Centre for Distance and Online Education
Punjabi University, Patiala

Class : B.A. II (Math) **Semester : III**
Paper : II (Linear Programming) **Unit–1 & 2**
Medium : English

Lesson No.

Department website : www.pbidde.org

B.A. / B.Sc. (MATHEMATICS) Semester - III

PAPER-II: LINEAR PROGRAMMING

Maximum Marks: 50 **Pass Percentage: 35%**
Maximum Time: 3 Hrs

INSTRUCTIONS FOR THE PAPER SETTER

The question paper will consist of three sections A, B and C. Sections A and 8 will have four questions each from the respective sections of the syllabus and Section C will consist of one compulsory question having eight short answer type questions covering the entire syllabus uniformly. Each question in sections A and B will be of 7.5 marks and Section C will be of 20 marks.

INSTRUCTIONS FOR THE CANDIDATES

Candidates are required to attempt five questions in all selecting two questions from each of the Section A and B and compulsory question of Section C.

Section-A

Linear Programming: Formation of LPP, Graphical method, Theory of the Simplex method, Standard form of LPP, Feasible solution to basic feasible solution, Improving BFS, Optimality condition, Unbounded solution, Alternative optimal solution, Correspondence between BFS and extreme points, Simplex method, Simplex algorithm, Simplex Tableau, Simplex method case of degeneracy, Big-M method, Infeasible solution, Alternate solution, Solution of LPP for restricted variable.

Section-B

Transportation Problem: Formation of TP, Concepts of solution, Feasible solution, Finding initial basic feasible solution using North West corner method, Matrix minima method, Vogel's approximation method, Optimal solution by MODI method, Unbalanced and maximization type of TP.
Assignment Problem: Maximization, Minimization, Unbalances, With restriction assignment problems, Algorithm, Hungarian method.

RECOMMENDED BOOKS:

1. G. Hadley : Linear Programming, Narosa Publishing 2002.
2. J. K. Sharma : Operations Research, Theory and Applications, 2006.

LAST UPDATED: MAY, 2023

B.A. II (SEMESTER-III) **PAPER-II**

LINEAR PROGRAMMING

LESSON NO. 1.1 **AUTHOR : DR. BHAWDEEP SINGH**

LINEAR PROGRAMMING-I

1.1.1 Objectives :

The main objectives of this lessons are :

- To study the concept of linear programming problem (LPP) and its formulation.
- To understand graphical method for solving LPP.
- To discuss various types of solutions associated with LPP.

1.1.2 Introduction :

In any business or economic activity, the stress is always laid on the optimum use of resources. Because the resources are scarce, and its optimum use can manage the results to the targets.

In management, the production manager has the job to perform in such a way, so as the cost of production must be the minimum. What is needed in decision-making, a mathematical devise, that incorporates all the constraints like : machinery, labour, money, time, warehouse space or raw-materials, etc. and finally presents a mathematical suit to the targets.

As Kantorovich had stated, "There are two ways of increasing efficiency of work of a shop, an enterprise or a whole branch of industry. One way is by improvement in technology, that is, new attachments for individual machines, changes in technological processes and discovery of better kind of raw-materials. The other way thus far much less used, is by improvement in the organisation of planning and production. Here are included such questions as distribution of work among individual machines or among mechanics, or such other factors." Linear programming is one such technique which deals with the problem of improvement in efficiency. Linear programming is such a mathematical devise which helps in decision-making regarding the direction, use, ways and means of the scarce resources.

1.1.3 General Structure

Linear Programming deals with the optimization (maximization or minimization) of a function of variable known as objectives function, subject to a set of linear equalities and or inequalities known as constraints. The objective function may be profit, cost, production capacity or any other measures of effectiveness, which is to be obtained in the best possible or optimal manner. The constraints may be imposed by different sources such as market demand, production process and equipment, storage capacity, raw material availability etc.

I. The team linear indicates that the relation amongst interdependent activities should be linear, while the term programming refers to determine the line of action to achieve the desired objective.

II. General linear programming problem : Let Z be a linear function given by optimise (maximise & Minimise)

$Z = C_1 X_1 + C_2 X_2 + ...+C_n X_n$

Subject to Constraints

$a_{11}\ x_1 + a_{12}\ x_2 + ... a_{1n}\ x_n\ (\leq = \geq) b_1$

$a_{21}\ x_1 + a_{22}\ x_2 + ... a_{2n}\ x_n\ (\leq = \geq) b_2$

$am_1\ x_1 + am_2\ x_2$ _ -------amn xn $(\leq = \geq)\ b_m$

Where $(x_1, x_2,$ --------- $x_n) \geq 0$

- non - negative constraints

(i) C_1, C_2 ------ C_n are constants (Profit of cost Coefficients)

(ii) aij for i = 1, 2, ----- m, and J = 1, 2, -----n are structural constants

(iii) b_1, b_2 ---- bm are inequalities Parlimentry Secreton

The method of Linear Programming can be divided into following three parts :

(i) Objective function

There is a linear objective function, which is to be maximised or minimised depending upon the nature of the function : Profit Function or Cost Function or Quantity Function, etc.

(ii) Structural Constraints

Further, there is a set of structural constraints subject to the objective function, which contains the technical specifications of the objective function in relation to the given resources or requirements.

(iii) Non-negativity Constraints

The third set is of the non-negativity constraints, which specifies that there is no question of negative solutions of the problem. Linear Programming is one of the most versatile, powerful, and useful techniques for making managerial decisions. It has been employed in solving a broad range of problems in business, government, industry, hospitals, libraries, and education. As a technique of decision-making, it has demonstrated its value in such diverse areas as optimal product-mix, transportation schedule, plant location, assignment of personnel and other type of problems that can be solved by linear programming.

1.1.4 Development of Linear Programming

The first use of Linear Programming was in business by L.V. Kantorovich who was interested in the applications of mathematical models. Following Kantorovich, another Soviet outstanding mathematician, A.N. Kolmogorov applied the linear programming in the diverse fields of business activity. Later on, Stigler in 1945, applied it to the diet problems in Health Economics.

George D. Dantzig, a pioneer in the development of the simplex alogorithm had applied it to the logistic problems. After this, many eminent mathematicians contributed a lot for the development of linear programming like : E.M.L. Beale, W. Orchard-Hays, P. Wolfe, etc. An Indian mathematician Narender Karmakar has advanced a solution for solving complex linear programming problems. His work is far more effective and efficient than the simplex method.

1.1.5 Requirements of a Linear Programming Problem

In every organisation, all the resources at its disposal, resources being limited, the management always tries its best to use the resources more economically and efficiently, so that it may get the maximum profits or minimise, the losses or utilise the production capacity to the optimum. Now the linear programming has very wide applications, not only in business but almost in every field. All the linear programming problems have four properties in common :

(i) There must be a well-defined objective function like profit maximisation, cost minimisation of quantities to be produced and which must be expressed in linear form of decision variable.

(ii) There must be some constraints or restrictions which presents the specifications to the objective function.

(iii) There must be the presence of alternative courses of action like in the problems of production of wide range of products, there is a problem of allocation of resources; and

(iv) All the objective functions and the constraints must be expressed in linear equations or inequalities, that is, in first degree. Further, decision variables should be interrelated and non-negative. The non-negativity condition shows that linear programming deals with real life situations for which negative quantities are generally illogical.

1.1.6 Assumptions of Linear Programming

In addition to the above four requirements, there is a set of five additional conditions of a linear programming problem, like :

(i) We assume that the manager here is completely certainty regarding technology, sources, strategies and their respective consequences.

(ii) The condition of proportionality exists in the objective function and constraints.

(iii) The production requirements are fixed during the planning horizon.

(iv) All decision variables are continuous, that is, we can produce products in fractional units.

(v) All variables assume non-negative values.

1.1.7 Formulating Linear Programming Problem

For formulating a linear programming problem, one should follow the following steps in order :

Step 1 Study the situation to find the key decision to be made. In this connection identify the variables helps considerably.

Step 2 Select symbols for variable quantities identified in step 1.

Step 3 Express feasible alternatives mathematically in terms of the variables. These feasible alternatives are those which are physically, economically and financially possible.

Step 4 Identify the objective quantitatively and express it as a linear function of variables.

Step 5 Express in words the influencing factors or constraints which occur generally because of the constraints on availability (resources) or requirements (demands). Express these restrictions also as linear equalities/inequalities in terms of variables.

1.1.8 Illustrations

Example : 1.

A manufacturer prepares two products A and B. The times of preparation. capacity available at each work centre and net revenue are given below :

The formulation of a product-mix problem into linear programming form is as follows :

Work centre/ Product	Cutting (hrs.)	Fabrication (hrs.)	Assembly (hrs.)	Net revenue per unit (Rs.)
A	1	4	2	150
B	2	5	3	180
Total Capacity	500	1400	700	

Formulate the Linear Programming model.

Solution :

Step 1 : The main decision to be taken by the manufacturer is to determine the number of units of product A and B to be produced.

Let x = number of units of product A

Step 2 : The total net revenue received after selling products A and B is given by the objective function.

Z = 150x + 180 y

Step 3 : To prepare the two products A and B. the total number of hours required at the cutting centre is x + 2y.

Total number of hours required at assembly centre is 2x + 3y.

Now the cutting centre is not available more than 500 hrs., fabrication centre is available only for 700 hours, so we have :

$$x + 2y \leq 500$$
$$4x + 5y \leq 1400$$
$$2x + 3y \leq 700$$

Step 4 : As the manufacturer cannot produce negative number of products, so we have also $x \geq 0$ $y \geq 0$.

Step 5 : Summary of above.

For two real number x and y

Max. Z = 150x + 180 y (objective function is to be maximised, subject to

Sub To $x + y \leq 500$ ....(i)

$4x + 5y \leq 1400$ ....(ii) (constraints)

$2x + 3y \leq 700$ ....(iii)

$x \geq 0, y \geq 0$ (non-negative restrictions)

Example 2 : A house wife desires to take maximum vitamins A, B and C for her family members, the minimum daily consumption of the vitamins A, B, C for the family are respectively 40, 30 and 22 units for these requirements, the housewife depends on fresh available foods. The first food provides 5, 4, 3 units of the three vitamins per gram respectively while second one provides 4, 5, 7 units of the same vitamins per gram of the foodstaff respectively. The first foodstaff costs Rs. 7 per gram and second Rs. 5 per gram. The problem is how many grams of each foodstaff should the house-wife buy daily to keep her food bill aslow as possible ?

Formulate the problem.

Solution :

Step 1 : Let the two foodstaffs be P and Q and the respective number of units purchased be x and y. The data can be tabulated as follows :

Variable	*Food*	*Contents of vitamins*			*Cost per unit*
		A	*B*	*C*	*Rs.*

x	p	5	4	3	7
y	Q	4	5	7	5
Minimum vitamin required		40	30	22	

Step 2 : Objective function is Z = 7x + 5y, which is to be minimised.

Step 3 : As the minimum amount of vitamins required in A, B, and C are respectively 40, 30 and 22, the constraints of the problem are $5x + 4y \geq 40$, $4x + 5y \geq 30$, $3x + 7y \geq 22$.

Step 4 : As the number of units cannot be negative so $x \geq 0$. $y \geq 0$

Step 5 : Summary :
Minimize Z = 7x + 5y subject to the following :
Constraints : $5x + 4y \geq 40$, $4x + 5y \geq 30$, $3x + 7y \geq 22$; and restrictions $x \geq 0$, $y \geq 0$.

1.1.9 Graphic Methods

Step 1 : Formulate the linear programming problem.

Step 2 : Each inequality in the constraints may be written as equality.

Step 3 : Plot the constraints linear considering them as equalities.

Step 4 : Identitfy the feasible solution region. Feasible solution is a solution for which all the constraints are satisfied. Conversly an infeasible solution is a soultion for which at least one constraint is violated.

Step 5 : Locate the corner points of the feasible region.

Step 6 : Calculate the values of the objectives function on the corner points.

Step 7 : Choose the point where the objective function has optimal value. An optional solution is a feasible solution that has the most favourable value of the objecitve function. The most favourable value is the largest value if the objective function is to be maximize, whereas it is the smallest value if the objective function is to be minimised.

1.1.10 Illustration

The above idea will be clear from the following example :

Maximisation Example :

Example 3 : A factory manufactures two products A and B. To prepare the product A, a certain machine is to be used for 1 hr. and then a craftsman has to work for 2.5 hrs. per unit. For the product B, the machine is to be used for 2 hrs. and a craftsman has to work 1.5 hrs. per unit. In a week the machine is available for 40 hrs., while the craftsman's work is 60 hrs. for the same products. The profit for A is Rs. 4 per unit and that of B is Rs. 6 per unit. Assuming all the products being sold out, find how many of each kind should be produced to earn the maximum profit per week.

Solution : *Step 1. Let x = number of units in product A*

any y = number fo units is product B.

Now we shall furmulate the given problem as follows :

Decision variables	*Product*	*Hours (per unit)* machine	craftsman	*Profit per unit (Rs.)*
x	A	1	2.5	4
y	B	2	1.5	6
Maximum hrs. available per week		40	60	

Objective function is Z = 4x + 6y to be maximized, subject to following constraints :-

$x + 2y \leq 40$ .... (i) (machine hours)

$2.5x + 1.5y \leq 60$..................(ii) (craftsman hours constraint)

$x \geq 0.\ y \geq 0$....................... (iii) (non-negativity)

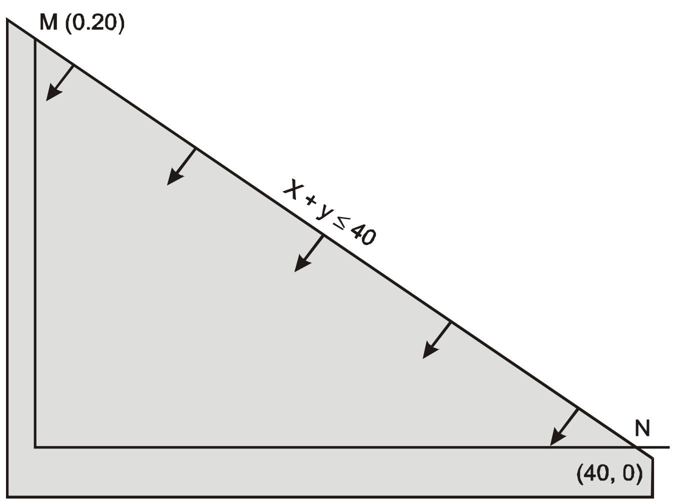
Step 2 : Construction of graph.

