_i or B_i for i = 1, 2, 3,......., n.

For Example : The minsets generated by two sets B_1 & B_2 are

$A_1 = B_1 \cap B_2'$, $A_2 = B_1 \cap B_2$, $A_3 = B_1^C \cap B_2$, $A_4 = B_1^C \cap B_2^C$.

Normal form (or Canonical form) :

A set F is said to be in minset normal (or cannonical) form when it is expressed as the union of distinct non-empty minsets or it is ϕ

i.e., either $F = \phi$ or $F = \bigcup_{\lambda \in \Delta} A_i$, where A_λ's are non-empty minsets.

Principle of Duality for Sets :

Let S be any identity in set theory involving the operation union ($\cup$), intersection ($\cap$). Then the statement S* obtained from S by changing union to intersection to union and empty set ϕ to universal set U is also an identity called the dual of the statement S.

Remark : The number of minsets generated by n sets is 2^n.

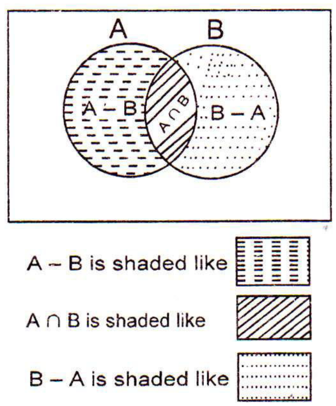
1.1.8 The Inclusion-Exclusion Principle

It is the most general form of addition principle for enumeration. As we know that number of elements of a finite set A is denoted by n (A) or |A|, so following results regarding number of elements should be kept be kept in mind for doing problems :

1. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
2. $n(A \cup B) = n(A) + n(B) \Leftrightarrow$ A, B are disjoint sets.
3. $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$
4. $n(A) = n(A - B) + n(A \cap B)$
5. $n(B) = n(B - A) + n(A \cap B)$
6. $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B)$
   $-n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$
7. $n(A' \cup B') = n((A \cap B')) = n(U) - n(A \cap B)$
8. $n(A' \cap B') = n((A \cup B)') = n(U) - n(A \cup B)$
9. $n(A \cap B' \cap C') = n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$

Proof : (1) We know that $A \cup B$ is the union of three disjoint sets

$A = B$, $A \cap B$ and $B - A$.



$\therefore \quad n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$

Again A is union of $A - B$ and $A \cap B$, which are disjoint sets

$\therefore \quad n(A) = n(A - B) + n(A \cap B)$

Similarly, $n(B) = n(B - A) + n(A \cap B)$

Adding (2) and (3), we get

$$n(A) + n(B) = n(A - B) + n(B - A) + 2n(A \cap B)$$
$$= [n(A - B) + n(B - A) + n(A \cap B)] + n(A \cap B)$$

$\therefore \quad n(A) + n(B) = n(A \cup B) + n(A \cap B)$ $[\because$ of $(1)]$

$\Rightarrow \quad n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(2) Since A and B are disjoint sets

$\therefore$ $A \cap B = \phi \Rightarrow n(A \cap B) = 0$

$\therefore$ $n(A \cup B) = n(A) + n(B) - 0$ or $n(A \cup B) = n(A) + n(B)$.

(6) L.H.S. $= n(A \cup B \cup C) = n[(A \cup (B \cup C)]$

$= n(A) + n(B \cup C) - n[A \cap (B \cup C)]$

$= n(A) + n(B) + n(C) - n(B \cap C) - n[(A \cap B) \cup (A \cap C)]$

$= n(A) + n(B) + n(C) - n(B \cap C) - [n(A \cap B) + n(A \cap C)$

$- n[(A \cap B) \cap (A \cap C)]$

$= n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$

$\therefore$ $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C)$

$- n(C \cap A) + n(A \cap B \cap C)$.

Example 10 : In a group of 50 persons, 14 drink tea but not coffee and 30 drink tea. Find

(i) How many drink both tea and coffee ?

(ii) How many drink coffee but not tea ?

Sol. Let T denote the set of persons drinking tea and C denote the set of persons drinking coffee.

$\therefore$ $n(T \cup C) = 50$, $n(T) = 30$, $n(T \cap C^C) = 14$

(i) now $n(T \cap C^C) = n(T) - n(T \cap C)$

$\therefore$ $14 = 30 - n(T \cap C) \quad \Rightarrow n(T \cap C) = 16$

$\therefore$ number of persons drinking both tea and coffee = 16

(ii) Also $n(T \cup C) = n(T) + n(C) - n(T \cap C)$

$\therefore$ $50 = 30 + n(C) - 16 \quad \Rightarrow n(C) = 36$

$\therefore$ number of persons drinking coffee but not teat

$= n(C \cap T^C) = n(C) - n(C \cap T)$

$= n(C) - n(T \cap C) = 36 - 16 = 20$

Example 11 : A survey of 500 television watchers produced the following information:

285 watch football, 195 watch hockey, 115 watch basketball, 45 watch football and basketball, 70 watch football and hockey, 50 watch hockey and basketball, 50 do not watch any of the three games.

How many watch all the three games ? How many watch exactly one of the three games ?

Sol. Let F, H, B denote the sets of viewers who watch football, hockey, basketball respectively.

$\therefore$ $n(F) = 285$, $n(H) = 195$, $n(B) = 115$, $n(F \cap B) = 45$,

$n(F \cap H) = 70$, $n(H \cap B) = 50$, $n(F \cup H \cup B)^c = 50$

Also total number of viewers = 500

Now $n(F \cup H \cup B)^C = 50$

$\Rightarrow$ $500 - n(F \cup H \cup B) = 50$

$\Rightarrow$ $n(F \cup H \cup B) = 450$

$\Rightarrow$ $n(F) + n(H) + n(B) - n(F \cap H) - n(H \cap B) - n(B \cap F)$

$+ n(F \cap H \cap B) = 450$

$\Rightarrow$ $285 + 195 + 115 - 70 - 50 - 45 + n(F \cap H \cap B) = 450$

$\Rightarrow$ $n(F \cap H \cap B) = 20$

$\therefore$ number of viewers watching all the three games = 20.

Number of viewers watching football alone = $n(F \cap H^C \cap B^C)$

$= n(F) - n(F \cap H) - n(F \cap B) + n(F \cap H \cap B)$

$= 285 - 70 - 45 + 20 = 190$

Number of viewers watching hockey alone = $n(H \cap F^C \cap B^C)$

$= n(H) - n(H \cap F) - n(H \cap B(+ n(F \cap H \cap B)$

$= 195 - 70 - 50 + 20 = 95$

Number of viewers watching basket - ball alone = $n(B \cap H^C \cap F^C)$

$= n(B) - n(B \cap H) - n(B \cap F) + n(F \cap H \cap B)$

$= 115 - 50 - 45 + 20 = 40$

$\therefore$ number of viewers watching exactly one of the three games

$= 190 + 95 + 40 = 325.$

Example 12 : Find how many integers between 1 and 60 are not divisible by 2 nor by 3 and nor by 5 ?

Sol. Try yourself

1.1.9 Mathematical Induction

Principle of Mathematical Induction :

Let P(n) be the given statement. Then to prove the validity of P(n) we have to perform following three steps :

(i) Basis : First we prove the given statement is true for n = 1 i.e. P(1) is true.

(ii) Assumption : We assume result is true for n = k.

(iii) Induction : We prove that the given statement is true for n = k + 1 i.e. P(k + 1) is true.

Then we conclude by principle of mathematical induction that statement is true for all $n \in N$.

