Centre for Distance and Online Education

Punjabi University, Patiala

Class : BBA-Part-I **Semester : I**

Paper : BBA-103 (Business Mathematics) Unit : I

Medium : English

Lesson No.

Department website : www.pbidde.org

BBA-103 : BUSINESS MATHEMATICS

Maximum Marks: 100 **Time Allowed: 3 Hrs**
Theory: 60
Internal Assessment: 40

Instructions for Paper-setters/Examiners

The question paper covering the entire course shall be divided into three sections as follows:

Section-A

It will consist of four essay type questions (two numerical and two theoretical) set by the examiner from Part-I and the candidate shall be required to attempt two. Each question shall carry ten marks; total weight of the section is 20 marks.

Section-B

It will consist of four essay type questions (two numerical and two theoretical) set by the examiner from Part-I and the candidate shall be required to attempt two. Each question shall carry ten marks; total weight of the section is 20 marks.

Section- C

It will consist of ten short answer questions. All questions are compulsory. Each question shall carry two marks; total weight of the section is 20 marks.

Course Input :

Part-I

Functions: Introduction, Characteristics of a Function, Linear Function, Exponential Function, Logarithmic Function.
Matrices: Introduction, Types of Matrices, Operations on Matrices, Transpose and Inverse of a Matrix. Solutions of a System of Linear Equations: Cramer's Rule and Matrix Inverse Method.
Differentiation: Concept of Limit, Simple Derivatives Excluding Trigonometric Functions; Partial Differentiation, Homogenous Functions. Euler's Theorem, Applications of Differentiation in Business. Maxima and Minima of up to Two Independent Variables.

Part-II

Linear Programming: Graphic and Simplex Methods; Transportation Problem : Methods and Tests; Assignment Problem and Game Theory.

Recommended Readings:

1. Ajay Goel and Alka Goel : Mathematics and Statistics (Taxmann's)
2. Qazi Zameeruddin, et al. : Business Mathematics (Vikas)
3. G.S. Monga : Mathematics for Management and Economics (Vikas)
4. Tara Yamane : Mathematics for Economists (Prentice Hall)
5. Render and Stair : Quantitative Analysis for Management

LAST UPDATED ON : JUNE, 2023

BBA PART- I **PAPER : BBA-103**

SEMESTER-I **BUSINESS MATHEMATICS**

LESSON NO. 1.1

AUTHOR : Mrs. AMANPREET KAUR

FUNCTIONS AND THEIR DERIVATIVES

1.1.1 Objectives

In this lesson, the main objectives are :

(i) to define y as function of x i.e. y = f(x) where change in y is due to change in x

(ii) to discuss types of functions

(iii) to understand the concept of limit of a function.

(iv) to understand the concept of derivative of a function and various rules for finding the derivatives of different types of functions.

1.1.2 Introduction to Functions

The concept of relations leads us to the concept of functions, denoted by f(x).

We have to deal with several variables. The way in which one variable depends on other variables is described by means of functions.

A function is thus a relation which associates any given number with another number.

A function is normally represented by the letter f.

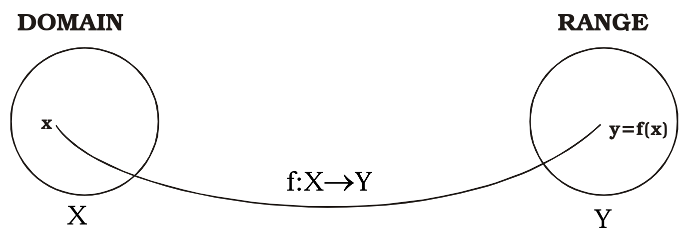
We define a function from the set X into the set Y as a set of ordered pairs, (x,y) where x is an element of X and y is an element Y, such that for each x in X there is only one ordered pair (x,y) in the function. (i.e. $f : X \rightarrow Y$ such that f(x) = y)

A function is a mapping or transformation of X into Y or f(x).

The variable x presents elements of the domain and is called the independent variable.

The variable y represents elements of the range and is called the dependent variable becuase y is dependent on x.

The function y = f(x) is often called a single valued function since there is a unique y in the range for each specified x.



For example, if p is the price and x the quantity of a commodity demanded by the consumers, we write the demand functions as: x = f (p)

i.e. quantity demanded of a commodity depends upon its price.

1.1.3 Classification of Functions:

Some types of function are:

(i) **Real valued functions:** A function whose domain and range are set of real numbers is called a real valued function of a real variable.

(ii) **Constant function:** A function is constant function if the range consists of a single element. It may be written as : y = k, or f(x) = k, where k is a constant.

(iii) **Explicit function:** In an explicit function, one variable is expressed directly in terms of other variable,

For Example: -

$$y = 5x+3, \qquad y = 6x + 2 \qquad \text{and } y = 7x+4$$

(iv) **Implicit function: -** In an implicit function, the relation between the variables is given by an equation containing all the variables, such that dependent variables cannot be distinguished.

For Example :-

$$5xy + 6x + 2y = 20$$

(v) Single valued/uniform function:- If to each value of x there corresponds one and only one vlaue of y, the function is said to be single valued or uniform function,

For Example: -

$$y = 3x + 2$$

(vi) Multi-valued or multiform function: - If more than one vlaue of y corresponds to one vlaue of x, the function is said to be multivalued.