Consider the inequality $x + 2y \leq 40$. Assume that only the first variable has been produced i.e. the whole machine capacity has been utilised for it.

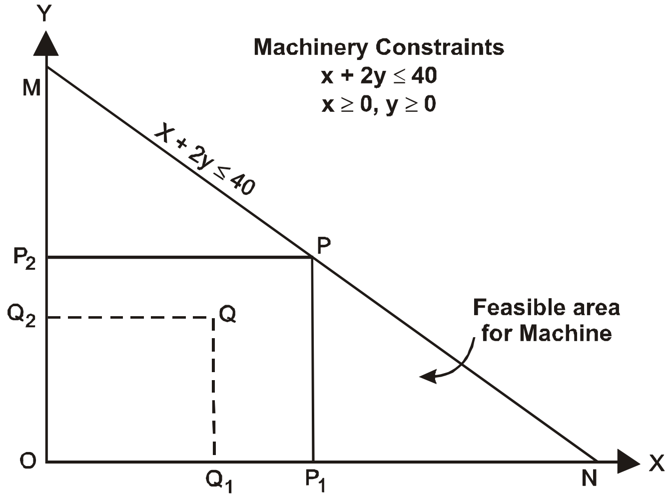
$\therefore$ x + 0.y = 40 (as the maximum) or, x = 40, when y = 0.

Again if y variable of product B has been produced, the whole machinery work has been utilised for it, then

0.x + 2y = 40 or, y = 20, when x = 0.



On plotting these points we get the line MN (as shown in the figure) to represent the equation x + 2y = 40. Our purpose is to represent the inequality $x + 2y \leq 40$, so the region of reference will be shaded area below the line MN (as indicated by arrows) the shaded area also includes negative production of x and y (as the shaded area covers four quadrants). But we have constraints $x \geq 0$, $y \geq 0$, so our area remains restricted only for first quadrant.

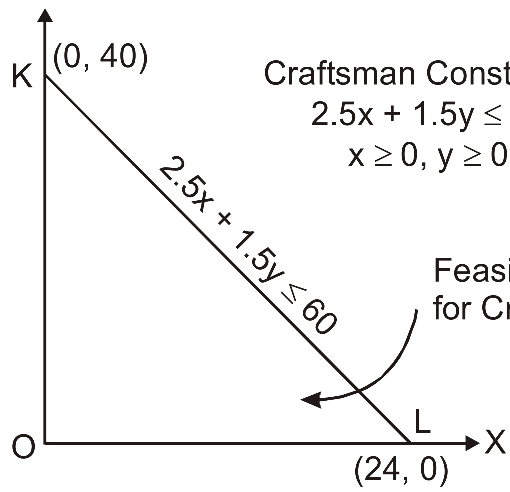


Any point P on the line MN satisfies the equation $x + 2y = 40$ while any point Q below the MN satisfies the inequality $x + 2y < 40$. The point P on MN involves such amounts of production of two commodities ($x=OP_1$, $Y = OP_2$) that machinery capacity gets fully utilised, while for the point Q indicates such amount of two commodities where the machinery capacity are not used completely.

Similarly we can plot the inequalities of craftsman

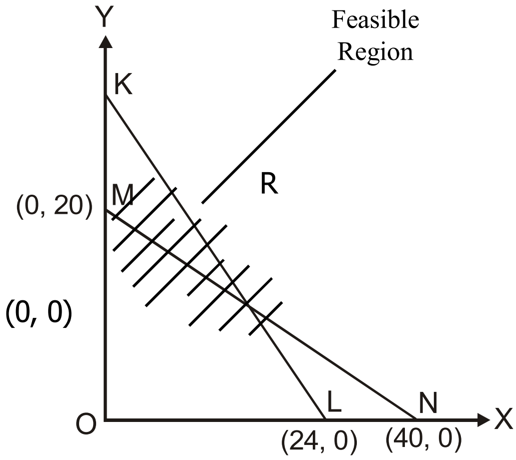
$2.5x + 1.5y \leq 60$ which is shown below :

x	0	24
y	40	0



Step 3 : To identify feasible region :

The feasible region i.e., solution space is the area of the graph satisfying all the constraints together. This means feasible region will be bounded by the two axes, and the two lines x + 2y = 40 and 2.5x + 1.5y = 60 and will be the common area that falls to the left of these constraint equations as both the constraints are less than equal to type.



Step 4 : *To locate solution points :*

Area shown by networks in the figure OLRM represents the set of feasible solutions. The four corners of the above area are O (0, 0), L (24, 0), R (17.2, 11.4) and M (0, 20).

Note : The point R is the intersection of the lines MN and KL. The co-ordinates are obtained by solving the equations

x + 2y = 40 and 2.5x + 1.5y = 60.

Step 5 : *Evaluation of objective functions :*

The optimal solution to linear programming problem (L.P.P.) occurs at one or more of the corner points. Let us find the values as these corners.

Corner Point	*Objective function Z = 4x + 6y*	*Value*
O (0.0)	4.0 + 6.0 = 0	f (O) = 0
L (24, 0)	4.24 + 6.0 = 96	f (L) = 96
R (17.2, 11.4)	4 × 17.2 + 6 × 11.4 = 137.2	f (R) = 137.2
M (0, 20)	4.0 + 6.20 = 120	f (M) = 120

Conclusion : Optimal solution is that corner at which the objective function has the largest value. So the optimal solution lies at the point R (17.2, 11.4) i.e. at x = 17.2, y = 11.4 with the maximum profit of Rs. 137.20.

Hence to maximize profit the company should manufacture 17 units of article A and 11 units of article B, (neglecting decimal parts).

Minimisation Example

Example 4 : A person requires a certain minimum amount of C and D vitamins per day in order to maintain his health. These two types of vitamins are available in two types of food items. What amount food items must be purchased so that daily vitamins requirements are met and also the cost of the food items purchased is minimum according to following table :

Vitamins	Food items containing vitamins per unit *P*	*Q*	Minimum daily Vitamins requirements
C	5	2	20
D	2	4	16
Price per unit (Rs.)	2	3	–

Solution :

Let x = quantity in food P and y = quantity in food Q satisfy minimum requirement of vitamins.

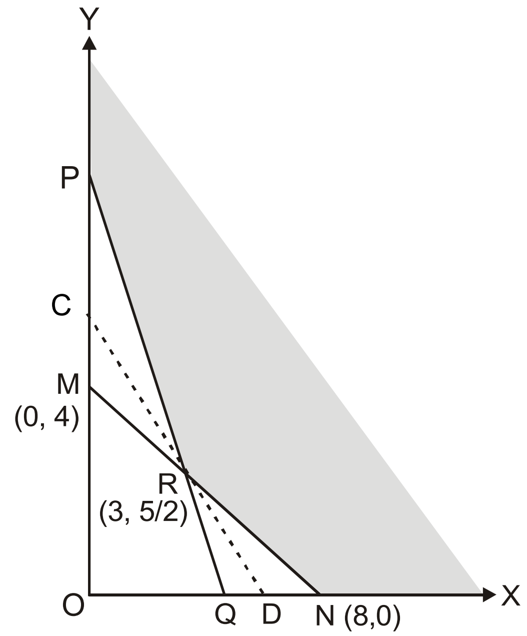
Objective function : $Z = 2x + 3y$ Subject to $5x + 2y \geq 20$; $2x + 4y \geq 16$; $x \geq 0, y \geq 0$.

Graph is drawn by usual method with the help of following tables :

x	4	0
y	0	10

x	8	0
y	0	4

Each constraint is ≥ type, so solution will lie above the lines of constraints. As such shaded area above the lines i.e. MRN (See Fig.) will be our feasible region and each point will contain feasible solution.



To minimise cost we must select the lowest possible isocost line lying in the feasible solution. Now the tangent line CD at R will be the lowest isocost line. to find R, intersection of the line PQ and MN, we solve the equation

$5x + 2y = 20$, $2x + 4y = 16$; and we get $x = 3$, $y = .\frac{5}{2}$

So the person should purchase 3 units of food P and 5/2 units of food Q. These quantities will minimise the cost of consumer.

The cost is $2.3+3.\ \frac{5}{2} =$ Rs. 13.50.

Example 5. A firm produces two products Jam and Jelly. The amounts of material, labour and equipment to produce each product and their resources are as follows :

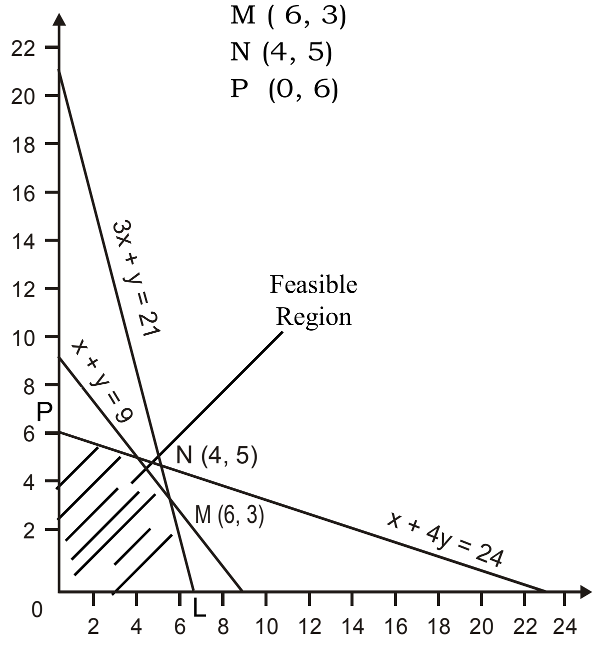
	Jelly	*Jam*	*Available resources*
Labour (man hours)	1	4	24
Equipement (machine hour)	3	1	21
Material (units)	1	1	9

One unit each of Jelly and Jam has profit margin of Rs. 2 and Rs. 5 respectively. Determine the mathematical formulation and find the optimum solution.

Solution : Let x = amount of Jelly and y = amount of Jam. Objective function is $z = 2x + 5y$, subject to constraints $x + 4y \le 24$; $3x + 4y \le 21$; $x + y \le 9$; $x \ge 0$; $y \ge 0$. The above linear equations can be represented graphically as follows :

Again the value of objective function of the extreme points is given below :

Corner points	value of $z = 2x + 5y$
L (7,0)	14
M (6, 3)	27
N (4, 5)	33
P (0, 6)	30



Value of Z is maximum at N (4, 5), so 4 units of Jelly and 5 units of Jam are to be produced to get maximum profit.

Example 6 :

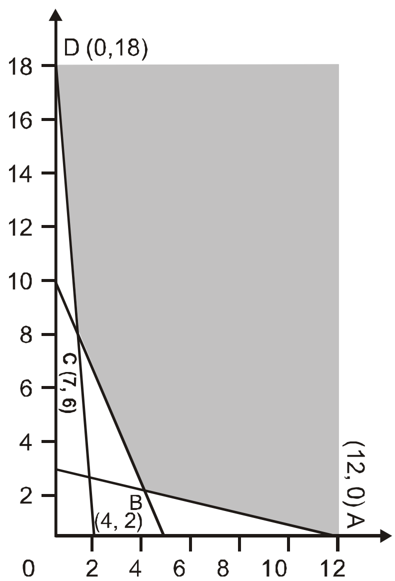
Minimize : $z = 20x + 40y$

Subject to : $6x + y \geq 18;$

$x + 4y \geq 12$

$2x + y \geq 10; \quad x, y \geq 0$

Solution : We draw the lines $6x + y = 18$, $x + 4y = 12$, $2x + y = 10$ and $x = 0$, $y = 0$. The feasible region is shown by the shaded region in the figure. The feasible region is bounded from below only. The extreme points are A (12,0) B (4, 2), C (2, 6) and D (0,18).



The value of the objectives function z at the extreme points are :

Z = 20.12 + 40.0 = 240 at A (12,0)

= 20.4 + 40.2 = 160 at B (4,2)

= 20.2 + 40.6 = 280 at C (2,6)

= 20.2 + 40.18 = 720 at D (0,18)

So z is the minimum at x = 4, y = 2 and the minimum value is 160.

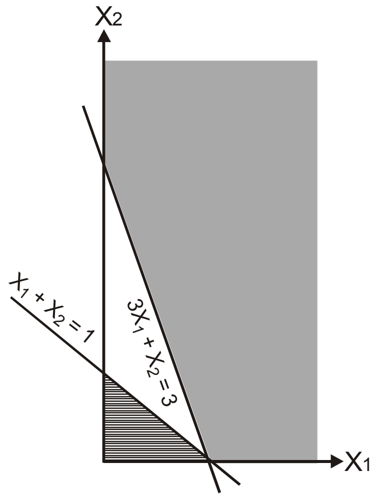
Note : In the example the objective function has no maximum. This is because as x and y tend to infinity in the feasible region, the value of z also tends to infinity (i.e., no definite maximum value).

Further there may be some L.P.P. which have no optimal solutions. This is because either the constraints are inconsistent or there exists no point which simultaneously satisfies the given constraints and the non-negative restrictions of the problem. The idea will be clear from the next example.

Example 7 : Mark the feasible region represented by the constraint equations :

$x_1 + x_2 \leq 1$; $3x_1 + x_2 \geq 3$; $x_1 \geq 0$, $x_2 \geq 0$ of a linear optimizing function $z = x_1 + x_2$

Solution : The solution spaces are shaded in the following figure. From the diagram it appears that there is no feasible region.



1.1.11 Merits of Linear Programming Problem

1. Scientific Approach : Linear Programming Problem in studying the information of an organisation in such a way that it depicts the clear picture of the problem. This scientific approach to the problem is as valuable and necessary as is the solution.

2. Number of Possible Solution : Management problems are So tedious that it is very difficult to arrive at the best alternative solutions. With the use of Linear Programming Problem technique managers sure that he is considering the best optimal solution with the help of Liner Programming Problem.

3. Cost-Benefit Analysis : Linear Programming Problem helps the managers to paln and execute the policies of the top management in such a way that costs or penalties involved are minimum. Management always put restrictions under which the manager must oprate and Linear Programming Problem helps managers to make maximum use of limited resources available with him.

4. Flexibility : After the plans are prepared it can be re-evaluated for changing conditions. Linear Programming Problem is one of the best techniques to be used under the changing circumstances.

5. Removing Bottlenecks : Linear Programming Problem higlights the various bottlenecks in problem process e.g. when bottleneck occurs, some machines cannot meet demands while other remian idle and Liner Programming Problem helps to remove all such bottleneck.

6. Optimum use of Scare Resources : Other advantages of Liner Programming Problem include optimal use of productive factors by indicating the best use of existing facilities.