Example 13 : Use Mathematical Induction to show that $1 + 2 + 4 + \ldots\ldots 2^n = 2^{n+2} - 1$

Sol. Let P (n) be the statement

$$1 + 2 + 4 \ldots\ldots\ldots 2^n = 2^{n+1} - 1 \text{ or } 2^0 + 2^1 + 2^2 \ldots\ldots 2^n = 2^{n+1} - 1$$

Basis : First we show that P(1) is true, so put n = 1

L.H.S R.H.S

$2^0 + 2^1 = 1 + 2 + 3$ $\quad 2^{14} - 1 = 4 - 1 = 3$

3 = 3 :P (1) is true

Assumption : Suppose that P(k) is true, so taking n = k

$2^0 + 2^1 + 2^2 \ldots\ldots\ldots 2^k = 2^{k+1} - 1$

Induction : Now we prove P (k + 1) is true, taking n = k + 1

$2^0 + 2^1 + 2^2 \ldots\ldots\ldots 2^k + 2^{k+1} = 2^{k+1+1} - 1$

L.H.S. $2^0 + 2^1 + 2^2 \ldots\ldots 2^k + 2^{k+1} = (2^0 + 2^1 + 2^2 \ldots\ldots 2^k) + 2^{k+1}$

$= 2^{k+1} - 1 + 2^{k+1}$

$= 2^{k+1} + 2^{k+1} - 1 = 2.2^{k+1} - 1 = 2^{k+1+1} - 1 = \text{R.H.S}$

$\therefore$ P (k + 1) is true.

Hence P(n) is true by Induction.

Example 14 : Prove by induction $a + ar + ar^2 \ldots\ldots\ldots ar^{n-1} = \frac{a(1-r^n)}{1-r}, r \neq 0$

Sol. Basis : for n = 1

L.H.S R.H.S

$a \,.\, r^{1-1}$ $\quad \frac{a(1-r^1)}{1-r}, a$

a = a

$\therefore$ Result is true for n = 1

Assumption : Let result is true for n = k

$$a + ar + ar^2 \ldots\ldots + ar^k = a\,\frac{(1-r^k)}{1-r}, r \neq 0 \quad \ldots (1)$$

Induction : Put n = k + 1

$$a + ar + ar^2 \ldots\ldots\ldots\ldots + ar^k = a\,\frac{(1-r^{k+1})}{1-r}$$

L.H.S. $= a + ar + ar^2 \ldots\ldots\ldots\ldots + ar^k = a + ar + ar^2 \ldots\ldots\ldots ar^{k-1} + ar^k$

$$= a\,\frac{(1-r^k)}{1-r} + ar^k = a\left[\frac{1-r^k+r^k-r^{k+1}}{1-r}\right] = a\left[\frac{1-r^{k+1}}{1-r}\right] = \text{R.H.S.}$$

So, result is true for n = k + 1.

Example 15 : Prove by induction that 21 divides $4^{n+1} + 5^{2n-1}$.

Sol. Basis : For n = 1

$4^{1+1} + 5^{2.1-1} = 4^2 + 5^1 = 16 + 5 = 21$ which divides 21

∴ P(1) is true.

Assumption : Let result is true for n = k.

i.e. 21 divides $4^{k+1} + 5^{2k-1}$ $\Rightarrow 4^{k+1} + 5^{2k-1} = 21m$

$\Rightarrow$ $5^{2k-1} = 21\ m - 4^{k+1}$ … (1)

Induction : Put n = k + 1

$4^{k+1+1} + 5^{2\,(k+1)-1} = 4^{k+1}\,.\,4^1 + 5^{2k-1}\,.\,5^2$

$= 4^{k+1}\,.\,4 + (21m - 4^{k+1})\ 25$ [Using (1)]

$= 4^{k+1}\,.\,4 + 21m\,.\,25 - 4^{k+1}.25 = 4^{k+1}\,(4-25) + 21m\,.\,25 = 4^{k+1}\,(-21) + 21m.25$

$= 21\,(-4^{k+1} + 25m)$ which is divisible by 21.

1.1.10 Summary

In this lesson, we have defined about sets, subsets, equal and equivalent sets, power set, universal set, order of a set etc.Further, we have illustrated the basic operations on sets using Venn diagrams. Some fundamental laws of set theory are stated and their proofs have been discussed. Moreover, the concepts of partitioning of sets, principal of duality and inclusion-exclusion principle have been elaborated. At last, the Principle of Mathematical Induction is explained.

1.1.11 Key Concepts

Set, Tabular form, Set-builder form, Singleton set, Finite set, Infinite set, Void set, Subset, Superset, Power set, Universal set, Equal sets, Equivalent sets, Comparable sets, Non-comparable sets, Order, Cardinality, Union, Intersection, Complement, Difference, Symmetric difference, Venn diagram, Disjoint sets, Idempotent law, Identity law, Commutative law, Associative law, Distributive laws, De-Morgan's laws, Cartesian product, Partitioning, Ordered pair, Minset, Normal form, Duality, Inclusion-Exclusion principle, Principle of Mathematical Induction

1.1.12 Long Questions

1. If A and B be non-empty subsets, then show that A × B = B × A iff A = B.
2. A class has a strength of 70 students. Out of it 30 students have taken Mathematics and 20 have taken Mathematics but not Statistics. Find
   (a) The number of students who have taken Mathematics and Statistics?
   (b) How many of them have taken Statistics but not Mathematics ?

3. In a town of 10,000 families, it was found that 40% families buy newspaper A, 20% buy newspaper B and 10% buy newspaper C. 5% families buy A and B, 3% buy B and C, and 4% buy A and C. If 2% families buy all the newspapers, find the number of families which buy (i) A only (ii) B only (iii) none of A, B, and C.
4. Among integers 1 to 300, how many of them are divisible neither by 3, nor by 5, nor by 7 ? How many of them are divisible by 3 but not by 5, nor by 7 ?

1.1.13 Short Questions

1. If A, B are two sets, then show that $A \cup B = \phi \Leftrightarrow A = \phi, B = \phi$.
2. Is it true that power set of $A \cup B$ is equal to union of power sets of A and B ? Justify.
3. Prove the following :
   (i) $A \cap (A^C \cup B) = A \cap B$
   (ii) $A - (B \cap C) = (A-B) \cup (A-C)$
   (iii) $A - (B \cup C) = (A - B) \cap (A - C)$
   (iv) $A \cap (B - C) = (A \cap B) - (A \cap C)$
4. Let A = [+, –], and B = [00, 01, 10, 11]
   (a) List the elements of $A \times B$
   (b) How many elements do A^4 and $(A \times B)^3$ have ?

1.1.14 Suggested Readings

1. Dr. Babu Ram, Discrete Mathematics
2. C.L. Liu, Elements of Discrete Mathematics (Second Edition), McGraw Hill, International Edition, Computer Science Series, 1986.
3. Discrete Mathematics, S. Series.
4. Kenneth H. Rosen, Discrete Mathematics and its Applications, McGraw Hill Fifth Ed. 2003.

LESSON NO. 1.2 **Author : Dr. Chanchal**
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LOGIC

Structure :

1.2.1 Objectives

In this lesson, our prime objectives are:

- To study about propositions/logical statements and basic operations on them with truth tables
- To discuss conditional and bi-conditional statements
- To study about an argument and its validity
- To discuss the concept of quantifiers

1.2.2 Introduction

Firstly, we introduce some basic terms associated with prepositional calculus.

Def. Sentence : It is sensible combination of words.

For example : Sun is a heavely body.

Def. Statement or Proposition : A statement is a declarative sentence which is either true or false but not both. The truth or falseness of a statement is called its truth value. In simple words, a statement is a sentence in the grammatical sense conveying a situation which is neither imperative, interrogative nor exclamatory.

For Example : (i) "May God bless you with happiness !". This sentence is not a statement because of its exclamation mark.

(ii) "$(x-1)^2 = x^2 = 2x + 1$". This is a statement and its truth value is T or 1. It should be noted that a mathematical identity is always a statement.

Now, a statement or preposition is of two types : simple and compound. Any statement whose truth or otherwise does not explicitly depend on another statement is said to be simple but a compound statement is combination of two or more simple statements. Moreover, the phrases or words which connect two simple statements are called logical connectives or simply connectives and some of these are "and", "or", "not", "if then", "if and only if".

For Example : "8 is an even number" is a simple statement while "If you work hard, then you will pass" is a compound statement.

The simple statements which are combined to form compound statements, are called components. Our problem is to determine the truth value of a compound statement from the truth values of their components and for this purpose, we draw truth table consisting of columns and rows. The number of columns depends upon the number of simple statements and relationships among them but number of rows depends only upon the number of simple statements. The truth tables are very helpful in finding out the validity of a report.

1.2.3 Logical Statements

There are various types of logical compound statements, which are discussed below :

I. Conjunction of Original Statements

Any two statements can be combined by the connective "and" to form compound statement called the "conjunction" of original statements.

For Example : The conjunction of "He is practical" and "He is sensitive" is "He is practical and sensitive". In symbols, if two statements are denoted by p, q then, their conjunction is denoted by p ^ q (read as "p and q").

Rule : p ^ q is true when p and q are true.

Truth Table for ^

p	q	p^q
T	T	T
T	F	F
F	T	F
F	F	F

Table 2.1

Example : Let p : 5 + 7 = 12 and q : 2 is a prime number.

∴ p ^ q : 5 + 7 = 12 and 2 is a prime number. Now, p and q both are true, therefore p ^ q is true.

II. Disjunction of Original Statements

Any two statements can be conbined by the connective "or" to form compound statement called the "Disjunction" of original statements.

For Example : The disjunction of "I shall watch the game on television" and "I shall go to college" is "I shall watch the game on television or go to college".

In symbols, the disjunction of two statements p and q is denoted by p v q (read as "p or q")

Rule : p v q is false when both p and q are false otherwise it is true.

Truth Table for V

p	q	p v q
T	T	T
T	F	T
F	T	T
F	F	F

Example : Let p : 5 < 12 and q : 8 + 3 = 12.

∴ p v q : 5 < 12 or 8 + 3 = 12.

Here p is true and q is false. Therefore, p v q is true.

III. Negation (or Denial) of a Statement

To every statement, there corresponds a statement which is its negation that refers to contradiction. The best way to write the negation of given statement is to put in the word "not" at the proper place or to put the phrase "It is not the case that" in the beginning. Negation of a statement p is denoted by "~p".

For Example : If p : He is a good student. Then, ~p : He is not a good student or It is not the case that he is a good student. We cannot say that "He is a bad student" is the negation of p.

Rule : If p is true, then ~p is false and vice versa.

Truth Table for ~

p	~ p
T	F
F	T

IV. Conditional Statement

Let p and q be two statements. Any statement of the form "if p then q" is called a conditional statement. It is denoted by $p \to q$ (read as p conditional q or p implies q). Here, p is sufficient for q but not essential i.e. there can be q, even without p.

For example : Let p : you work hard and q : you will pass. Now, $p \to q$: If you work hard, then you will pass.

Rule : $p \to q$ is true in all cases except when p is true and q is false.

Truth Table for →

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

V. Biconditional Statement or Equivalence

Let p and q be two statements. Any statement of the form "p if and only if q" is called a biconditional statement, denoted by $p \leftrightarrow q$.

Rule : $p \leftrightarrow q$ true if both p and q have the same truth value and false if p and q have apporite truth values.

Truth Table of ↔

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

VI. Converse, Inverse and Contrapositive

If $p \to q$ is a direct statement, then

(i) $q \to p$ is called its converse,

(ii) $\sim p \to \sim q$ is called its inverse

and (iii) $\sim q \to \sim p$ is called its contrapositive

Note : Since $p \to q = \sim q \to \sim q$, $\therefore$ contrapositive ≡ direct statement

and $q \to p = \sim p \to \sim q$, $\therefore$ converse ≡ inverse

Exercise : Write truth tables for $q \to p$, $\sim p \to \sim q$ and $\sim q \to \sim p$.

VII. Dual of a Statement

As we know that dual relationship between 'line' and 'point' exists through the interchange of the words 'meet' and 'join'.

For Example : Dual of "A line is the join of two points" is "A point is the meet of two lines". Similarly, to find the dual of any statement in logic, we first interchange v and ^.

For Example : Dual of $\sim (p \vee q) = (\sim p) \wedge (\sim q)$ is $\sim(p \wedge q) = (\sim p) \vee (\sim q)$.

VIII. Tautologies and Contradictions (or Fallacies)

A tautology is a proposition which is true for all the truth values of its components and a contradiction is a porposition which is false for all the truth values of its components.

For Example : $p \vee \sim p$ is a tautology and $p \wedge \sim p$ is a contradiction, as shown below.

Truth Table

p	~p	p v ~p	p ^ ~p
T	F	T	F
F	T	T	F

Art 1 : Prove De-Morgan's laws using conjuction and disjunction.

Or

Prove that (i) $\sim (p \wedge q) = \sim p \vee \sim q$

(ii) $\sim (p \vee q) = \sim p \wedge \sim q$

Proof : **Truth Table**

(i)

p	q	~ p	~ q	p ^ q	~ (p ^ q)	~ p v ~ q
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

As the last two columns of truth table are same, therefore $\sim (p \wedge q) = \sim p \vee \sim q$.

(ii) Do Yourself.

Art 2 : Prove that : (i) $p \rightarrow q = (\sim p) \vee q$

(ii) $\sim (p \rightarrow q) = p \wedge \sim q$

Proof : (i)

p	q	~p	$p \to q$	(~p) v q
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Since the last two columns are same, therefore $p \to q$ = (~p) v q

(ii) Do Yourself.

Example 1 : Write down the truth table for the statement $(p \to q) \leftrightarrow (\sim p \vee q)$

Sol.

p	q	$p \to q$	~p	~pvq	$(p \to q) \leftrightarrow (\sim p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Example 2 : (i) If p stands for the statement 'I do not like chocolates' and q for the statement 'I like ice-cream', then what does ~p^q stand for ?

(ii) If p stands for the statement, 'I will not go to school' and q for the statement, 'I will watch a movie', then what does ~p v q stand for ?

Sol. : (i) p : I do not like chocolates, q : I like ice-cream, ~p^q: I like chocolates and ice-cream.

(ii) p : I will not go to school, q : I will watch a movie, ~p v q : Either I will go to school or I will watch a movie.

Example 3 : Write the following statement in symbolic form and give its negation : If it rains, he will not go to school.

Sol. : Let p : It rains, q : He will go to school

$\therefore$ symbolic expression is $p \to \sim q$

Its negation is $\sim(p \to \sim q) = \sim(\sim p \vee \sim q) = \sim(\sim p) \wedge \sim(\sim q) = p \wedge q$

In words : Even if it rains, he will go to school.

Example 4 : Prove that if $p \to q$ and $q \to r$, then $p \to r$.

Sol. : Here we are given that $p \to q$, $q \to r$ and we have to prove that $p \to r$. The result will be established if we show that $[(p \to q) \wedge (q \to r)] \to (p \to r)$ is a tautology.