1.1.12 Summary

In this lesson, we have elaborated the concept of LPP along with its various components. We have also learnt to formulate a LPP from the given data. One of the methods to solve LPP i.e. Graphical method has been clearly explained step by step and various possible solutions have been discussed. The concepts are made more clear with the help of various suitable examples.

1.1.13 Key Concepts

Linear Programming, Objective function, Structural constraints, Non-negativity constraints, Feasible region, Feasible solution, Optimal solution, Unbounded solution, Multiple optimum solution, Infeasible solution, Graphical method.

1.1.14 Long Questions

1. What is Linear Programming problem ? State the assumptions and limitations of a linear programming problem.
2. Linear Programming sometimes is defined as an effective search procedure for best solution in the given situation. Explain the meaning and significance of this statement.
3. Discuss linear programming as a technique of resource allocation.
4. Explain the statement that an optimal solution to a linear programming problem can always be found at one of the extreme points of the convex solution set. Illustrate this concept graphically.
5. Consider this linear programming formulation :

   Minimize Cost = 1x + 2y

   Subject to :

   $Y + 3Y \geq 90$

   $8x + 2y \geq 160$

   $3x + 2y \geq 120$

   $Y \geq 70$

   $x, y \leq 0$

   (x = 25.71, y = 21.43 and Cost = 68.57)
6. An advertising company is planning a media campaign for client willing to spend Rs. 50 lakh to promote a new product directed at middle aged men and women.

   The client wishes to limit his compaign media to radio and prime time television. A staff researcher has complied the following cost effectiveness data :

Advertising Media	Price per Unit (Thousand Rs.)	Estimate number of middleaged men and women exposed to each advertising unit (in lakhs)
Radio	40	5
Television	200	10

The client wishes to maximise the auidence exposed to the advertisement. In addition, at least 6 million people should be exposed to the television advertising. Also, the investment in radio advertising must not exceed Rs. one million.

Formulate the problem as a linear programming problem and solve it graphically (Radiation = 25 units, TV = 20 units and Exposure = 32.8 million)

7. A firm is manufacturing two types of electrical items. A and B each make a profit of Rs. 15 per unit of A and Rs. 240 per unit of B. Both A and B have a common process of a Motor and A Transformer. Each unit of A requires 3 motors and 2 transformers, whereas each unit of B requires 2 motors and 4 transformers. Type B is an export model requiring a voltage stablizer, which has a supply restricted to 65 units per month. Solve it graphically and determing product mix of A and B to (A = 30 units, B = 60 units and profits = Rs. 19200)

8. A company has two plants, one at Delhi and the other at Calcutta. It makes three products A, B and C at these plants. The following data matrix is given to us :

Products	Delhi Plant (per day capacity)	Calcutta Plant (per day capacity)	Demand (per day)
A	3000	1000	24000 units
B	1000	1000	16000 units
C	2000	6000	48000 units
Operating Cost Rs.	**600**	**400**	

For how many days should each of the two plants work to meet the demand of the three products at minimum cost ?

(Delhi = 4 Calcutta = 12 and Cost = Rs. 7200)

9. A minining company is taking a certain kind of ore from two mines A and B The one is divided into three quality groups x, y and z. Every week the company has to deliver 240 tons of x, 160 tons of y and 440 tons of z. The cost per day for running mine A is Rs. 3000 and for running B is Rs. 2000. Each day, A will produce 60 tons of x, 20 tons of y and 40 tons of z. While B will produce 20 tons of x, 20 tons of y, and 80 tons of z. Using graphical method, find the most economical production plan.

(A = 14 tons, B = 42 tons and Minimum Cost = Rs. 19200)

1.1.15 Short Questions

1. Explain the graphical approach for solving the linear programming problem.
2. Solve graphically :

Maximize Profits = 2x + 5y

Subject to :

$4x - 3y \geq 12$

$x + y \leq 2$

$x \geq 4$

$x, y \geq 0$

($x = 4$, $y = 6$ and Profits = 38)

1.1.16 Suggested Readings

Chaing, Alpha C.	Fundamental Methods of Mathematical Economics (Singapore, McGraw-Hill, 1984)
Gass, S.I.	Linear Programming (New York, McGraw-Hill, 1975)
Gupta, P.K. & Man Mohan	Linear Programming And Theory of Games (New Delhi, Sultan Chand, 1991)
Hadley, G.	Linear Programming (New Delhi, Oxford IBH, 1964)
Loomba, N.P.	Linear Programming (New York, McGraw-Hill, 1964)
Loomba, N.P.	Management-A Quantitative Perspective (New York, Macmillan, 1978)
Render, B & Stair R.M.	Quantitative Analysis for Management (Allyn And Bacon. Inc. 1988)
Taha, Handy A.	Operations Research-An Introduction (New York, Macmillan, 1987)

LESSON NO. 1.2 AUTHOR : DR. BHAWDEEP SINGH

LINEAR PROGRAMMING-II

1.2.1 Objectives

The main objectives of this lessons are :

- To study in detail about the simplex method for solving LPP.
- To understand the concept of dual of a LPP.
- To discuss about degeneracy in LPP.

1.2.2 Simplex Method and Applications

In the previous chapter, we have studied the graphical method of Linear Programming which is applicable in a situation that involves two independent variables. However, in practice we need to handle simultaneously a number of variables. Most production problems reach a level of complexity involving a dozen products and the same number of departments, where the graphical approach fails to provide the results quickly and efficiently, and the new approach of linear programming is the simplex method, which deals with these production mix problems and provides the optimal results. The simplex technique was developed by G.B. Dantzig for dealing with such complex problems, where all alternatives fail. The computational procedure used is an iterative process, sometimes referred to as an algorithm, that is, the same basic computational routine is used over and over again. This results in a series of successively improved solutions until the best one is found. A basic characteristic of the simplex method is that the last solution yields a contribution as large as or larger than the previous solution in a maximization problem. In a minimization problem, the simplex method determines a cost that is the same or lower than the previous one.

This feature assures that the optimal answer can finally be reached.

To fix ideas and to facilitate comparison with the graphical method, we apply the simplex method to the same problem, as it was solved in chapter 2 by graphical method. The problem can be put into a tabular form as :

Table 3.1 Process Time Date per Unit of Product

	Product		Capacity
Department	X	Y	
Wiring	3	2	240
Drilling	2	1	140
Profit per Unit	25	15	

As first we convert the data in Table 3.1 into inequalities, by the objective function and the constraints and the non-negativity constraint.

Maximize Profits : $Z = 25x + 15y$

Subject to Constraints :

$3x + 2y \leq 240$

$2x + 1y \leq 140$

$x, y \geq 0$

In the next step, these inequalities are transformed into equalities, for this, the surplus variables have to be incorporated into the objective function, and in constraints also. The surplus variable shows the unutilized capacity in producing the products through the various departments, and will contribute nothing to the per unit profit.

The transformed problem is as follows :

Maximize Profit : $Z = 25x + 15y + Os_1 + Os_2$

Subject to the constraints :

$3x + 2y + Is_1 + Os_2 = 240$

$2x + 1y + Os_1 + Is_2 = 140$

$x, y, S_1, S_2 > 0$

The objective function and the constraints are presented in a matrix or tabuler form, which is called the simplex table. The above problem can be presented as follows in matrix form :

Table 1.2 Initial Simplex Table of Electrical Company

Contribution Per Unit	C_j		25	15	0	0	
Product Quantity	Product						
C_B	Mix		x	y	S_1	S_2	Variables Coefficients
O	S_1	240	3	2	1	0	
O	S_2	140	2	1	0	1	
			Body Matrix		Identity Matrix		
	Zj	0	0	0	0	0	

Contribution					Losts per Unit
$C_j – Z_j$	25	15	0	0	Net contribution per Unit

Explanation of the Parts of Table 1.2

C_B Column : This column contains the contribution per unit for the slack variable S_1 and S_2. The zero indicates that the contribution per unit is zero, because profits are not made on unused time in a department, but on time used.

Product-Mix : This column contains the variables in the solution, which are used to determine total contribution.

Quantity : The third column indicates the quantity of time available for production of products. In the initial solution, no products are being made. So this will be the unused time in the beginning.

C_j Row : This row represents the coefficients of the objective function.

Body-Matrix : Body-matrix contains the coefficients of the basic variables of the constraint equations.

Identity-Matrix : The identity matrix in the first simplex tableau represents the coefficients to the slack variables that have been added to the original inequalities to make them equations.

Zj row : The Zj row value under the quantity column indicates an initial solution of zero contribution and under basic (X, Y) and non-basic (S_1, S_2) variables, is the contribution losts per unit.

Cj-Zj Row : This row indicates the net contribution per unit made by the basic variables. This row also provides the test of optimality and the criteria for tests of optimality is as :

For Maximization Problems :

The solution will be optimal, when all the elements of Cj-Zj row are zero or negative.

For Minimization Problems :

The solution will be optimal, when all the elements in the Cj-Zj row are positive or zero.

1.2.3 Methodology of Simplex Method

For proceeding from first or initial simplex tableau to 2nd simplex tableau, we have to follow the following steps :

Step I

Identify the column with the highest plus (positive) value in the Cj-Zj row. This will be the pivot column and will indicate the basic variable to enter the solution mix in the first.

Step II

Now the basic variable is going to enter to the product mix, it will replace the non-basic variable from the product mix column. For this, identify the row with the smallest positive calculated quantity, that is, to calculate the smallest value by dividing the elements of the quantity column by their respective elements in the pivot column. This will be pivot row.

Step III

Computed new values for the pivot row. For this, we simply divide every number in the row by the pivot number. The pivot number is the elements at the intersection of the pivot row and pivot column.

Step IV

Compute new values for the remaining rows. The procedure for calculating the values for each row is :

New Row Numbers	=	Numbers in Old Rows	–	Numbers above or below pivot number	x	Corresponding number in the new row (the row of Step-III)

Step V

Calculate the values for the contribution losts per unit row (Zj) and for the Cj-Zj row. Apply the test of optimality. If there is a need to develop another tableau, then apply all the steps in toto.

The above defined steps are applied to the problem of Electrical Company manufacturing products X and Y.

Step I

Identify the column having the highest positive values :

Since the present problem has two non-basic variables X and Y, in the net evaluation row X has 25 and Y has 15 as the positive values. Out of which X has the highest positive value. Column of X is the Pivot column. In other words, X will be the first variable, which will enter into the solution mix.

Step II

Identify the row having the smallest quantity ratio :

Now X is going to enter the solution mix, it will replace the variable, which has the smallest quantity ratio. The quantity ratio is calculated through dividing the quantity column by the respective elements of the pivot column. S_1 has the quantity ratio 80 and S_2 has the quantity ratio 70. So S_2 will be replaced by X. S_2 will be the pivot row.

Step III

Now we have pivot column and pivot row. The element at the intersection of the pivot column and the pivot row is the Pivot Number or key element which is 2. Divide the S_2 row by 2, we get the new values for the pivot row.

Like : X, 25, 70, 1, 1/2' 0 & 1/2'

Step IV

The calculations for the other row, based upon the new pivot row are as follows :

S_1 Row :	30	=	240	—	3	x	70
	0	=	3	—	3	x	1
	1/2	=	2	—	3	x	1/2
	1	=	1	—	3	x	0
	3/2	=	0	—	3	x	1/2

Step V

Calculations for Zj Row :

Zj for column X	=	0	x	0	+	25	x	1	=	25
Zj for column Y	=	0	x	1/2	+	25	x	1/2	=	25/2
Zj for column S_1	=	0	x	1	+	25	x	0	=	0
Zj for column S_2	=	0	x	3/2	+	25	x	1/2	=	25/2
Zj for column Co	=	0	x	30	+	25	x	70	=	1750

Step VI

Calculations for the Net Contribution (Cj-Zj) Row :

	X	Y	S_1	S_2
Cj for column	25	15	0	0
Zj for column	25	25/2		25/2
Cj-Zj for column	0	5/2	0	- 25/2

By presenting all the calculations from step I to VI in tableau, we can apply the test of optimality, if there is any positive number then move to develop the next tableau until we get the optimal results. Since we have 5/2 positive number under column Y, so Y will enter into the solution mix and then apply all the steps for calculating the maximum profits.

These two Tableaus 3.3 and. 3.4 are developed as follows :

Table 1.3 Second Simplex Table

Cj			25	15	0	0
	Product Mix	Quantity	X	y	S_1	S_2
0	S_1	30	0	1/2	1	3/2
25	X	70	1	1/2	0	1/2
	Zj	1750	25	25/2	0	25/2
	Cj-Zj		0	5/2	0	25/2

Table 1.4 Third Simplex Table

Cj			25	15	0	0
	Product Mix	Quantity	X	y	S_1	S_2
15	Y	60	0	1	2	- 3
25	X	40	1	0	- 1	2
	Zj	1900	25	15	5	5
	Cj-Zj		0	0	- 5	- 5

As can be ascertained that all the number in the new contribution per unit row (Cj-Zj) are negative or zero, which indicates the optimality of the solution of the problem of the electrical company. At this final outcome, the firm is producing 40 units of X (air conditioners) and 60 units of Y (electric fans) and further earning the maximum profits of Rs. 1900/-. Hence we get the solution mix through the simplex method.

1.2.4 The Simplex Method for Minimization Problems

Sometimes the objective of the firm is to minimize the cost function rather than the maximization of the profit function. Linear Programming provides very useful solutions to the maximization problems. The minimization problems can be solved by linear programming by any of the following ways :

(i) The first method to solve the minimization problem is of Charne's M-Method;

(ii) The second method to solve such problems is to convert the minimization problem into a maximization problem. This can be done by charging the objective function through multiplying it by - 1, like
Minimize z = Maximize (- z)

(iii) The third method is to change the objective function from Primal (minimization) to Dual (maximization) and then solve it like the maximization problem.

These are three methods employed to solve the minimization problems, out of which the first two Charne's M-Method and the Convertion function will be discussed here, but the third method of primal to Dual change will be discussed in the next chapter.

1.4 Illustration : A chemical company produces two compounds A and B. The following table gives the units of ingredients C and D per Kg. of Compound A and B as well as minimum requirements of C and D and costs per kg. of A and B. Using the simplex method, find the quantities of A and B; which would give a supply of C and D at a minimum.