Truth Table

p	q	r	$p \to q$	$q \to r$	$p \to r$	$(p \to q) \wedge (q \to r)$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
T	T	T	T	T	T	T	T
T	F	T	F	T	T	F	T
F	T	T	T	T	T	T	T
F	F	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	F	F	T	F	F	T
F	T	F	T	F	T	F	T
F	F	F	T	T	T	T	T

So, if $p \to q$ and $q \to r$, then $p \to r$.

Example 5 : State the converse and contrapositive of the implication "If it snows tonight, then I will stay at home."

Sol. : Let p : It snows tonight, q : I will stay at home. Converse of statement $p \to q$ is $q \to p$ i.e., "If I stay at home then it snows tonight."

Contrapositive of statement $p \to q$ is $\sim q \to \sim p$ i.e., "If I do not stay at home then it will not snow tonight".

1.2.4 Validity of Arguments

Firstly, we define an argument as :

Def. Argument : An argument is a statement which assets that given set of propositons $p_1, p_2, p_3, \ldots\ldots\ldots, p_n$ taken together gives another proposition P. These are expressed as $p_1, p_2, p_3, \ldots\ldots\ldots p_n /\!-P$. The sign "/–" is spoken as turnstile. The porpositions $p_1, p_2, p_3, \ldots\ldots\ldots\ldots p_n$ are called "premises" or "assumptions" and P is called the "conclusion".

Valid Argument :- An argument $p_1, p_2, p_3, \ldots\ldots\ldots\ldots p_n /\!-P$ is true whenever all the premises $p_1, p_2, p_3, \ldots\ldots\ldots\ldots p_n$ are true, otherwise the argument is false. A true argument is called valid argument and a false argument is called a fallacy. The validity can also be judged by the relationship $p_1 \wedge p_2 \wedge p_3 \ldots\ldots\ldots\ldots\ldots \wedge p_n \to P$ provided it is a tautology.

Example 6 : Test the validity of :

Unless we control population, all advances resulting from planning will be nullified. But this must not be allowed to happen. Therefore we must somehow control population.

Sol. : Let the symbols for the statements be :

p : we control the population

q : all advances resulting from planning are nullified.

∴ the argument is ~p → q, ~ q/ –p

Truth Table

p	q	~ p	~ q	~ p → q	(~ p → q) ^ ~ q	[(~p → q) ^ ~ q] → p
T	T	F	F	T	F	T
T	F	F	T	T	T	T
F	T	T	F	T	F	T
F	F	T	T	F	F	T

Since [(~p → q) ^ ~ q) → is tautology.

∴ the given argument is valid.

Example 7 : Check the validity of argument :

If I work, I cannot study. Either I work or pass mathematics.

I passed mathematics. Therefore, I study.

Sol. : Let p : I work, q : I sudy, r : I pass mathematics

The given statement is [(p → ~ q) ^ (p V r) ^ (r)] → q

Truth Table

						I	II
p	q	r	~q	p → ~q	p V r	(p→ ~q) ^ (p V r) ^ (r)	1→q
T	T	T	F	F	T	F	T
T	T	F	F	F	T	F	T
T	F	T	T	T	T	T	F
T	F	F	T	T	T	F	T
F	T	T	F	T	T	T	T
F	T	F	F	T	F	F	T
F	F	T	T	T	T	T	F
F	F	F	T	T	F	F	T

The given statement is not a tautology.

So, argument is not valid.

1.2.5 Proposition Generated by a Set

Let S be any set of propositions. A proposition generated by S is any valid combination of propositions in S with conjunction, disjunction and negation.

Note : The conditional and biconditional operators are not included as they can be obtained from conjunction, disjunction and negation.

Equivalence

Let S be a set of propositions and p, q be propositions generated by S. p and q are equivalent if $p \leftrightarrow q$ is a tautology. The equivalence of p and q is denoted by $p \leftrightarrow q$.

Implication

Let S be a set of propositions and p, q be propositions generated by S. p implies q if $p \rightarrow q$ is a tautology. $p \Rightarrow q$ is written to indicate the implication.

Laws of Logic

Here 0 stands for cortradiction, 1 for tautology.

Commutative Laws

$p \vee q \Leftrightarrow q \vee p$ $\qquad$ $p \wedge q \Leftrightarrow q \wedge p$

Associative Laws

$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$ $\qquad$ $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

Distributive Laws

$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $\qquad$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

Identity Laws

$p \wedge 0 \Leftrightarrow p$ $\qquad$ $p \vee 1 \Leftrightarrow p$

Negation Laws

$p \wedge \sim p \Leftrightarrow 0$ $\qquad$ $p \vee \sim p \Leftrightarrow 1$

Idempotent Laws

$p \vee p \Leftrightarrow p$ $\qquad$ $p \vee p \Leftrightarrow p$

Null Laws

$p \wedge 0 \Leftrightarrow 0$ $\qquad$ $p \vee 1 \Leftrightarrow 1$

Absorbtion Laws

$p \wedge (p \vee q) \Leftrightarrow p$ $\qquad$ $p \vee (p \wedge q) \Leftrightarrow p$

DeMorgan's Laws

$\sim (p \vee q) \Leftrightarrow (\sim p) \wedge (\sim q)$ $\qquad$ $\sim (p \wedge q) \Leftrightarrow (\sim p) \vee (\sim q)$

Involution Laws

$\sim (\sim p) \Leftrightarrow p$

Common Implication and Equivalence

Detachment

$(p \rightarrow q) \wedge p \Rightarrow q$

Contrapositive

$(p \rightarrow q) \wedge \sim q \Rightarrow \sim p$

Disjunctive Additon

$p \Rightarrow (p \vee q)$

Conjunctive Simplification

$(p \wedge q) \Rightarrow p$ and $(p \wedge q) \Rightarrow q$

Disjunctive Simplification

$(p \vee q) \wedge \sim p \Rightarrow q$ and $(p \vee q) \wedge \sim q \Rightarrow p$

Chain Rule

$(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow p \rightarrow r$

Conditional Equivalences

$(p \rightarrow q) \Leftrightarrow (\sim q \rightarrow \sim p) \Leftrightarrow (\sim p \vee q)$

Biconditional Equivalences

$(p \leftrightarrow q) \Leftrightarrow ((p \rightarrow q) \wedge (q \rightarrow q)) \Leftrightarrow ((p \wedge q) \vee (\sim p \wedge \sim q))$.

Note : All the above laws, implications and equivalences can be proved very easily with the help of truth tables.

1.2.6 Proposition Over a Universe

Let U be a non-empty set. A proposition over U is a sentence that contains a variable that can take on any value in U and which has a definite truth value as a result of any such substitution.

For Example : Consider $7x^2 - 6x = 0 \Rightarrow x(7x - 6) = 0$

$$\Rightarrow \quad x = 0, \frac{7}{6}$$

If we take **Q** as universe, then truth set (i.e., solution set) of $7x^2 - 6x = 0$ is $\left\{0, \frac{7}{6}\right\}$.

If we take **Z** as universe, then truth set of $7x^2 - 6x = 0$ is $\{0\}$.

Truth Set If p (n) is a proposition over U, then the truth set of p (n) is

$$T_{p(n)} = \{a \in U / p(a) \text{ is true}\}$$

Consider the set {1, 2, 3, 4}

Its power set is $\{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$

Let proposition be $\{1, 2\} \cap A = \phi$

$\therefore$ truth set of proposition taken over the power set of {1, 2, 3, 4} is $\{\phi, \{3\}, \{4\}, \{3, 4\}\}$.