Ingredient	Compound		Minimum Requirements
	A	B	
C	1	2	80
D	3	1	75
Cost per kg.	4	6	

Charne's M-Method :

The compounds A and B are denoted by X and Y. The above problem can be stated as :

Objective Function

Minimize Cost (z) = 4X + 6Y

Subject to the constraints :

$1X + 2Y \geq 80$

$3X + 1Y \geq 75$

$X, Y \geq 0$

Transforming the inequalities into equalities :

Since the left side is greater than the right side of the given constraints, we have to introduce the surplus variables in the given constraints like S_1 and S_2 we get :

Minimize $Z = 4X + 6Y + OS_1 + OS_2$

Subject to the constraints :

$X + 2Y - S_1 = 80$

$3X + Y - S_2 = 75$

$X, Y, S_1, S_2 > 0$

The constraint equations do not fulfil the linear programming of non-negativity, because if X and Y are zero or less than the minimum requirements of 80 and 75, then S_1 and S_2 will be negative, which violates the linear programming condition. For avoiding this, we introduce other artificial slack variables (A's) to problem. The artificial slack variables are assigned very high positive value, say M, so that these artificial slack variables will not enter into the final solution because of very high cost. Now the problem can be rewritten as :

Objective Function :

Minimize Cost (z) = $4x + 6y + OS_1 + OS_2 + MA_1 + MA_2$

Subject to the constraints :

$1X + 2Y - IS_1 - OS_2 + A_1 + OA_2 = 80$

$3X + 1Y - OS_1 - IS_2 + OA_1 + A_2 = 75$

Where $X, Y, S_1, S_2, A_1, > 0$

Now the problem is similar to the maximization problem irrespective of the conditions of tests of optimality. The simplex, tableau will be developed in the same way and all the steps defined are employed in toto. The simplex tableau from first to the final tableau are as :

Table 1.5 **First Simplex Table**

Cj	Product Mix	Quantity	4 X	6 Y	0 S_1	0 S_2	M A_1	M A_2	Quotient
M	A_1	80	1	2	- 1	0	1	0	85
M	A_2	75	3	1	0	- 1	0	1	25
	Zj	155 M	4M	3M	- M	- M	M	M	
		Cj-Zj	4-4M	6-3M	M	M	0	0	

Where : X-Column-Pivot Column

A-2 Row-Pivot Row

3-Pivot Number

Table 1.6 **Second Simplex Table**

Cj			4	6	0	0	M	M	
	Product Mix	Quantity	X	Y	S_1	S_2	A_1	A_2	Quotient
M	55		0	5/3	-1	1/3	1	- 1/3	
4	25		1	1/3	0	-1/3	0	1/3	
	Zj	55M + 100	4	$5M+\frac{4}{3}$	–M	$\frac{M}{3}-\frac{4}{3}$	M	$\frac{-M}{3}+\frac{4}{3}$	
	Cj-Zj	$0\frac{14}{3}+5M$	M	$\frac{4}{3}-\frac{M}{3}$	0	$\frac{2M}{3}-\frac{4}{3}$			

Where :

Y - Column-Pivot Column

A_1 - Row-Pivot Row

5/3 - Pivot element

Table 1.7 Third and Final Tableau

Cj			4	6	0	0	M	M
	Product	Quantity	X	Y	S_1	S_2	A_1	A_1
6	Y	33	0	1	- 3/5	1/5	3/5	- 1/5
4	X	14	1	0	1/5	- 2/5	- 1/5	2/5
	Zj	254	4	6	- 14/5	- 2/5	- 14/5	2/5
	Cj-Zj		0	0	14/5	2/5	M-14/5	M-2/5

Now the solution is optimal, because it fully satisfies the criteria of tests of optimality, that is, all the numbers in the Net Contribution Row (Cj-Zj) are positive or zero. Hence to minimize the costs, the chemical manufacturing company must produce 14 units (kgs) of Compound A, and 33 kgs. of Compound B and the minimum costs are Rs. 254/-.

Second Method

The second method to solve the problem of minimization is to convert the problem to a maximization problem. This can be possible by multiplying the objective functon by - 1, and then the problem can be solved by the method already applied in maximization problem. The student should take it as an exercise. However, the problem can be presented in the following way :

Objective Function

Minimize Cost = 4x + 6y

Convert it into maximization :

Maximize (-Cost) = -4x - 6y

The constraints will remain the same for the problem.

1.2.6 Application of Linear Programming : The linear programming can be applied to the diverse fields of Economics, Management, Social Sciences, Physical Sciences and even in earth sciences, that is, where there is a need of decision-making and for getting the optimum results. Linear Programming has proved as very useful in Media Selection and Marketing Research of Marketing Management Production-Mix and Production Scheduling of Manufacturing applications Labour Planning; Portfolio selection; shipping problems, blending problems; health services, etc.

1.2.7 Duality

Every linear programming problem must have an alternative programming problem, if the problem is of maximization then the alternative will be of minimization, and vice-versa. The first problem is the primal and its's alternative is called dual. The concept of the dual problem is of much significance for the economists as well as for the manager. Because it helps them for formulating future plans by providing important information regarding the utilisation of the resources and it also helps in reducing the time and cost, that is, the dual problem is the most efficient and quick in supplying information. In other words, the dual problem requires us to find the very smallest valuation of the company's stock of inputs which completely accounts for all of the profits of each of the outputs.

The main objective of changing from primal to dual is to minimise the number of constraints. After changing the primal to dual the steps required to find the optimal solution are :

(i) If the primal problem is of maximization, it's dual problem will be of minimization, and vice-versa.

(ii) The constraints values at the right hand side will form the objective function of the dual problem;

(iii) It follows from the second, that the objective function coefficients will become the right hand side values of the constraints of the dual problem;

(iv) The transpose of the primal constraints coefficients become the dual constraint coefficients; and

(v) The inequality signs of the constraints are reversed.

The procedure and result of changing the primal of a problem to a dual will be clear from the following example :

The problem is already stated in Section. Table 3.1, the original linear programming problem is ;

Maximize $z = 25x_1 + 15x_2$

Subject to the constraints :

$3x_1 + 2x_2 \leq 240$

$2x_1 + 1x_2 \leq 140$

Where $X_1, X_2 \geq 0$

It's dual problem will be as under :

Minimize z = 240 Y_1 + 140 Y_2

Subject to the following constraints :

$3Y_1 + 2Y_2 \geq 25$

$2Y_1 + 1Y_2 \geq 15$

Where $Y_1, Y_2 \geq 0$

Now we can apply the simplex algorithm to the dual problem as :

The above dual problem is transformed and stated for solution as per simplex procedure for minimization case as :

Minimize Z = 240 Y_1 + 140Y_2 + OS_1 + OS_2 + MA_1 + MA_2

Subject to the constraints :

$3Y_1 + 2Y_2 - IS_1 - OS_2 + 1A_1 + OA_2 = 25$

$2Y_1 + 1Y_2 - OS_1 - IS_2 + OA_1 + 1A_2 = 15$

The various solution stages are summarized in Table 3.1 to 3.3. Because all the elements in the Cj-Zj row are positive or zero, hence third tableau represents the optimal program. The Tables 3.1 to 3.3 are as under :

Table 1.8 First Tableau

Program (Basic Variable)	Cost per Unit	Cj Quantity	240 Y_1	140 Y_1	0 S_1	0 S_2	M A_1	M A_2	Replacement Qty.
A_1	M	25	3	2	- 1	0	1	0	25/3
A_2	M	15	2	1	0	-1	0	1	15/2
Zj		40 M	5 M	3 M	-M	-M	M	M	
Cj-Zj			240-5M	140-3M	M	M	0	0	

Incoming Variable Outgoing Variable

Second Table

Programme (Basic Variable)	Cost per Unit	Cj Quantity	240 Y_1	140 Y_2	0 S_1	0 S_2	M A_1	M A_2	Replacement
A_1	M	2.50	0	1/2	1	3/2	1	3/2	5
y_1	240	7.50	1	1/2		1/2	0	1/2	15
Zj	1800 + 2.5 M		240	M/2 + 120-M3/2M - 120			M	3/2 M + 120	
Cj-Zj	0	20-M/2M-120-3/2MO 5/2 M-120)							

Incoming Variable Outgoing Variable

Third and Optimal Tableau

Programme	Cost	Cj	240	140	0	0	M	M

Replacement (Basic Quantity Variable)	Per Unit	Quantity	Y_1	Y_2	S_1	S_2		A_1	A_2
Y_2	140	5	0	1	- 2	3	2	- 3	
Y_1	240	5	1	0	1	- 2	- 1	2	
Zj		1900	240	140	- 40	- 60	4	60	
Cj-Zj			0	0	40	60	1	- 40	M - 60

The optimal program is :

$Y_1 = 5, Y_2 = 5$

The meaning of this program is as follows :

Y_1 = marginal worth of 1 unit of the wiring department = 5

Y_2 = marginal worth of the drilling department of 1 unit = 5

The comparison of the optimal tableaus of the primal (15.4) and its Dual (15.8) clearly demonstrates that the primal is maximising the profits and its dual is just minimising the opportunity cost.

1.2.8 Degeneracy

A situation of degeneracy in linear programming problem by simplex method, where a tie occurs between one or more variables for leaving variable which is known as degeneracy. This occurs when the replacement quantity of the two or more rows have equal least non-negative value. In this situation, there may be more than one optimal solutions. Degeneracy could lead to a situation known as cycling, in which the simplex algorithm alternates back and forth between the same non-optimal solutions, that is, it puts a new variable in, then takes it out in the next tableau, puts it back in and so on. One simple way of dealing with the issue is to select either row arbitraily. If we are unlucky, and cycling does occur, we simply go back and select the other row. By this we can get the optimal solution and the problem of degeneracy will be resolved.

Some authors suggest that in case of tie the row nearest to the top may be selected. Otherwise, the alternative method to decide the key row and leaving variable may be based on the following steps;

(i) Divide each element of the disputed row by the respective values of in the key column;

(ii) The values so obtained are compared step by step from left to right, priority being given to basic variable in case the two values are equal, and

(iii) The row having smaller ratio is to be considered as the key row and the leaving variable is selected on this basis.

1.2.9 Summary

In this lesson, we have studies the methodology for solving LPP with the help of simplex method. All the terms involved are explained and the method is elaborated step by step and a situation of degeneracy is also explained. Further, we have studied about the formulation of dual of a LPP. The concepts are made more clear with the help of some appropriate examples.

1.2.10 Key Concepts

Simplex method, Slack variable, Surplus variable, Basic variable, Artificial variable, Pivot row, Pivot column, Pivot number, Simplex table, Cost function, Profit function, Primal, Dual, Degeneracy.

1.2.11 Long Questions

1. Explain the conditions and procedure of simplex method.
2. Explain the uses of slack variables, surplus variables, Basic variables and artificial variables.
3. Solve the following problem by simplex method. Maximize Profits (Z) = 3x + 4y

   Subject to the constraints :

   $2x + 5Y \geq 19$

   $3x + 37 \geq 17$

   $X, Y \geq 0$

   (X = 3, Y = 3 and Maximum Profit Z = 18)
4. Solve the following linear programming problem by simplex method :

   Minimise Costs (Z) = 3x + 8y

   Subject to constraints :

   $X + Y = 200$

   $X \geq 80$

   $Y \leq 60$

   $X, Y \geq 0$

   (X = 200, Y = 0 and Z = 600)
5. A firm manufactures three types of products from three types of materials. The matris is given as :

	X	Y	Z
M_1	1	2	5
M_2	2	1	1
M_3	5	2	1

   The contribution per unit profit of the three products is Rs. 3/-, Rs. 4/- and Rs. 5/- respectively. The problem is to determine the product mix that will maximize profits. The quantities of the materials are given as $M_1 = 36$, $M_2 = 40$, $M_3 = 50$.

   (X = 2688/70, Y = 232/5 and Minimum Cost Z = 5936/70)
6. A pharmaceuticals company is about to launch production of three new drugs. An objective function designed to minimize ingredient costs, and three product constraint, are shown below :

   Minimize Cost X = 50X + 10 Y + 75Z

   Subject to : $X - Y = 1000$

   $2Y - 2Z = 2200$

   $X \leq 1500$

   $X, Y, Z \geq 0$

What is the optimal solution and cost ?

1.2.12 Short Questions

1. Explain the formulation of dual of a LPP.
2. Write a short note on degeneracy.

1.2.13 Suggested Readings

(i) Bhardwaj & Sabharwal - Mathematics for the Students Economics

(ii) Frank S. Budnick - Applied Mathematics

(iii) Mehta & Madnani - Mathematics for Economics

(iv) A. M. Natarajan, P. Bala Subramani and A. Tamilurasi - Operations Research

LAST UPDATED: MAY, 2023

B.A. PART-II (SEMESTER-III) **PAPER-II**

LINEAR PROGRAMMING

LESSON NO. 2.1 **AUTHOR : DR. BHAWDEEP SINGH**

TRANSPORTATION PROBLEM

2.1.1 Objectives

The main objectives of this lessons are :

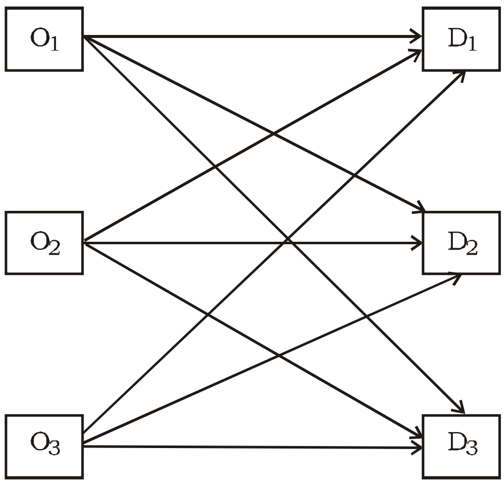
- To minimize total transportation cost for transporting goods from various origins to destinations.
- To study the mathematical formulation of transportation problem.
- To understand various methods for finding the initial basic feasible solution to transportation problem.
- To discuss methods for testing the optimality of transportation problem.

2.1.2 Introduction

The transportation model deals with a special class of linear programming problems in which the objective is to transport a homogenous commodity from various origins to different destinations at a minimum total cost. Given the information regarding the total capacities of the origins, the total requirements of the destinations, and the shipping cost per unit of goods for available shipping routes, the transportation model is used to determine the optimal shipping program that results in minimum total shipping cost. The transportation model can be extended to solve problems related to topics such as production planning, machine assignment, and plant location.

Transportation models can also be used when a firm is trying to decide where to locate a new facility. Before opening a new warehouse, factory, or sales office, it is good practice to consider a number of alternative sites. Good financial decisions concerning facility location also attempt to minimize total transportation and production costs. The eminent scholars like Weber, Polander Losch, Hotelling, etc. emphasised to attain the point of minimum transportation costs so as to achieve to maximum profits for the location of new industrial units.

2.1.3 Mathematical Formulation of the Problem



Formula : Let there are m origins 0_1, 0_2..........0_m having respective capacity of production a_1, a_2...............a_n and Let there are n destinations D_1, D_2...........D_n having respective demand b_1, b_2.........b_n. The cost of transportation of one unit of commodity from $O_i D_j$ be c_{ij}, which is known as cost coefficient (i = 1, 2, ...m; J = 1, 2, 3,...........n). Again if Xij be the number of units to be transported from Oi to Dj, then the problem is to find XiJ so that

$$Z=\sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} X_{ij} \text{ is minimum -................1}$$

Under the Constraints :

$$\sum_{j=1}^{n} X_{ij} \times ai, \quad i = 1, 2......m \text{ (Capacity Constraint) -................. 2}$$

$$\sum_{i=1}^{m} X_{ij} = b_{ij}; \; j = 1, 2.......n \quad \text{(requirement constraints) -........... 3}$$

$X_{ij} \geq 0$ for all i, j.

for, a feasible solution to exist it is necessary that total capacity equal total requirement i.e.,

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \quad \text{(R i m Condition) -................. 4}$$

Transportation Table

Destination ↓ D_1 Origin →		D_2	....	Dn	Available Supply
O1	C_{11}	C_{12}	....	C_1n	a_1
O2	C_{21}	C_{22}	....	C_2n	a_2
......	......	......	....	....	....
Om	Cm_1	Cm_2	....	Cmn	am
Requirement or Demand	b_1	b_2		.... bn	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

Solution Set

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}$$

Observations :

(i) Total supply from ith origin to all destinations is equal to total quantity produced at the ith origin i.e.

$X_{1j} + X_{2j} +$ .... Xin = ai, j=1, 2............., m

(ii) Total quantity transported at jth destination from various origins is equal to the quantity required at the jth destination; ie.,

$X_1j + X_2j +$ .... + Xmj = bj, j = 1, 2, ...n

(iii) A set of non-negative values

Xij, i = 1, 2, - m, j = 1, 2 .....n

that satisfies (i) and (ii)

and $Z = \sum_{i=1}^{m} \sum_{j=1}^{n}$ Cij Xij is called a feasible solution to the transportation problem.

(iv) An initial feasible solution with (m + n - 1) number of variables Xij, i = 1, 2 .... m; j = 1, 2.............n is called a basic feasible solution.

(v) A feasible solution (may not be basic) is said to be optimum if it minimises total transportation cost.

(vi) If $\Sigma ai = \Sigma bj$. T. P. is balanced otherwise, unbalanced.

(vii) Transportation table is ordered into mn, number of boxes arranged in m rows and n column. Each box is known as a cell. The cell situated at ith row and jth column is known as (i, j) Cell.

2.1.4 Assumptions of the Model

The transportation model depends upon the following conditions to be satisfied :

(i) The supply of the items must be equal to the demand/requirements of the consumption centres, that is, there must be a balance between the demand and supply of the items. Otherwise, if unbalance is there, it must be transformed into the balance, by adding a dummy row or column, as it needs, which is significant for solving the problem, and it will not enter into the solution mix.

(ii) Items can easily be transported from every production centre to consumption centres.

(iii) There must be the complete knowledge of the transport costs of every centre to and from.

(iv) The basic objective of the transport model is to minimize the total transport costs not the transport cost of an individual route.

2.1.5 Methods of the Transportation Problems

The transportation method consists of the following three steps :

First Step : It involves making the initial shipping assignment in such a manner that a basic feasible solution is obtained. This means that m + n - 1 cells (routes) of the transportation matrix are used for shipping purposes.

Second Step : This step is to test the optimality of the solution.

Third Step : It involves determining a new and better basic feasible solution. All the steps outlined above are applied in the systematic manner in the transportation method.

2.1.6 Initial Basic Feasible Solution

There are five important methods of developing an initial feasible solution, these are :

(i) North-West Corner Method

(ii) Least-Cost Entry Method

(iii) Vogel's Approximation Method

(iv) Row Minima Method

(v) Column Maxima Method

The detailed description of first three methods is presented as under :

2.1.6.1 North-West Corner Method (Rule)

According to this rule, first allocation is made to the cell occupying the upper left hand (north-west) corner of the matrix. Further, this allocation is of such a magnitude that either the capacity exhausted, or the requirements are satisfied, then we move to the right in the same row, so on so forth, till the capacities and requirements of the problem are not satisfied. In this process, we have no accountability of the transport costs of the different routes. In brief :

(i) Exhaust the supply at each row before moving down to the next row.

(ii) Exhaust the requirements of each column before moving to the right to the next column, and

(iii) Check that all supply and demands are met.

We now apply the north-west corner rule to the transportation problem of Table 4.1.

Table 4.1

Origin		Destination			Capacity
	D_1	D_2	D_3	D_4	
O_1	5	3	6	7	35
O2	2	8	1	9	60
O3	1	4	8	3	25
Destination Requirements	30	45	25	20	120

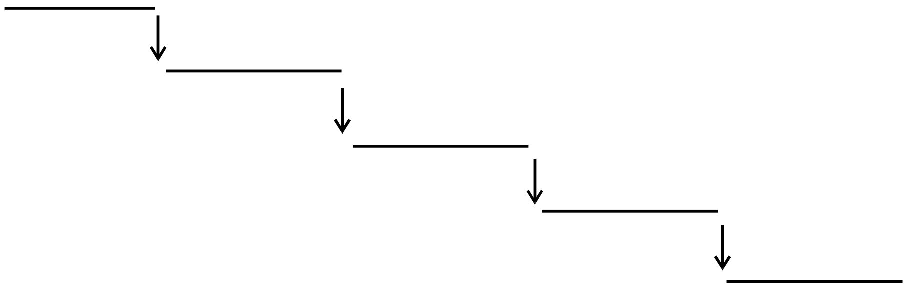
The solution of the above problem by north-west corner rule is as follows :

Table 4.2

Origin		Destination			Capacity
	D_1	D_2	D_3	D_4	
O_1	5 (30)	3 (5)	6	7	35
O2	2	8 (40)	1 (20)	9	60
O3	10	4	(5) 8	(20) 3	25
Destination Requirements	30	45	25	20	120

Detailed Analysis of Allocations to Different Cells

We move from the north-west corner (O_1D_1) allocated 30 units to O_1D_1 which satisfied the requirements of the first destination (D_1), but the capacity of 5 units is in surplus, which is allocated to the next column of D_2 which has a requirement of 45 units and the balance of 40 units is provided from the next capacity origin (O_2), now the requirements of the D_2 destination is satisfied, and similarly we allocate the other capacity to the requirements and move to next until the capacity is exhausted or the requirements satisfied. By this way, we get the complete solution by the North-West Corner Rule. Diagrammatically, the rule can be presented by the following way :



Thus, if x_{ij} is the transportation quantity from origin O_i to destination D_j, then the feasible solution is $x_{11} = 30$, $x_{12} = 5$, $x_{22} = 40$, $x_{23} = 20$, $x_{33} = 5$, $x_{36} = 20$. This is the first basic feasible solution.

Basic Feasible Solution

A basic feasible solution indicates the number of positive allocations (number of occupied cells) which equal to m + n - 1 where m = no. of rows and n is the no. of columns.

The present example satisfiess the criteria, i.e.

Number of Occupied Cells =

m + n - 1 = Number of Columns plus Number of Rows minus 1 = 4 + 3 - 1 = 6

Number of Occupied Cells = m + n - 1 = 6

Hence the initial feasible solution is the basic feasible solution.

Total Transportation Cost

As per the above route of North-West Corner Method, the Total Transportation Cost is as under :

Total Transport Cost for the route = O1D1 + O1D2 + O2D2 + O2D3 + O3D3 + O3D4

is (TTC) = 30 x 5 + 5 x 3 + 40 x 8 + 20 x 1 + 5 x 8 + 20 x 3

= 150 + 15 + 320 + 20 + 40 + 60

= 605

Hence the total transportation cost by North West Corner rule is Rs. 605/-.

2.1.6.2 Least-Cost Method (Lowest Cost Entry Method)

The basic approach of this method is to select that cell in the cost matrix which gives the lowest cost entry and allocate the capacity as per requirements of the cell. This process will go on, till the total capacity is not exhausted and all the requirements are not satisfied. That will be the transportation route by the least cost entry method. This method, generally, provides the lower cost than the north-west corner method. This method is clarified by the example of 4.1. The solution is as under :

Table 1.3

Origins	Destination				Capacity
	D_1	D_2	D_3	D_4	
O_1	5	3 (35)	6	7	35
O2	2 (30)	8 (5)	1 (25)	9	60
O3	10	4 (5)	8	3 (20)	25
Destination Requirements	30	45	25	20	120

In the above table, the Cell O_2D_3 has the least cost of one, its requirements are of 25 units, which are fully allocated out of the capacity of 60 units. The next Cell O_2D has the least cost, its requirements of 30 units are fully allocated out of the remaining 35 units. Both the cells have been allocated the capacity as per constraints. The next least cost cells are O_1D_2 and O_3D_4 with cost Rs. 3. We allocate here according to requirements and avail abality i.e. quantity 35 in O_1D_2 and 20 in O_3D_4. The next Cell O_3D_2 has the least cost of Rs. 4, the remaining five units is allocated.

And the last surplus of the capacity of O_2 of five units is allocated to Cell O_2D_2. Hence this is the way of least cost method to be applied.

Basic Feasible Solution

The initial feasible solution is the basic feasible solution, because the number of the occupied cells is equal to the number of columns plus no. of rows minus one, which is equal to six.

Total Transportation Cost (TTC) for this solution :

TTC = $(O_1D_2 + O_2D_1 + O_2D_2 + O_2D_3 + O_3D_2 + O_3D_4)$ route :
= 35x3 + 30x2 + 5x8 + 25x1 + 5x4 + 20x3
= 105 + 160 + 40 + 25 + 20 + 60
= 310

2.1.6.3 Vogel's Approximation Method (VAM)

Vogel's approximation method is another way of finding the initial feasible solution to the transportation problems. Indeed, this is the best technique and has an edge of superiority over the earlier technique of North-West Corner Rule and the Least Cost Entry Method, because it takes into account the costs associated with each route alternative. To apply VAM, we first compute for each row and column the penalty costs, if we should skip over the second best route instead of the least cost route. The following are six steps to be followed for applying the VAM :

Step 1 For each column and row of the Transportation table, find the difference between the two lowest unit shipping costs.

Step 2 Identify the row or column with the greatest opportunity cost or difference.

Step 3 Assign as many units as possible to the lowest cost square in the row or column selected on the basis of Step 2.

Step 4 Eliminate any row or column that has just been completely satisfied by the assignment just made as per the requirements or the capacity constraint. This can be done by placing X's in each appropriate square.

Step 5 Recompute the cost difference of the remaining rows and columns and revise the Steps from Step 2 to Step 4.

Step 6 The process of calculating the cost differences until the initial feasible, solution can be found out.

These all six steps are now applied to the problem presented in Table 3.1, as follows :

Table 4.4

Origin	Destination				Capacity	Cost
	D_1	D_2	D_3	D_4		Difference
O_1	5	3	6	7	35	2
O2	2	8	1	9	60	1
O3	10	4	8	3	25	1
Destination Requirements	30	45	25	20		
Cost Diference	3	1	5	4		

The greatest cost difference is of column D_3, at least cost cell in this column is O_2D_3 now assign 25 units of requirements out of the capacity of 60 units of cell O_2. As the requirements of this column are satisfied, cross the other cells in this column and recompute the cost difference of the remaining rows and columns :

Table 1.5

Origin	Destination		Capacity		Cost
	D_1	D_2	D_3	D_4	Difference
O_1	5 X	3	6 ×	7	35 (2)
O2	2 (30)	8	1 (25)	9	60 (6)
O3	10 X	4	8 ×	3	25 1
Requirements	30	45	25	20	120
Cost Difference	3	1		4	

The greatest cost difference is of Row O_2, the least cost cell in this row is O_2D_1, which has a requirement of 30 units, this can be allocated out of the remaining capacity of 35 units of O_2 row. The requirement of this column are fully met, so this column is also deleted. And now recompute the cost differences of the remaining rows and column :

Table 1.6

Origin			Destination		Capacity	Cost
	D_1	D_2	D_3	D_4	Difference	
O_1	5 X	3	6 X	7	35	4
O2	2 (30)	8	1 (25)	9	60	(3)
O3	1 X	4	8 X	3 (20)	25	1
Requirements	30	45	25	20	120	
Cost Difference		1		4		

The greatest cost difference is of column D_4 and row O_1, both have equal cost difference of 4, any of these can be selected. Suppose, we are selecting column D_4, and allocate the requirements of 20 units to the least cell, which is O_3D_4, and the column has fully satisfied requirements. Now any one column is left, and there is no choice of allocations of the remaining balances of the capacities may be allocated to the different cells, which will have to satisfy the requirements of the column D_2. In this way, the cell O_1D_2 is allocated 35 units, O_2D_2 is allocated 5 units and the least O_3D_2 is allocated 5 units. So we get the final tableau as per this method, which is presented as follows :

Table 1.7

Origin	Destination				Capacity
	D_1	D_2	D_3	D_4	
O1	5	3 (35)	6	7	35
O2	2 (30)	8 (5)	1 (25)	9	60
O3	10	4 (5)	8	3 (20)	25
Requirements	30	45	25	20	120

The initial feasible solution provided by the above Table 4.7 is basic feasible solution also, because the number of occupied cells is equal to the number of columns plus number of rows minus one. Which is 6.

Total Transportation Cost (TTC) :

$$\text{TTC} = O_1D_2 + O_2D_1 + O_2D_2 + O_2D_3 + O_3D_2 + O_3D_4$$
$$= 35x3 + 30x2 + 5x8 + 25x1 + 5x5 + 20x3$$
$$= 105 + 160 + 40 + 25 + 20 + 60$$
$$= 310$$

Hence the total transportation cost is of Rs. 310/- as computed by the routes of by the Vogel's approximation method, which is equal to the Lowest cost entry method, but less than the North-West Corner Method.

2.1.7 Test of Optimality

The optimality of the routes provided by the above methods can be tested through two methods :

4.5.(1) The Stepping Stone Method

4.5 (2) The Modified Distribution Method (Modi)

Design a New And Better Program

THE THIRD AND THE LAST STEP is to design a new and better program, if the second step shows that the routes provided by the initial feasible solution are not optimal. In this step, the new program and the new routes provided and chalked out by the two methods are used for checking optimality.