Tautology and contradiction : A proposition over U is a tautology if its truth set is U. It is a contradiction if its truth set is empty.

Equivalence : Two propositions are equivalent if $p \leftrightarrow q$ is a tautology. In other words, p and q are equivalent if $T_p = T_q$.

Example : $x + 7 = 12$ and $x = 5$ are equivalent propositions over the integers.

Implication : If p and q are propositions over U, then p implies q if $p \rightarrow q$ is a tautology. In other words $p \rightarrow q$ when $T_p \subseteq T_q$.

Example : Over the natural numbers,

$$n \leq 3 \rightarrow n \leq 8 \text{ as } \{0, 1, 2, 3\} \subseteq \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$$

Truth Set of Compound Propositions

The truth sets of compound propositions can be expressed in terms of the truth sets of simple propositions. The following list gives the connection between compound and simple truth sets :

1. $T_{p \wedge q} = T_p \cap T_q$
2. $T_{p \vee q} = T_p \cup T_q$
3. $T_{\sim p} = T_p^{\ c}$
4. $T_{p \leftrightarrow q} = \left(T_p \cap T_q\right) \cup \left(T_p^{\ c} \cap T_q^{\ c}\right)$
5. $T_{p \rightarrow q} = T_p^{\ c} \cup T_q$.

1.2.7 Quantifiers

If p(n) is a propositions over U with $T_{p(n)} \neq \phi$, then we say "There exists an n in U such that p (n) is true." We abbreviate this sentence as $(\exists\ n)_t$, (p(n)). $\exists$ is known as **existential quantifier**.

It is clear that if p (n) is a proposition over a universe U, its truth set $T_{p\ (n)}$ is a subset of U.

For Examples :

(i) $(\exists\ k)_z$, $(5k = 100)$ means that there is an integer k such that 100 is a multiple of 5. This is true.

(ii) $(\exists\ x)_Q$, $(x^2 - 3 = 0)$ means that there is a rational number x such that $x^2 = 3$. This is false as the solution set of the equation $x^2 - 3 = 0$ over **Q** is empty. We write it as $((\nexists\ x)_Q, (x^2 - 3 = 0)$

If p (n) is a proposition over U, with $T_{p(n)} = U$. Them we say "for all n in U, p(n) is true." We abbreviate this as $(\forall\ n)_t$, (p (n)). $\forall$ is known as universal quantifier.

Negation of Quantified Proposition

When we negate a quantified proposition, then the universal and existential quantifiers become complement of one another. In simple words, negation of an existentially quantified proposition is a universally quantified proposition and negation of a universally quantified proposition is an existentially quantified proposition. In symbols,

$\sim (\forall\, n)_U\ (p\,(n)) \Leftrightarrow (\exists\, n)_U\ (\sim p\,(n))$

and $\sim (\exists\, n)_U\ (p(n)) \Leftrightarrow (\forall\, n)_U\ (\sim p\,(n))$.

Example 8 : Over the universe of positive integers : p(n) : n is prime and n < 32, q (n) : n is power of 3, r (n) : n is a divisor of 27.

(a) What are the truth sets of these propositions ?

(b) Which of the three propositions implies one of the others ?

Sol. We have

(a) $T_p = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31\}$

$T_q = \{1, 3, 6, 9, 12, 15, 18, 21, ...\}$

$T_r = \{1, 3, 9, 27\}$

(b) Since $T_r \subseteq T_q$

$\therefore$ r implies q.

Example 9 : Translate in your own words and indicate whether it is true or false that : $(\exists\, x)_Q\,(3x^2 - 12 = 0)$

Sol. : Consider $3x^2 - 12 = 0$

$\therefore$ $x^2 - 4 = 0 \Rightarrow x^2 = 4 \Rightarrow x = -2, 2$ which are rational numbers

$\therefore$ the equation $3x^2 - 12 = 0$ has a solution in rationals is true.

Example 10 : Use quantifier to say that $\sqrt{3}$ is not a rational number.

Sol. : $\sim (\exists\, x)_Q\,(x^2 = 3)$.

1.2.8 Summary

In this lesson, we have discussed about propositions and its basic operations such as conjunction, disjunction, negation, conditional statements and bi-conditional statements. We have also defined an argument and checked whether its logically valid or invalid. An important structure i.e. quantifier and its types are also discussed in brief. We tried to elaborate the concepts with the help of suitable examples.

1.2.9 Key Concepts

Proposition, Logical statement, Simple statement, Compound statement, conjunction, disjunction, negation, conditional statements, bi-conditional statements, Argument, Tautology, Contradiction, Truth Table, Quantifier.

1.2.10 Long Questions

1. Prove that $(p \leftrightarrow q) \leftrightarrow r = p \leftrightarrow (q \leftrightarrow r)$
2. Prove that $p \to (\sim q \vee r) \equiv (p \wedge q) \to r$.
3. Test the Validity of : "If my borher stands first in the class, I will give him a watch. Either he stood first or I was out of station. I did not give my brother a watch this time. Therefore I was out of station."

1.2.11 Short Questions

1. Write down the truth table for

   (i) $p \wedge (q \rightarrow p)$ (ii) $p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$

   (iii) $[p \rightarrow (q \vee r)] \vee [p \leftrightarrow \sim r]$

2. If p stands for the statement, 'I like tennis', and q stands for the statement, 'I like football', then what does $\sim p \wedge \sim q$ stand for ?

1.2.12 Suggested Readings

1. Dr. Babu Ram, Discrete Mathematics
2. C.L. Liu, Elements of Discrete Mathematics (Second Edition), McGraw Hill, International Edition, Computer Science Series, 1986.
3. Discrete Mathematics, S. Series.
4. Kenneth H. Rosen, Discrete Mathematics and its Applications, McGraw Hill Fifth Ed. 2003.

Bachelor of Computer Application Part-II **PAPER : BCAB2102T**

DISCRETE MATHEMATICS

LESSON NO. 1.3 **Author : Dr. Chanchal**

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RELATIONS

Structure :

1.3.1 Objectives

In this lesson, we are going to study:

- Relations and its types describing an equivalence relation
- Representations of relations using different illustrative technique
- Composition of relations

1.3.2 Introduction

As we have already studied about the cartesian product of two sets in set theory, discussed in Lesson No. 1.1. In continuation to that, we can define a relation as :

Def. Relation : A relation from a set A to a set B is defined as a subset of A × B. Therefore each subset of A × B is a relation from A to B. If R is a relation from a set A to set B and if (a, b) ∈ R for some a ∈ A and b ∈ B, then we say that a is related to b and we write it as a R b. If (a, b) ∉ R then we say that a is not related to b and we write it as a $\not R$ b.

Domain and Range of a Relation

If R is a relation from a set A to a set B. Then the set of the first components of the elements of R is called the domain of R and the set of the second components of the elements of R is called the range of R.

Thus, domain of R = { a : (a, b) ∈ R}, and range of R = {b : (a, b) ∈ R}.

If R is a relation from a set A to the set A, then R is called a relation on A. Thus a relation on a set A is defined as any subset of A × A.

For Example : For any a, b ∈ N, the set of natural numbers, define a relation R by a R b if a divides b.

Then, R = { (1, 1), (1, 2), (1, 3), ..., (2, 2), (2, 4),.......(3, 3), (3, 6),........}

R is clearly a subset of N × N and hence a relation on N.

Here, (1, 2) ∈ R since 1 divides 2 but (2, 1) ∉ R since 2 does not divide 1.

1.3.3 Types of Relation

I. Reflexive Relation

Def. : A relation R on a set A is called a reflexive relation if (x, x) ∈ R for all x ∈ A i.e., if x R x for every x ∈ A.

For Example : Let A = {1, 2}.

Then A × A = {(1, 1), (1, 2), (2, 1), (2, 2)}.

Let R = {(1, 1), (2, 2), (1, 2)}.

Clearly R ⊆ A × A and so R is a relation on the set A.

Since (x, x) ∈ R ∀ x ∈ A, so R is a reflexive relation on A.

II. Symmetric Relation

Def. : A relation R on a set A is called a symmetric relation if a R b ⇒ b R a where a, b ∈ A i.e., if (a, b) ∈ R ⇒ (b, a) ∈ R where a, b ∈ A.

For Example : Let A = {1, 2, 3}

Then A × A = { (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3,), (3, 2), (3, 3)}

Let R = {(1, 1), (1, 3), (3, 1)}

Clearly, R ⊆ A × A and therefore, R is a relation on A.

Since (x, y) ∈ R ⇒ (y, x) ∈ R, therefore R is a symmetric relation on A.

III. Transitive Relation

Def. : A relation R on a set A is called a transitive relation if

aRb, b R c ⇒ a R c ∀ a, b, c, ∈ R,

i.e, if (a, b) ∈ R and (b, c) ∈ R ⇒ (a, c) ∈ R where a, b, c ∈ A.

For Example : For a, b ∈ N, the set of natural numbers, define a R b if 2a + b=10.

The natural numbers a and b satisfying the relation 2a + b = 0 are given by :

a = 1, b = 8, a = 2, b = 6, a = 3, b = 4, a = 4, b = 2

∴ R = {(1, 8), (2, 6), (3, 4), (4, 2)}

Since (3, 4) ∈ R and (4, 2) ∈ R but (3, 2) ∉ R. Therefore R is not a transitive relation.

IV. Anti-Symmetric Relation

Def. : A relation R on a set A is called an anti-symmetric relation if a R b and b R a implies that a = b.

i.e., if $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$.

Note : Identity relation on a set is symmetric as well as anti-symmetric.

For Example : For $a, b \in N$, the set of natural numbers define a R b if $a \leq b$.

Let $a, b \in N$ such that a R b and b R a.

$\therefore$ $a \leq b$ and $b \leq a$. $\Rightarrow$ $a = b$.

$\therefore$ R is an anti-symmetric relation.

V. Equivalence Relation

Def. : A relation R on a set A is called an equivalence relation if R is reflexive, symmetric and transitive.

For Example : Let X be the set of all triangles in a plane.

For any two triangles Δ_1 and Δ_2 in X define $\Delta_1 R \Delta_2$, if Δ_1 and Δ_2 are congruent triangles. Then

(i) R is Reflexive: Since each triangle is congruent to itself, so $\Delta R \Delta$ for each Δ in X.

(ii) R is Symmetric : Let Δ_1 and $\Delta_2 \in X$ such that $\Delta_1 R \Delta_2$. Then Δ_2 and Δ_1 are congruent triangles. Hence $\Delta_2 R \Delta_1$.

(iii) R is Transitive : Let $\Delta_1, \Delta_2, \Delta_3 \in X$ such that Δ_1 and Δ_2 and $\Delta_2 R \Delta_3$. Then Δ_1, Δ_2 are congruent triangles and so are Δ_2 and Δ_3. This implies that the Δ_1 and Δ_3 are also congruent triangles. Hence $\Delta_1 R \Delta_3$.

So, R is reflexive, symmetric and transitive.

Therefore, R is an equivalence relation on X.

VI. Partial-Order Relation : A relation R on a set A is called partial order relation if it is reflexive, anti-symmetric and transitive.

For Example : For $a, b \in N$, the relationR defined by a R b if $a \leq b$, is partial-order relation.

VII. Some other Relations on a Set

Def. Void Relation : Since ϕ is a subset of $A \times A$, therefore the null set ϕ is also a relation in A, called the void relations in a set A.

Universal relation in a set: Let A be any set and R be the set $A \times A$. Then R is called the universal relation in A.

Identity relation in a set : Let A be any set. Then the relation R defined by

$R = \{(a, a) : \text{for all } a \in A\}$ is called identity relation in A. It is usually denoted by I_A.

Compatible Relation. A relation R in A is said to be compatible relation if it is reflexive and symmetric.

VIII. Inverse of a Relation

The inverse of a relation R, denoted by R^{-1}, is obtained from R by interchanging the first and second components of each ordered pair of R.

Therefore, $R^{-1} = \{ (a, b) : (b, a) \in R\}$.

If R is a relation from a set A to set B, then R^{-1} is relation from the set B to the set A.

$\therefore$ Domain of R^{-1} = Range of R and Range of R^{-1} = Domain of R.

For Example : Let A = {1, 2, 3} and Let R = { (1, 2), (1, 3), (2, 3), (3, 2)}.

Then R is a relation on the set A, since $R \subseteq A \times A$.

$\therefore$ $R^{-1} = \{ (2, 1), (3, 1), (3, 2), (2, 3)\}$.

Example 1 : Give an example of a relation which is reflexive but neither symmetric nor transitive.

Sol. Let A = {2, 3, 4}.

Then $A \times A = \{ (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)\}$

Let R = {(2, 2), (3, 3), (4, 4), (2, 3), (4, 3), (3, 4)}

Since $R \subseteq A \times A$, therefore R is a relation on A.

R is reflexive since $(a, a) \in R \ \forall \ a \in A$.

R is not symmetric since $(2, 3) \in R$ but $(3, 2) \notin R$.

R is not transitive since (2, 3) and $(3, 4) \in R$ but $(2, 4) \notin R$.

Further R is not anti-symmetric since (3, 4) and $(4, 3) \in R$ but $3 \neq 4$.

Example 2 : Give an example of a relation which is symmetric but neither reflexive nor transitive.

Sol. Let A = {1, 2}.

Then $A \times A = \{ (1, 1), (1, 2), (2, 1), (2, 2)\}$.

Let R = { (1, 2), (2, 1)}.

Then $R \subseteq A \times A$ and hence R is a relation on the set A.

R is symmetric since $(a, b) \in R \Rightarrow (b, a) \in R$.

R is not reflexive since $1 \in A$ but $(1, 1) \notin A$.

R is not transitive since $(1, 2) \in R$, $(2, 1) \in R$ but $(1, 1) \in R$ is notanti-symmetric since $(1, 2) \in R$ and $(2, 1) \in R$ but $1 \neq 2$.

Example 3 : The relation $R \subseteq N \times N$ is defined by $(a, b) \in R$ if and only if 5 divides b – a. Show that R is an equivalence relation.

Sol. The relation $R \subseteq N \times N$ is defined by $(a, b) \in R$ if and only if 5 divides b – a.

This means that R is a relation on N defined by, if $a, b \in N$ then $(a, b) \in R$ if and only if 5 divides b – a.

Let a, b, c belongs to N. Then

(i) $a - a = 0 = 5 . 0.$

$\therefore$ 5 divides a – a.

$\Rightarrow$ $(a, a) \in R$. $\Rightarrow$ R is reflexive.

(ii) Let $(a, b) \in R$.

$\therefore$ 5 divides $a - b$.

$\Rightarrow$ $a - b = 5\,n$ for some $n \in N$. $\Rightarrow$ $b - a = 5\,(-n)$.

$\Rightarrow$ 5 divides $b - a$ $\Rightarrow$ $(b, a) \in R$.

$\therefore$ R is symmetric.

(iii) Let (a, b) and $(b, c) \in R$.