Now we are applying the above methods of checking the optimality upon the initial feasible solutions computed by the Lowest Cost Entry Methods or the Least Cost Method.

2.1.7.1 Stepping-Stone Method - A Test For Optimality

The Stepping-Stone Method is a very useful technique for testing the optimality of the transportation problem. This method is applicable when the initial feasible solution is the basic feasible solution, that is, the number of the occupied cells must be equal to the number of rows plus the number of columns minus one. In the next step, we calculate the opportunity cost of each and every empty cell by framing its closed path. If the opprtunity cost of a single cell is positive, then the initial feasible solution is non-optimal and we have to design a new program which will be optimal. Now we are testing the optimality of the North-West Corner Method, as results provided in Table 3.2. The application of the Stepping-Stone Method is as follows :

Table 1.8

Origin	Destination				Capacity
	D_1	D_2	D_3	D_4	
O1	5 30	3 (5)	6 +10	7 +16	35
O2	2 –8	8 (40)	1 (20)	9 +13	60
O3	10 –7	4 (–11)	8 5	3 (20)	25
Requirements	30	45	25	20	120

A closed path is that only squares currently used for shipping can be used in turning the corners of the route being traced. The closed path of O_2D_1 is O_2D_1 - O_2D_2 - O_1D_2 - O_1D_1, similarily we can trace the closed path of the other unused squares, the closed path of O_1D_4 as shown in the above Table 15.8. In the next step, we put a plus sign in the unused square of the closed path and alternative signs to other corner used squares as shown in the Table. All the closed paths of unused squares and their opportunity costs are presented in the following :

Table 1.9

Empty Cell	Closed Path	Net Cost Change	Opport- -unity	Action Cost
O_2D_1	O_2D_1-O_2D_2+O_1D_2-O_1D_1	+2-8+3-5=-8	8	Candidae for including next program
O_3D_1	O_3D_1-O_3D_3+O_2D_3-O_2D_2+O_1D_2 -O_1D_1	+10-8+1-8+3-5=-7	7	-do-
O_3D_2	O_3D_2-O_3D_3+O_2D_3-C_2D_2	+4-8+1-8=-11	11	-do-
O_1D_3	O_1D_3-O_1D_2+O_2D_2-O_2D_3	+6-3+8-1=10	-10	Do not include
O_2D_4	O_2D_4-O_2D_3+O_3D_3-O_3D_4	+9-1+8-3=13	-13	Do not include
O_1D_4	O_1D_4-O_1D_2+O_2D_2-O_2D_3+O_3D_3 -O_3D_4	+7-3+8-1+8-3=16	-16	Do not include

In the Table 4.9, the opportunity cost of three empty cells is positive, which indicates the non-optimality of the solution provided by the North-West Method. For optimality, we move and in corporate the empty cell O_3D_2, because it has the highest opportunity cost. This cell must be included in our next program and we allocate maximum capacity of the minimum of the negative sign number of the closed path which is 5 units. Now we again revise the program and calculate the opportunity cost of each and every cell as we did previously and find out the positive opportunity cost, if any, if not, then the solution is optimal and we have designed a new program. But by calculating the opportunity cost of the revised program we find that the opportunity cost of O_2D_1 is positive, which is 8, we include this into our new program and allocate similarly the minimum of the negative sign figures units of the closed path, which is 30. Now by calculating the opportunity costs, all are negative, hence the new designed program is optimal, which is :

Table 1.10

Origin	Destination				Capacity
	D_1	D_2	D_3	D_4	
O_1	5	3 (35)	6	7	35
O2	2 (30)	8 (5)	1 (25)	9	60
O3	10	4 (5)	8	3 (20)	25
Requirements	30	45	25	20	

Table 4.10 designed a new and a better program and the total transportation cost worked out to be is Rs. 310/-.

2.1.7.2 The Modified Distribution Method (Modi Method)

The MODI method allows us to compute improvement indices quickly for each unused square without drawing all of the closed paths. Because of this, it can often provide considerable time savings over the Stepping-Stone Method for solving transportation problems.

The MODI method is applied after an initial and basic feasible solution is obtained. In this method, we calculate the values of the rows and columns, we apply the method in the following way :

Ui = value assigned to row i.

Vj = value assigned to column j.

Cij = cost in square ij.

For calculating the values of the rows and columns, we use the following formula :

Ui + Vj - Cij = Δij (Opportunity Cost)

Improvement Index = Cij - Ui - Vj

The improvement index is calculated for every unused square.

We shall illustrate the mechanics and rationale of the modified distribution method by solving the transportation problem for which a basic feasible solution is shown in Table 4.2.

From Table 4.2, we first write up an equation for each occupied square.

$U_1 + V_1 = 5$ $\quad$ $U_2 + V_3 = 1$

$U_1 + V_2 = 3$ $\quad$ $U_3 + V_3 = 8$

$U_2 + V_2 = 8$ $U_3 + V_4 = 3$

Let us presume $U_1 = 0$, and we can calculate the other values :

$V_1 = 5; V_2 = 3; U_2 = 5; V_3 = -4; U_3 = 12;$ and $V_4 = -9$

From these values of U's and V's we can compute the improvement index for the unused squares as follows :

Where Os are denoted by U's and D's by Vs.

	Opportunity Cost
$\Delta_{21} = U_2 V_1 = 2 - 5 - 5 = -8$	8
$\Delta_{31} = U_3 V_1 = 10 - 12 - 5 = -7$	7
$\Delta_{32} = U_3 V_2 = 4 - 3 - (12) = -11$	11
$\Delta_{14} = U_1 V_4 = 7 - 0 - (-9) = 16$	– 16
$\Delta_{24} = U_2 V_4 = 9 - 5 - (-9) = 13$	– 13
$\Delta_{13} = U_1 V_3 = 6 - (-4) - 0 = 10,$	– 10

Since the opportunity cost of the three empty cells is positive, which indicates that the solution is not optimal. For optimality, the current program must be revised starting from the cell, which has the maximum positive opportunity cost, which is U3V2 and the procedure is the same as we did in the Stepping-Stone Method. But now we need the specific closed paths for those cells, whose opportunity cost is positive.

2.1.8 Maximization Case

For maximization problems, we substract each and every profit element from the highest element including itself (profit), then solve the problem like the case of minimization.

2.1.9 Degeneracy

Degeneracy in the transportation problem is of very grave concern. It occurs when no. of occupied cells is less than $m + n - 1$. This can be overcome by placing a zero in the independent unused square, so that it may be treated as the occupied cell and proceed as earlier. The degeneracy problem is resolved.

2.1.10 Summary

In this lesson, we have learnt to formulate a transportation problem and understood various terms involved in it. A necessary and sufficient condition is given for the existence of feasible solution to this problem. Then we have studied some methods of finding the initial basic feasible solution to transportation problem out of which Vogel's approximation method is found to be the appropriate one. Further, two methods viz. Stepping Stone method and MODI method are explained to check for the optimality of initial basic feasible solution. The methods are made more elaborative with the help of some appropriate examples.

2.1.11 Key Concepts

Transportation problem, Mathematical Formulation, Origin, Destination, Production Capacity, Demand, North-West Corner method, Trasnportation cost, Least

Cost method, Vogel's Approximation method, Stepping Stone method, MODI method, Degeneracy.

2.1.12 Long Questions

I. Solve the following transportation problems, also apply the test of optimality and if necessary, design the new optimal program.

A Cement manufacturing Co. wishes to transport cement from its three factories P, Q and R to five distribution depots situated at A, B, C, D and E.The quantities produced at the factories per week, requirement at the depts per week and respective transportation cost in Rs. per ton are given in the table below :

Depots

Factories	A	B	C	D	E	Tons Available
P	4	1	3	4	4	60
Q	2	3	2	2	3	35
R	3	5	2	4	4	40
Tons Required	22	45	20	18	30	135

Determine the least cost distribution program for the company.

2. Solve the following transportation problem :

	D1	D2	D3	Available (Quantity)
Q1	8	7	3	60
Q2	3	8	9	70
Q3	11	3	5	80
Required (Quantity)	50	80	80	

(Ans. 750)

3. A company has three plants location A, B and C, which supply to warehouse located at D, E, F, G and H. Monthly plant capacities are 800, 500 and 900 units respectively. Monthly warehouse requirements are 400, 500, 400 and 300 units respectively. Unit transportation cost (in rupees) are given as :

To / From	D	E	F	G	H
A	5	8	6	6	3
B	4	7	7	6	5
C	8	4	6	6	4

Determine the optimum distribution for the company in order to minimize the total transportation cost. (Introduce a dummy plant)

4. Solve the following transportation problem :

	A	B	C	D	Stocks (Qty.)
X	9	5	8	5	225
Y	9	10	13	7	75
Z	14	5	3	7	100

Qty. required	225	80	95	100	400/500

(Ans. 2310)

5. Solve the following transportation problem using Vogel's Approximation Method in order to minimize in the total transportation Cost :

	A	B	C	D	E	Availability
X	3	5	8	9	11	20
Y	5	4	10	7	10	40
Z	2	5	8	7	5	30
Requirements	10	15	25	30	40	

(Ans. 525)

6. A company has three factories A, B and C which supply to four warehouses situated at P, Q, R and S. The monthly production capacity (Tons) of A, B and C are 120, 80 and 200 respectively. The monthly requirements (Tons) for the warehouses P, Q, R and S are 100, 75, 150 and 75 respectively. The transportation cost (Rs. per Ton) matris is as :

	Factories		
Warehouses	A	B	C
P	4	3	7
Q	5	8	4
R	2	4	7
S	5	8	4

Using Vogel's Method determine the optimum transportation distribution of products to warehouse to minimize the total transportation costs.

2.1.13 Short Questions

1. Write the mathematical formulation of transportation problem.
2. Explain the technique of North-West Corner method.
3. For which condition, a feasible solution is said to be optimum?

2.1.14 Suggested Readings

Cohen, S.S. *Operational Research* (London, ELBS, 1987)

Loomba, N.P. Management–A Quantitative Perspective (New York, Macmillan, 1978)

Render, B & Stair R.M. *Quantitative Analysis for Management* (London, Allyn and Bacon INC, 1988)

Sharma, K.K. *Quantitative Techniques and Operations Research* (New Delhi, Kalyani, 1992)

Taha, Hamdy A. *Operations Research–An Introduction* (New York, Macmillan, 1987)

Thierauf, Robert J. *An Introductory Approach to Operations Research* (New York, Wiley, 1978)

B.A. PART–II (SEMESTER-III) **PAPER-II**

LINEAR PROGRAMMING

LESSON NO. 2.2 **AUTHOR : DR. PARMOD KUMAR AGGARWAL**

ASSIGNMENT PROBLEMS

2.2.1 OBJECTIVES:

The main objectives of this lessons are :

- To minimize the total time required for the completion of project by assigning a particular task/job to particular person/machine.
- To study the mathematical formulation of assignment problem.
- To understand various steps involved in Hungarian method for solving assignment problem.
- To discuss about unbalanced problem and its solution.
- To solve travelling salesman problem.

2.2.2 INTRODUCTION :

The assignment problem is a particular case of transportation problem in which the number of jobs or origins or sources are equal to the number of facilities or destinations or machines or persons and so on. The objective is to maximise total profit of allocation or to minimise the total cost. An assignment problem is a completely degenerate form of transportation problem. The units available at each origin and the units demanded at each destination are all equal to one.

2.2.3 MATHEMATICAL FORMULATION OF AN ASSIGNMENT PROBLEM :

Given n jobs or activities and n persons and effectiveness (in terms of cost profit, time and others) of each person for each job, the problem lies in assigning each person to one and only one job so that the given measure of effectiveness is optimised. The data matrix for this problem is shown as :

Jobs (Activities)

Persons (Resources)

	J_1	J_2	. . .	J_n	Supply
W_1	C_{11}	C_{12}	. . .	C_{1n}	1
W_2	C_{21}	C_{22}	. .	C_{2n}	1
:	:	:	:	:	:
W_n	C_{n1}	C_{n2}	. .	C_{nn}	1
Demand	1	1	. . .	1	n

In the above table, C_{iJ} be the cost of assigning ith person to the Jth job.

Let X_{iJ} denote the assignment of person i to job j such that

$X_{iJ} = 1$ if person i is assigned to job j

The assignment problem can be stated as :

$$\text{Minimise } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{iJ} X_{iJ}$$

subject to the constraints

$$\sum_{j=1}^{n} X_{iJ} = 1, \text{ for } i = 1, 2 \ldots\ldots\ldots n$$

and $$\sum_{i=1}^{n} X_{iJ} = 1, \text{ for } j = 1, 2 \ldots\ldots\ldots n$$

and $X_{iJ} = 0$ or 1 for all i and j

C_{iJ} = Cost of assignment of person i to job j.

2.2.4 HUNGARIAN METHOD :

This method was developed by D. Konig, Hungarian mathematician. This method provides us with an efficient method of finding the optimal solution without having to make a direct comparison of every solution. Various steps of calculation of optimal solution can be summarised as :

Step 1: If the number of rows are not equal to number of columns and vice-versa, then a dummy row or column must be introduced with zero cost elements.

Step 2: Find the smallest cost element in each row of the cost matrix. Subtract this smallest cost element from each element in that row. Therefore, there will be atleast one zero in each row of this matrix which is called the Ist reduced cost matrix.

Step 3: In the reduced cost matrix, find the smallest element in each column, subtract the smallest cost element from each element in that column. As a result, there would be at least one zero in each row and column of the second reduced matrix.

Step 4: Determine an optimum assignment :

(i) Examine the row successively until a row with exactly one unmarked zero is obtained. Make an assignment to this single zero by making a square (□) around it.

(ii) For each zero value that becomes assigned, eliminate all other zeros in the same row or column.

(iii) Repeat steps 4 (i) and 4 (ii) for each column with exactly single zero value cell that has not been assigned or eliminated.

(iv) If a row or column has two or more unmarked zeros and one cannot select by inspection, then select the assigned zero cell arbitrarily.

(v) Continue this process until all zero in rows/columns are either enclosed (assigned) or struck off (X).

Step 5: An optimal assignment is found if the number of assigned cells equals the number of rows and columns. If a zero cell is arbitrarily selected, there may be an alternate optimum. If no optimum solution is found (some rows or columns without an assignment) then go to next step.

Step 6: Draw the minimum number of horizontal and vertical lines through all the zeros as follows :

(i) Mark $(\surd)$ to those rows where no assignment has been made.