$\therefore$ 5 divides $a - b$ and $b - c$ both

$\therefore$ $a - b = 5n_1$ and $b - n = 5\,n_2$ for some n_1 and $n_2 \in N$

$\therefore$ $(a - b) + (b - c) = 5n_1 + 5n_2$ $\Rightarrow a - c = 5\,(n_1 + n_2)$

$\Rightarrow$ 5 divides $a - c$

$\Rightarrow$ $(a, c) \in R$

$\therefore$ R is transitive relation in N.

Example 4 : Prove that the intersection of two equivalence relations on a non-empty set is again an equivalence relation on that set.

Sol. Suppose that R_1 and R_2 are two equivalence relations on a non-empty set X.

First we prove that $R_1 \cap R_2$ is an equivalence relation on X.

(i) $R_1 \cap R_2$ is reflexive :

Let $a \in X$ arbitrarily.

Then $(a, a) \in R_1$ and $(a, a) \in R_2$, since R_1 , R_2 both being equivalence relations are reflexive.

So, $(a, a) \in R_1 \cap R_2$

$\Rightarrow$ $R_1 \cap R_2$ is reflexive.

(ii) $R_1 \cap R_2$ is symmetric :

Let $a, b \in X$ such that $(a, b) \in R_1 \cap R_2$

$\therefore$ $(a, b) \in R_1$ and $(a, b) \in R_2$

$\Rightarrow$ $(b, a) \in R_1$ and $(b, a) \in R_2$, since R_1 and R_2 being equivalence relations are also symmetric.

$(b, a) \in R_1 \cap R_2$

$(a, b) \in R_1 \cap R_2$ implies that $(b, a) \in R_1 \cap R_2$.

$\therefore$ $R_1 \cap R_2$ is a symmetric relation.

(iii) $R_1 \cap R_2$ is transitive :

Let $a, b, c \in X$ such that $(a, b) \in R_1 \cap R_2$ and $(b, c) \in R_1 \cap R_2$.

$(a, b) \in R_1 \cap R_2 \Rightarrow$ $(a, b) \in R_1$ and $(a, b) \in R_2$ ... (i)

$(b, c) \in R_1 \cup R_2$ $\Rightarrow$ $(b, c) \in R_1$ and $(b, c) \in R_2$ ... (ii)

(i) and (ii) $\Rightarrow$ (a, b) and $(b, c) \in R_1$

$\Rightarrow$ $(a, c) \in R_1$, since R_1 being an equivalence relation is also trasnsitive.

Similarly, we can prove that $(a, c) \in R_2$.

$\therefore$ $(a, c) \in R_1 \cap R_2$

So, $R_1 \cap R_2$ in transitive.

Thus $R_1 \cap R_2$ is reflexive, symmetric and also transitive. Thus $R_1 \cap R_2$ is an equivalence relation.

Example 5 : If R is an equivalence relation on a set A, then so is R^{-1}

Sol. Let a, b, c $\in$ A. Then

(i) (a, a) $\in$R, since R being equivalence relation is also a symmetric relation

$\Rightarrow$ (a, a) $\in R^{-1}$ $\Rightarrow$ R^{-1} is reflexive

(ii) Let (a, b) $\in R^{-1}$ $\Rightarrow$ (b, a) $\in$ R

$\Rightarrow$ (a, b) $\in$ R, since R is symmetric

$\Rightarrow$ (b, a) $\in R^{-1}$

$\therefore$ (a, b) $\in R^{-1}$ $\Rightarrow$ (b, a) $\in R^{-1}$.

So, R^{-1} is also symmetric

(iii) Let (a, b) and (b, c) $\in R^{-1}$

$\therefore$ (b, a), (c, b) $\in$ R.

$\Rightarrow$ (c, b), (b, a) $\in$ R.

$\Rightarrow$ (c, a) $\in$ R, since R is transitive.

$\Rightarrow$ (a, c) $\in R^{-1}$

$\therefore$ R^{-1} is also transitive.

$\therefore$ R^{-1} is an equivalence relation.

1.3.4 Composition of Relations

Def. : Let A, B and C be sets and let R be a relation from A to B and let S be a relation from B to C. That is, R is a subset of A × B and S is a subset of B × C. There R and S give rise to a relation from A to C denotes by RoS and defined by

a(RoS) c if for some b $\in$ B we have aRb and bSc

That is RoS = {(a, c) : there exists b $\in$ B for which (a, b) $\in$ R and (b, c) $\in$ S}

The relation RoS is called the composition of R and S ; it is sometimes denoted simply by RS, RoR is denoted by R^2, R^3 = RoRoR.

For Example : Let R and S defined on A be

R = {(1, 1), (3, 1), (3, 4), (4, 2), (4, 3)}

S = {(1, 3), (2, 1), (3, 1), (3, 2), (4, 4)}

Now, RoS = }(1, 3), (3, 3), (3, 4), (4, 1), (4, 2)}

R^3 = {(1, 1), (3, 1), (3, 4), (4, 1), (4, 2)}.

1.3.5 Closures of Relation

Let R be a relation in a set A. R may not satisfy particular property like reflexivity, symmetry or transitivity. The new relation, obtained after adding least number of new pairs so that R satisfies particular property, is called closure of R. The types of closures are discussed below :

Reflexive Closure : Let R be a relation on A. A reflexive closure of R is the smallest reflexive relation that contains R.

Symmetric Closure : Let R be a relation on A which is not symmetric.

$\therefore$ there exists $(a, b) \in R$ but $(b, a) \notin R$

Now $(b, a) \in R^{-1}$

$\therefore$ to make R symmetric, we add all pairs of R^{-1}.

$\therefore$ $R \cup R^{-1}$ is symmetric closure of R.

If R is a relation on A which is not symmetric. Then $R \cup R^{-1}$ is symmetric closure of R.

Transitive Closure : Let A be a set and R be a relation on A. The transitive closure of R, denoted by R^+, is the smallest relation which contains R as a subset and which is transitive.

Another Definition : Let A be a set and R be a relation on A. The relation $R^+ = R \cup R^2 \cup R^3 \ldots\ldots\ldots$ in A is called the transitive closure of R in A.

Example 6 : Let R be a relation on a set A = {1, 2, 3} defined by R = {(1, 1), (1, 2), (2, 3)}. Find the reflexive closure of R and symmetric closure of R.

Sol. A = {1, 2, 3}

R = {(1, 1), (1, 2), (2, 3)}

R^{-1} = { (1, 1), (2, 1), (3, 2)}

$R \cup R^{-1}$ = {(1, 1), (1, 2), (2, 1), (2, 3), (3, 2)}

Reflexive closure of R is {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)}

Symmetric closure of R is $R \cup R^{-1}$ = {(1, 1), (1, 2), (2, 1), (2, 3), (3, 2)}.

Example 7 : Let R be a relation on set A = {1, 2, 3, 4} defined by

R = {(1, 2), (2, 3), (3, 4), (2, 1)}. Find transitive closure of R.

Sol. A = {1, 2, 3, 4}

R = {(1, 2), (2, 3), (3, 4), (2, 1)}

$$\therefore \quad M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ where m is matrix of R}$$

$$M^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M^3 = M^2M = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M^4 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \quad M^4 = M + M^2 + M^3 + M^4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ R^+ = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4)}
which is transitive closure of R.

1.3.6 Equivalence Class

Consider, an equivalence relation R on a set A. The equivalence class of an element a $\in$ A, is the set of elements of A to which element a is related. It is denoted by [a].

For Example : Let A = {4, 5, 6, 7} and R = {(4, 4), (5, 5), (6, 6), (7, 7), (4, 6), (6, 4)} be an equivalence relation on A.

Now, equivalence classes are as follows :

[4] = [6] = {4, 6}

[5] = {5}

[7] = {7}

Results : (i) Suppose that R is an equivalence relation on a set X,

Then (I) a $\in$ [a] $\forall$ a $\in$ X.

(II) a $\in$ [b] iff [a] = [b] $\forall$ a, b $\in$ X

(III) [a] = [b] or [a] $\cap$ [b] = ϕ $\forall$ a, b $\in$ X, i.e. any two equivalence classes are disjoint or identical.

(ii) The distinct equivalence classes of an equivalence relation on a set form a partition of that set.

1.3.7 Representing Relations

In order to represent a relation, there are numerous ways such as matrix representation, graphical representation, arrow diagram, diagraph or directed graph and Hasse diagram. All these may be understood from the following examples :

Example 8 : If A = {1, 2, 3, 4} and B = {x, y, z}. Let R be the following relation from A to B : R = (1, y), (1, z), (3, y), (4, x), (4, z).

(a) Determine the matrix of the relation

(b) Draw the arrow diagram of R

Sol. (a) From the fig. 1 Observe that rows of the matrix are labeled by the elements of A and the columns by the elements of B. Also observe that entry in the matrix corresponding to a ∈ A and b ∈ B is 1 if a is related to b and 0 otherwise.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array}$$

Fig. 1

(b) From fig. 2, Observe that there is an arrow from a ∈ A to b ∈ B iff a is related to b i.e. iff (a, b) ∈ R.

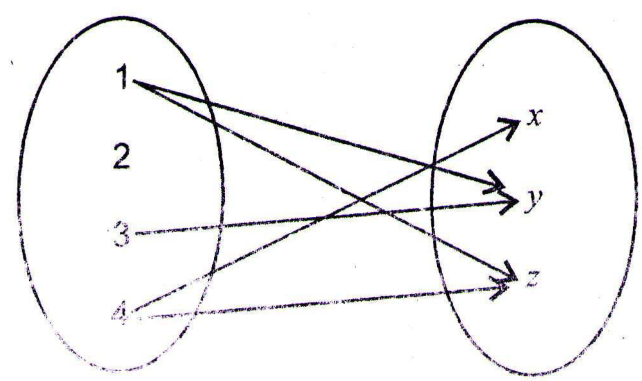


Fig. 2

Example 9 : If A = {1, 2, 3, 4}. Consider the following relation in A

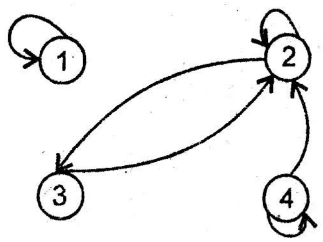
R = {(1, 1), (2, 2), (2, 3), (3, 2), (4, 2), (4, 4)}

(a) Draw its directed graph.

(b) Is R (i) reflexive, (ii) symmetric (iii) transitive or (iv) antisymmetric

(c) R^2 = RoR

Sol. (a)



(b) (i) R is not reflexive $3 \in A$ but $3 \not R\ 3$ i.e. $(3, 3) \notin R$.

(ii) R is not symmetric because 4 R 2 but $2 \not R\ 4$

i.e. $(4, 2) \in R$ but $(2, 4) \notin R$.

(iii) R is not transitive because 4 R 2 and 2 R 3 but $4 \not R\ 3$ i.e. $(4, 2) \in R$ and $(2, 3) \in R$ but $(4, 3) \notin R$.

(iv) R is not anti-symmetric because 2 R 3 and 3 R 2 but $2 \neq 3$.

(c) For each pair $(a, b) \in R$, find all $(b, c) \in R$ since $(a, c) \in R^2$

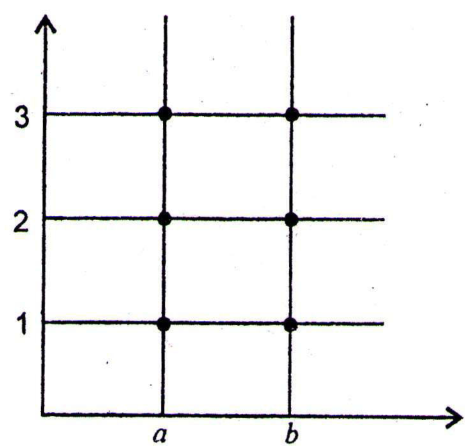
$R^2 = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 2), (4, 3), (4, 4)\}$.

Example 10 : Let A = {1, 2, 3} and B = {a, b}. Represent A × B graphically.

What is $|A \times B|$?

Sol. A × B = {(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)}

Graphically A × B is shown below :

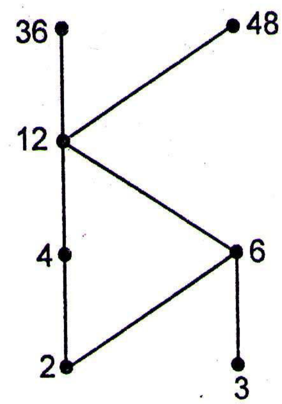


$|A \times B| = |A| \cdot |B| = 3.2 = 6$.

Now, we define **Hasse diagram** as :

Def. : The Hasse diagram of a relation R defined on a set X is a directed graph whose vertices are the elements of X and there is an undirected edge from a to b whenever (a, b) ∈ R (Instead of drawing an arrow from a to b, we some times place b higher than a and draw a line between them). An arrow from a vertex to itself is drawn whenever (a, a) ∈ R.

For Example : Let B = {2, 3, 4, 6, 12, 36, 48} and S be the relation/"divide" on B. Then, Hasse diagram of S is



1.3.8 Summary

In this lesson, we have defined relations and studied reflexivity, symmetry, anti-symmetry, transitivity to understand an equivalence relation and partial order relation. We have also discussed about the inverse of a relation, composition of relations and closures of relations. The representation of relations has been also discussed.

1.3.9 Key Concepts

Relation, Domain, Range, Reflexive, Symmetric, Anti-symmetric, Transitive, Equivalence relation, Partial order relation, Composition of relations, Inverse, Closure, Equivalence class, Arrow diagram, Digraph, Hasse diagram.

1.3.10 Long Questions

1. Show that $R_1 \cup R_2$ may not be an equivalence relation on a set X if R_1, R_2 are equivalence relations on X.
2. Let X = {1, 2, 3, 4} and R = {(x, y) : x > y}. Draw the diagraph and matrix of R.
3. R is a relation on set of positive integers s.t. R = {(a, b) : a – b is an odd integer}. Is R an equivalence relation ?

1.3.11 Short Questions

1. In N × N, show that the relation defined by (a, b) R (c, d) if ad = bc is an equivalence relation.
2. How many relations are possible from a set A of m elements of another set B of n elements ? Why ?
3. Let R and S be the relations on X = {a, b, c} defined by
   R = {a, b), (a, c), (b, a)}, S = {(a, c), (b, a), (b, b), (c, a)}
   (i) Find M_R and M_S (ii) Find RoS (iii) Find SoR

4. Let A = {1, 2, 3, 4} and relation on it R = {(a, b) : | a – b| = 2}. Find transitive closure of R.

1.3.12 Suggested Readings

1. Dr. Babu Ram, Discrete Mathematics
2. C.L. Liu, Elements of Discrete Mathematics (Second Edition), McGraw Hill, International Edition, Computer Science Series, 1986.
3. Discrete Mathematics, S. Series.
4. Kenneth H. Rosen, Discrete Mathematics and its Applications, McGraw Hill Fifth Ed. 2003.

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