(ii) Mark $(\surd)$ to those columns which have zeros in the marked rows.

(iii) Mark $(\surd)$ rows (not already marked) which have assignments in marked columns.

(iv) The process may be repeated until no more rows or columns can be checked.

(v) Draw straight lines through all unmarked rows and marked columns.

Step 7: If the minimum number of lines passing through all the zeros is equal to the number of rows or columns, the optimum solution is attained by an arbitrary allocation in the positions of the zeros not crossed in step 3. Otherwise go to the next step.

Step 8: Revise the cost matrix as follows :

(i) Find the elements that are covered by a line, select the smallest of these elements and subtract this element from all the uncrossed elements and add the same at the point of intersection of the two lines.

(ii) Other elements crossed by the lines removed unchanged.

Step 9: Go to step 4 and repeat the method till an optimum solution is obtained.

Example 1

The assignment cost of assigning any one operator to any one machine is given in the following Table :

Operators

		I	II	III	IV
	A	10	5	13	15
Machine	B	3	9	18	3
	C	10	7	3	2
	D	5	11	9	7

Find the optimal assignment.

Solution :

Step 1: Subtracting the smallest element of each row from every element of the corresponding row, we get the reduced matrix as :

5	0	8	10
0	6	15	0
8	5	1	0
0	6	4	2

Step 2: Subtracting the smallest element of each column of the reduced matrix from every element of the corresponding column, we get the following reduced matrix :

5	0	7	10
0	6	14	0
8	5	0	0
0	6	3	2

Example 2

A department has five employees with five jobs to be performed. The time (in hours) each man will take to perform each job is given in the cost matrix.

Employees

		I	II	III	IV	V
	A	10	5	13	15	16
	B	3	9	18	13	6
Jobs	C	10	7	2	2	2
	D	7	11	9	7	12
	E	7	9	10	4	12

How should the jobs be the allocated, one per employee, so as to minimise the total man-hours?

Solution :

Applying step 1 and 2 of the algorithm, the reduced time matrix is shown below:

5	0	8	10	11
0	6	15	10	3
8	5	0	0	0
0	4	2	0	5
3	5	6	0	8

Step 3: Starting with row 1, box a single zero, if any, and cross all other zeros in its column. Thus, we get

5	$\boxed{0}$	8	10	11
$\boxed{0}$	6	15	10	3
8	5	$\boxed{0}$	~~0~~	0
~~0~~	4	2	~~0~~	5
3	5	6	$\boxed{0}$	8

The solution is not optimal since only four assignments are made.

Step 4: Since row D does not have any assignment we tick this row (√)

	I	II	III	IV	V
A	5	$\boxed{0}$	8	10	11
B	$\boxed{0}$	6	15	10	3
C	8	5	$\boxed{0}$	0	~~0~~
D	~~0~~	4	2	~~0~~	5
E	3	5	6	$\boxed{0}$	8

(i) Now, there is a zero in the Ist and IVth column of the ticked row. So, we tick I and IVth column.

(ii) Mark (√) in the rows B and E since columns I and IV have an assignment in rows B and E, respectively.

(iii) Draw straight lines through all unmarked rows A and C and marked columns, I and IV as shown below :

	I	II	III	IV	V	
A	5	$\boxed{0}$	8	10	11	
B	$\boxed{0}$	6	15	10	3	√
C	8	5	$\boxed{0}$	~~0~~	~~0~~	
D	~~0~~	4	2	~~0~~	5	√
E	3	5	6	$\boxed{0}$	8	√
	√			√		

Step 5: In step 4, the minimum number of lines drawn is 4, which is less than the order of the cost matrix (order is 5), indicating that the current assignment is not optimum.

To increase the minimum number of lines, we generate new zeros in the modified matrix.

Step 6: Develop the new Table by selecting the smallest element among all uncovered elements by the lines. Here, the element is 2. Subtract this element from all the uncovered elements and add the same to all the elements lying at the intersection of the lines. We obtain the following new reduced cost matrix.

$$\begin{bmatrix} 7 & 0 & 8 & 12 & 11 \\ 0 & 4 & 13 & 10 & 1 \\ 10 & 5 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 3 \\ 3 & 3 & 4 & 0 & 6 \end{bmatrix}$$

Step 7: Repeat 3 to 6 to find a new solution. The new assignment is :

$$\begin{bmatrix} 7 & \boxed{0} & 8 & 12 & 11 \\ \boxed{0} & 4 & 13 & 10 & 1 \\ 10 & 5 & \not{0} & 2 & \boxed{0} \\ \not{0} & 2 & \boxed{0} & \not{0} & 3 \\ 3 & 3 & 4 & \boxed{0} & 6 \end{bmatrix}$$

Since the number of assignments (5) equals the number of rows (5), the solution is optimal. The optimum assignment is :

$A \rightarrow II$, $B \rightarrow I$, $C \rightarrow V$, $D \rightarrow III$, $E \rightarrow IV$

The minimum total time for this assignment schedule is 5 + 3 + 2 + 9 + 4 = 23 hours.

2.2.5 VARIATIONS OF THE ASSIGNMENT PROBLEM :

1. Non-square matrix (unbalanced assignment problem) :

The Hungarian method of assignment requires that the number of columns and rows in the assignment matrix must be equal. When the given cost matrix is not a square matrix, then the problem is called an *unbalanced problem.* In this case dummy row(s) or column(s) with zero cost is/are added to make it a square matrix. These cells are treated in the same way as the real cells during the process. Then, adopt the Hungarian method to find the solution.

2. Maximisation problem :

There may be problems of maximising the profit, revenue, and so on. Such problems may be solved by converting the given maximisation problem into a minimisation problem before the Hungarian method is applied. The transformation may be done in the following two ways:

(i) by subtracting all the elements from the highest element of the matrix.

(ii) by multiplying the matrix elements by – I.

3. Multiple optimal solutions

While making an assignment in the reduced assignment matrix, it is possible to have two or more ways to strike off certain number of zeros. Such situation leads to multiple solutions with the same optimal value of objective function. In such cases the most suitable solution may be considered by the decision-maker.

4. Restrictions on assignments (or) impossible assignment

Cells in which assignments are not allowed are assigned a very heavy cost (written as M or ∞). Such cells are prohibited to enter into the final solution.

2.2.6 UNBALANCED PROBLEM :

When demand total is not equal to supply total, it is called unbalanced transporation problem. In this case, dummy row/column is added with zero cost elements in order to get balanced transporation problem.

Example 3

In the modification of a plant layout of a factory four new machines M_1, M_2, M_3, M_4 are to be installed in a machine shop. There are five vacant places A, B, C, D and E that are available. Because of limited space, machine M_2 cannot be placed at C and M_3 cannot be placed at A. The cost of placing of machine *i* and place *j* (in rupees) shown below :

		Location				
		A	B	C	D	E
	M_1	9	11	15	10	11
Machine	M_2	12	9	–	10	9
	M_3	–	11	14	11	7
	M_4	14	8	12	7	8

Find the optimal assignment schedule.

Solution: As the cost matrix is unbalanced, add one dummy row with zero cost. Also, assign a high cost M to the pair (M_2, C) and (M_3, A). The cost matrix is shown below :

	A	B	C	D	E
M_1	9	11	15	10	11
M_2	12	9	M	10	9
M_3	M	11	14	11	7
M_4	14	8	12	7	8
M_5	0	0	0	0	0

Apply the Hungarian method to get the optimal solution. The optimal solution is shown below:

	A	B	C	D	E
M_1	[0]	2	6	1	2
M_2	3	[0]	M	1	~~0~~
M_3	M	4	7	4	[0]
M_4	7	1	5	[0]	1
M_5	~~0~~	~~0~~	[0]	~~0~~	0

The optimal solution is obtained. The schedule are :

$M_1 \rightarrow A, M_2 \rightarrow B, \quad M_3 \rightarrow C, \quad M_4 \rightarrow D, \quad M_5 \rightarrow C$

and the total minimum cost is Rs. : 9 + 9 + 7 + 7 + 0 = Rs. 32.

2.2.7 MAXIMISATION PROBLEM :

Example 4

A company has four territories open, and four salesman available for the assignment. The territories are not equally rich in their sales potential; it is estimated that a typical salesman operating in each territory would bring in the following annual sales:

Territory :	I	II	III	IV
Annual Sales (Rs) :	60,000	50,000	40,000	30,000

Four salesman are also considered to differ in their ability. It is estimated that working under the same conditions, their yearly sales would be proportional as follows:

Salesman	A	B	C	D
Proportion	7	5	5	4

If the criterion is maximise expected total sales, then the intuitive answer is to assign the best salesman to the richest territory, the next best salesman to the second richest and so on. Verify this answer by the assignment technique.

Solution :

Step 1: Construct the effectiveness matrix.

To avoid the fractional values of annual sales of each salesman in each territory, for convenience consider the yearly sales as 21 (the sum of the sale proportion = 7 + 5 + 5 + 4 = 21), taking Rs. 10,000 as one unit. Now, divide the individual sales in each territory by 21 to obtain the required annual sales by each salesman.

Thus, the maximum sales matrix is obtained as follows :

Sales in 10,000 Rupees

Sales proportion		6	5	4	3
		I	II	III	IV
7	A	42	35	28	21
5	B	30	25	20	15
5	C	30	25	20	15
4	D	24	20	16	12

Step 2: Convert maximisation into minisation problem.

Subtract from the highest element (i.e. 42) among all the elements of the given cost matrix. The resulting matrix is

	I	II	III	IV
A	0	7	14	21
B	12	17	22	27
C	12	17	22	27
D	18	22	26	30

Step 3: Apply Hungarian method to get the optimal solution.

(i) Subtract the smallest element in each row from every element in that row.

(i) Subtract the smallest element in each column. The reduced matrix is obtained as follows:

	I	II	III	IV
A	0	3	6	9
B	0	1	2	3
C	0	1	2	3
D	0	0	0	0

Step 4: Assignment is made in row A. All zeros in the assigned column I are crossed out. Column II has only one zero in cell (D, II). Assignment is made in this column and other zeros are crossed in row D. The reduced matrix is shown below :

	I	II	III	IV
A	[0]	3	6	9
B	~~0~~	1	2	3
C	~~0~~	1	2	3
D	~~0~~	[0]	~~0~~	~~0~~

Step 5: Since row B and C does not have any assignments, we tick this row. Since there is a zero in the I column of the ticked row, we tick I column. Further, there is an assignment in the first row of the ticked column, so we tick first row. Draw lines through all unmarked rows and marked columns.

	I	II	III	IV	
A	[0]	3	6	9	√
B	~~0~~	1	2	3	√
C	~~0~~	1	2	3	√
D	~~0~~	[0]	~~0~~	~~0~~	
	√				

The number of lines drawn is 2, which is less than order of cost matrix, therefore the current assignment is not optimal.

Step 6: Revised matrix is developed by selecting the minimum element (= 1) among all uncovered elements by the lines. Subract 1 from each uncovered element and add it to the element at the intersection of two lines. The revised Table is :

	I	II	III	IV
A	[0]	2	5	8
B	~~0~~	[0]	1	2
C	~~0~~	~~0~~	1	2
D	1	~~0~~	~~0~~	[0]

Again, the solution is not optimal. Repeat Step 1 to 5. Two alternative optimal assignments we get are :

	I	II	III	IV
A	[0]	2	4	7
B	~~0~~	~~0~~	[0]	1
C	~~0~~	[0]	~~0~~	1
D	2	1	~~0~~	[0]

	I	II	III	IV
A	[0]	2	4	7
B	~~0~~	~~0~~	[0]	1
C	~~0~~	[0]	~~0~~	1
D	2	1	~~0~~	[0]

The two possible solution are :

(i) A → I, B → III, C → II, D → IV

(ii) A → I, B → II, C → III, D → IV

with maximum sales of Rs. (42 + 20 + 25 + 12) = Rs. 99

[or Rs. (42 + 25 + 20 + 12) = Rs. 99.]

Both solution show the best salesman A is assigned to the richest territory I, the worst salesman D to the poorest territory IV. Salesman B and C are equally good and they may be assigned VI or III.

2.2.8 MULTIPLE SOLUTION :

Example 5

Solve the minimal assignment problem whose cost matrix is given below :

	1	2	3	4
I	2	3	4	5
II	4	5	6	7
III	7	8	9	8
IV	3	5	8	4

Solution :

(i) Subtract the smallest element in the row from each element in that row.

(i) Subtract the smallest element in the column from each element in that column. The reduced matrix is :

	1	2	3	4
I	0	0	0	2
II	0	0	0	2
III	0	0	0	0
IV	0	1	3	0

Since single zeros do not exist either in the columns or in the rows, we get the following alternative solutions :

	1	2	3	4
I	×0	[0]	×0	2
II	×0	×0	[0]	2
III	×0	×0	×0	[0]
IV	[0]	1	3	×0

	1	2	3	4
I	0	[0]	0	2
II	×0	×0	[0]	2
III	×0	×0	×0	[0]
IV	[0]	1	3	×0

	1	2	3	4
I	×0	×0	[0]	2
II	×0	[0]	×0	2
III	×0	×0	×0	[0]
IV	[0]	1	3	0

The possible optimal solutions with each of cost Rs. 20 are :

I → 2, II → 3, III → 4, IV → 1
I → 1, II → 2, III → 3, IV → 4
I → 3, II → 2, III → 1, IV → 4
I → 3, II → 2, III → 4, IV → 1
I → 2, II → 3, III → 1, IV → 4

2.2.9 RESTRICTIONS ON ASSIGNMENTS :

Example 6

Four new machines M_1, M_2, M_3, M_4 are to be installed in a machine shop. There are five vacant places A, B, C, D, E that are available. Because of limited space, machine M_2 cannot be placed at C and M_3 cannot be placed at A. The cost matrix is shown below :

	A	B	C	D	E
M_1	4	6	10	5	4
M_2	7	4	–	5	4
M_3	–	6	9	6	2
M_4	9	3	7	2	3

Find the optimal assignment schedule.

Solution : As machine M_2 cannot be placed at C, M_3 cannot be placed at A, assign a large cost M in cells (M_2, C) and (M_3, A). The assignment problem is unbalanced. So, balance it by adding a dummy row with cost 0 as shown below :

	A	B	C	D	E
M_1	4	6	10	5	4
M_2	7	4	M	5	4
M_3	M	6	9	6	2
M_4	9	3	7	2	3
M_5	0	0	0	0	0

Apply the Hungarian method to get the optimal solution. The optimal solution is shown below:

	A	B	C	D	E
M_1	[0]	2	6	1	2
M_2	3	[0]	M	1	~~0~~
M_3	M	4	7	4	[0]
M_4	7	1	5	[0]	1
M_5	~~0~~	~~0~~	[0]	~~0~~	~~0~~

The optimal assignment is :

$M_1 \rightarrow A$, $M_2 \rightarrow B$, $M_3 \rightarrow C$, $M_4 \rightarrow D$, $M_5 \rightarrow C$ (C will remain vacant)

Total assignment cost = Rs. (4 + 4 + 2 + 2) = Rs. 12.

2.2.10 CREW BASED ASSIGNMENT PROBLEM

The method of solution discussed in this section can be used to plan the assignment of crew members in different locations by a transport company.

Example 7

A trip from Chandigarh to Delhi takes six hours by bus. A typical time table of the bus service in both directions is given below :

Departure from Chandigarh	**Route Number**	**Arrival at Delhi**	**Arrival at Chandigarh**	**Route Number**	**Departure from Delhi**
06.00	a	12.00	11.30	1	05.30
07.30	b	13.30	15.00	2	09.00
11.30	c	17.30	21.00	3	15.00
19.00	d	01.00	00.30	4	18.30
00.30	e	06.30	06.00	5	00.00

The cost of providing this service by the transport company depends upon the time spent by the bus crew (driver and conductor) away from their places in addition to service time. There are five crews. There is a constraint that every crew should be provided with more than 4 hours of rest before the return trip again and should not wait for more than 24 hours for the return trip. The company has residential facilities for the crew at Chandigarh as well as at Delhi. Find the optimal service line connections.

Solution : As the service time is constant for each line it does not appear directly in the computation. If the entire crew resides at Chandigarh then the waiting times in hours at Delhi for different route connections are given in the following Table :

	1	2	3	4	5
a	17.5	21	M	6.5	12
b	16	19.5	M	5	6.5
c	12	15.5	21.5	M	6.5
d	4.5	8	14	17.5	23
e	23	M	8.5	12	17.5

If route a is combined with route 1, the crew after arriving at Delhi at 12 noon start at 5.30 next morning. Thus, the waiting time is 17.5 hrs.

Some of the assignments are infeasible. Route 3 leaves Delhi at 15:00 hrs. Thus, the crew of route a reaching Delhi at 12 noon are unable to take rest of 4 hours if they are asked to leave by route 3. Hence, (a, 3) is an infeasible assignment. Its cost is thus, M, a large positive number.

Similarly, if the crew are assumed to reside at Delhi (so that they start from and came back to Delhi with halt for minimum time at Chandigarh), then waiting time at Chandigarh for different service line connections are given by the following Table.

	1	2	3	4	5
a	18.5	15	9	5.5	M
b	20	16.5	10.5	7	M
c	M	20.5	14.5	11	5.5
d	7.5	M	22	18.5	13
e	13	7.5	M	M	18.5

As the crew can be asked to reside at Chandigarh or Delhi, minimum waiting time from the above operation can be computed for different route combination by choosing the minimum of the two waiting time. These values of the waiting time are shown below:

	1	2	3	4	5
a	17.5	15	9	5.5	12
b	16	16.5	10.5	5	10.5
c	12	15.5	14.5	11	5.5
d	4.5	8	14	17.5	13
e	13	9.5	8.5	12	17.5

Hungarian method can now be applied for finding the optimal route connections which gives minimum over all waiting time and hence the minimum cost of bus service operations. It consists of the following steps.

Step 1: Find the first and second reduced cost matrix.

	1	2	3	4	5
a	12	9.5	3.5	0	6.5
b	11	11.5	3.5	0	5.5
c	6.5	10	9	5.5	0
d	0	3.5	9.5	13	8.5
e	4.5	1	0	3.5	9

	1	2	3	4	5
a	12	8.5	3.5	0	6.5
b	11	10.5	5.5	0	5.5
c	6.5	9	9	5.5	0
d	0	2.5	9.5	13	8.5
e	4.5	0	0	3.5	9

Step 2: Check for optimality.

	1	2	3	4	5
a	12	8.5	3.5	[0]	6.5
b	11	10.5	5.5	~~0~~	5.5
c	6.5	9	9	5.5	[0]
d	[0]	2.5	9.5	13	8.5
e	4.5	[0]	~~0~~	3.5	9

The solution is not optimal, as the number of assignments is less than the number of rows or columns.

Step 3: Mark √ in row b since no assignment was made in this row. Note that row b has zero in column 4, therefore mark √ in column 4. We then mark √ in row a, since column 4 has an assigned zero in row a. Draw straight lines through all unmarked rows and marked columns. The resulting matrix is :

	1	2	3	4	5	
a	12	8.5	3.5	[0]	6.5	√
b	11	10.5	5.5	~~0~~	5.5	√
c	6.5	9	9	5.5	[0]	
d	[0]	2.5	9.5	13	8.5	
e	4.5	[0]	~~0~~	3.5	9	
				√		

Step 4: Develop a new revised matrix. Examine those elements that are not covered by the lines and among those elements take the smallest element. Here, 3.5 is the element. Subtract 3.5 from the uncovered elements and add 3.5 with the elements at the intersection of two lines, we get the new Table as :

8.5	5	0	0	3
7.5	7	2	0	2
6.5	9	9	9	0
0	2.5	9.5	16.5	8.5
4.5	0	0	7	9

Step 5: Check if optimal assignment can be made in the current feasible solution.

8.5	5	[0]	0̸	3
7.5	7	2	[0]	2
6.5	9	9	9	[0]
[0]	2.5	9.5	16.5	8.5
4.5	[0]	0̸	7	9

As the number of assignment is equal to the number of rows or columns, the current solution is optimal.

Therefore, the routes to be paired to achieve the minimum waiting time are a → 3, b → 3, c → 5, d → 1 and e → 2. We can obtain the waiting times of these assignments as well as the residence of the crew as

Crew	Resident at	Service number	Waiting time hours
1	Chandigarh	(d–1)	4.5
2	Delhi	(2–e)	9.5
3	Delhi	(3–a)	9.0
4	Chandigarh	(b–4)	5.0
5	Delhi	(5–c)	5.5

Total = 33.5

Total minimum waiting time is thus 33.5 hours.

2.2.11 TRAVELLING SALESMAN PROBLEM :

Suppose a salesman wants to visit a certain number of cities starting from his headquarters. The distances (or cost or time) of journey between every pair of cities, denoted by c_{ij}, that is, distance from city *i* to city *j* is assumed to be known. The problem is:

Salesman starting from his home city visited each city only one and returns to his home city in the shortest possible distance (or at the least cost or in the least time).

Given n cities and distance c_{ij}, the salesman starts from city 1, then any permutation of 2, 3, ..., n represents the number of possible ways for his tour. So, there are (n–1)! possible ways for his tour. The problem is to select an optimal route that could achieve his objective.

The problem may be classified as :

(i) *Symmetrical:* If the distance between every pair of cities is independent of the direction of his journey.

(ii) *Asymmetrical:* For one or more pair of cities the distance changes with the direction.

Example 8

A machine operator processes five types of items on his machine each week, and must choose a sequence for them. The set-up cost per change depends on the item presently on the machine and the set-up to be made according to the following table.

From item	To item				
	A	B	C	D	E
A	∞	4	7	3	4
B	4	∞	6	3	4
C	7	6	∞	7	5
D	3	3	7	∞	7
E	4	4	5	7	∞

If he processes each type of item once and only once each week, how should he sequence the items on his machine in order to minimise the total set-up cost?

Solution:

Step 1: Reduce the cost matrix using Step 1 and 2 of the Hungarian algorithm and then make assignments in rows and columns having single zeros as usual.

∞	1	3	[0]	1
1	∞	2	~~0~~	1
2	1	∞	2	[0]
~~0~~	[0]	3	∞	4
[0]	~~0~~	~~0~~	3	∞

Step 2: Note that row 2 is not assigned. So, mark √ to row 2. Since there is a zero in the 4th column of the marked row, we tick 4th column. Further, there is an assignment in the first row of 4th column. So, tick first row. Draw lines through all unmarked rows and marked columns. We can find the number of lines is 4 which is less than the order of the matrix. So, go to next step (see table).

∞	1	3	[0]	1	√
1	∞	2	~~0~~	1	√
2	1	∞	2	[0]	
~~0~~	[0]	3	∞	4	
[0]	~~0~~	~~0~~	3	∞	
			√		

Step 3: Subtract the lowest element from all the elements not covered by these lines and add the same with the elements at the intersection of two lines. Then we get the table as:

	1	2	3	4	5
1	∞	~~0~~	2	[0]	~~0~~
2	[0]	∞	1	~~0~~	~~0~~
3	2	1	∞	3	[0]
4	~~0~~	[0]	3	∞	4
5	~~0~~	~~0~~	~~0~~	4	∞

The optimum assignment is 1→ 4, 2 → 1, 3 → 5, 4 → 2, 5 → 3 with minimum cost as Rs. 20.

This assignment schedule does not provide us the solution of the travelling salesman problem as it gives 1→ 4, 4 → 2, 2 → 1, without passing through 3 and 5.

Next, we try to find the next best solution which satisfies this restriction. The next minimum (non-zero) element in the cost matrix is 1. So, we bring 1 into the solution. But the element '1' occurs at two places. We consider all cases separately until we get an optimal solution.

We start with making an assignment at (2, 3) instead of zero assignment at (2, 1). The resulting assignment schedule is

1→ 4, 4 → 2, 2 → 3, 3 → 5, 5 → 1

When an assignment is made at (3, 2) instead of zero assignment at (3, 5), the resulting assignment schedule is

1→ 5, 5 → 3, 3 → 2, 2 → 4, 4 → 1

The total set-up cost in both the cases is 21.

Example 9

Solve the travelling salesman problem given by the following data :

$c_{12} = 20$, $c_{13} = 4$, $c_{14} = 10$, $c_{23} = 5$, $c_{34} = 6$

$c_{25} = 10$, $c_{35} = 6$, $c_{45} = 20$ where $c_{ij} = c_{ji}$

and there is no route between cities *i* and *j* if the value for cij is not shown.

Solution : The cost matrix is :

∞	20	4	10	∞
20	∞	5	∞	10
4	5	∞	6	6
10	∞	6	∞	20
∞	10	6	20	∞

Repeating the steps as before using the Hungarian algorithm, the optimum table obtained is :

∞	12	[0]	~~0~~	∞
11	∞	~~0~~	∞	[0]
~~0~~	1	∞	[0]	1
[0]	∞	~~0~~	∞	9
∞	[0]	~~0~~	8	∞

The solution is

$$1 \rightarrow 3,\ 3 \rightarrow 4,\ 4 \rightarrow 1,\ 5 \rightarrow 2,\ 2 \rightarrow 5$$

which is not the solution of the travelling salesman problem as the sequence obtained is not in the cyclic order.

The next lowest number (other than 0) is 1. Therefore, make an assignment in the cell (3, 2) having the element 1. Consequently, make an assignment in the cell (5, 4) having element 8, instead of zero element in the cell (5, 2). The assignment Table is

∞	12	[0]	~~0~~	∞
11	∞	~~0~~	∞	[0]
~~0~~	[1]	∞	~~0~~	1
[0]	∞	~~0~~	∞	9
∞	~~0~~	~~0~~	[8]	∞

The shortest for the travelling salesman is

$$1 \rightarrow 3 \rightarrow 2 \rightarrow 5 \rightarrow 4 \rightarrow 1$$

2.2.12 SUMMARY

In this lesson, we have learnt to formulate an assignment problem and understood the various terms. It is found that for an assignment problem, number of jobs/sources must be equal to number of persons(machines)/destinations. Then we have explained the steps involved in the well known Hungarian method for solving an assignment problem. Further, the concept of maximization problem, multiple solution, unbalanced problem, assignment with restrictions, crew based assignment is also covered. Also, a solution the traveling salesman problem is explained. All these concepts are presented in an elaborative manner with the help of some suitable examples.

2.2.13 KEY CONCEPTS

Assignment problem, Mathematical Formulation, Assignment cost, Hungarian method, Unbalanced problem, Multiple solution, Travelling salesman problem.

2.2.14 LONG QUESTIONS

1. What is an assignment problem? Give two areas of its applications.
2. Explain the conceptual justification that an assignment problem can be viewed, as a linear programming problem.
3. Find the optimal solution for the assignment problem with the following cost matrix.

	I	II	III	IV	V
A	11	17	8	16	20
B	9	7	12	6	15
C	13	16	15	12	16
D	21	24	17	28	26
E	14	10	12	11	15

(Answer. A → 1, B → IV, C → V, D → III, E → II, minimum cost = 60.)

4. Five men are available to do five different jobs. From the past records, the time (in 2 hour) that each man takes to do each job is known and given in the following Table.

	I	II	III	IV	V
A	2	9	2	7	1
B	6	8	7	6	1
C	4	6	5	3	1
D	4	2	7	3	1
E	5	3	9	5	1

Find the assignment of men to jobs that will minimise the total time taken.

(Answer. A → III, B → V, C → I, D → IV, E → II, optimal value = 13 hours.)

5. Solve the following assignment problem.

	1	2	3	4	5
A	8	4	2	6	1
B	0	9	5	5	4
C	3	8	9	2	6
D	4	3	1	0	3
E	9	5	8	9	5

(Answer. A → 5, B → 1, C → 4, D → 3, E → 2, minimum cost = 9.)

6. For the following problem of assigning four sales persons to four different sales regions, find assignments in order to maximise sales.

Sales Region

		I	II	III	III
	A	10	22	12	14
Salesman	B	16	18	22	10
	C	24	20	12	18
	D	16	14	24	20

2.2.15 SHORT QUESTIONS

1. Explain the difference between a transportation problem and an assignment problem.
2. What is an unbalanced assignment problem and How it can be solved?.
3. What is travelling salesman problem?

2.2.16 SUGGESTED READINGS

1. Loomba, N.P. : Management – A Quantitative Perspective (New York, Macmillan, 1978).
2. Sharma, K.K. : Quantitative Techniques and Operation Research (New Delhi, Kalyani, 1992).
3. Taha, Hamdy A. : Operations Research – An Introduction (New York, Macmillan, 1987).
4. Swarup Kanti, Gupta, P.K. and Manmohan : Operations Research (New Delhi, Sultan Chand & Sons, 2005).

Mandatory Student Feedback Form

https://forms.gle/KS5CLhvpwrpgjwN98

Note: Students, kindly click this google form link, and fill this feedback form once.