For Example:-

$y = \sqrt{x}$ i.e. Positive and negative both.

(any value of x, say 16, gives two values of y i.e. -4 and 4)

(vii) Monotone functions: - A function is monotone if it is either increasing or decreasing. A function is increasing if there is an increase in the value of the dependent variable when value of the independent variable increases.

For Example: -

$y = x^2$ $x > 0$ (x should be positive)

A function is decreasing if with the increase in value of independent variable, the value of dependent variable decreases.

For Example: -

Demand = f (price).

(viii) Inverse function: If there exist an implict function between x and y, we can derive two explicit functions from it, y as an explicit function of x as

$$y = f^{-1}(x)$$

(ix) Linear function: A polynomial function of degree 1 is called a linear function.

For Example :

$f(x) = mx + c$ is a linear function.

The graph of linear function is a line.

Linear function is also called unit function. The value of x is linear.

(x) Quadratic Function: A polynomial function of degree 2 is called a quadratic function.

For Example : - $f(x) = ax^2 + bx + c$ (the value of x is 2)

1.1.4 Some Main Functions Used in Economics:

In Economics, we generally make use of the following main functions:-

(i) Demand Function: - Let q denote the quantity of a commodity and p be its price. Then the demand function may be written as:

$q = f(p)$, showing dependence of q on p.

The demand curve has a negative slope because with the increase in price, the quantity demanded decreases.

(ii) **Supply Function:-** The above function q = f(p) can also denote the supply function if we consider q to be quantity supplied.

In general, the slope of a supply curve is positive, because an increase in price leads to an increase in supply.

(iii) **Cost Function:** If a firm produces a quantity x at total cost C, we have the total cost function, C = f(x)

Average cost function is given by:

Average cost function = $\frac{C}{x}$ where C is total cost, x is quantity produced.

(iv) **Production Function:** A production function tells us the relationship between inputs and outputs.

For Example:

P = f (K,L), where P is production (output)

K is capital (input)

L is labour (input)

(v) **Revenue Functions:** A revenue function depicts the relation between revenue earned (R) and the sales and price of commodity.

R = f(p,x) where p is price

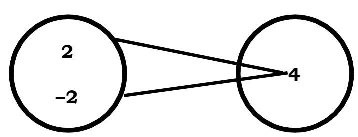
x is quantity of the commodity sold.

If $p(x) = 2x^2-3$ then revenue function is $x(2x^2-3) = 2x^3 - 3x$

1.1.5 One-One and Onto Functions :

A function which is not one-one, is called a many-one function.

e.g.f : IR → IR defined by f(x).

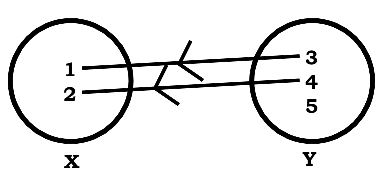


f (2) = 4 = f (–2)

So, f is not 1–1.

A function which is not onto is called on into function.

e.g.



for 3 in Y, ∃ 1 in X

for 4 in Y, ∃ 2 in X

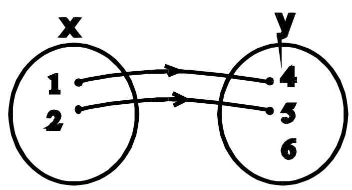
But for 5 in Y, there is no element in X.

So, this is an into function.

One-One Function (1-1 function) : A function is said to be one-one, if for each element in x there is at least one element in y

$$f : X \rightarrow Y$$

e.g.



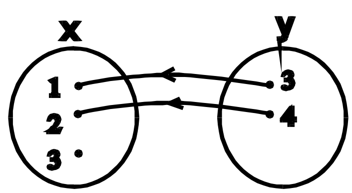
for 1 in x 4 in y

for 2 in x 5 in y

Onto-function : A function is said to be onto-function, if for each element in y, there is at least one element in x.

$$f : x \rightarrow y$$

e.g.



for 3 in y, ∃ 1 in x

4 in y, ∃ 2 in x

1.1.6 Limit of a Function :

The process of obtaining a derivative is called differentiation of the function. Differentiation mean breakdown.

The process of obtaining derivative make use of the concept of limit of a function. So now we will have to define the limit of a function.

Let y = f(x) be a function.

We want to study the behaviour of this function f(x), as x approaches a particular value say 'a'. When we say x approaches a, it means that x gets arbitrary close to 'a', but as x never equals a.

Symbolically we write $x \rightarrow a$ or Lt x = a. For example let $y = \frac{1}{2x}$ is single valued function of x.

i.e. to each value of x there corresponds one and only one vlaue of y. From the above function, we get.

x	:	1	2	3	4	5	1000
y	:	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{10}$	.... $\frac{1}{2000}$

To the x sequence, there corresponds a y sequence. y sequence has been cosntructed according to some rule. It is not a collection of some arbitrary numbers. The idea one gets is that as x becomes larger and larger, y or f(x) becomes smaller and smaller. As x tends to infinity, y tends to zero. We can never make it equal to zero by making x larger and larger, but we can make it very close to zero. Thus when a sequence of values is alloted to x according to some law, the corresponding sequence of values of f(x) determines the limit of which the function approaches.

Definition: Let f(x) be a function defined for all values of x close to c except possibly at point c. Then L is said to be the limiting value of f(x) as x approaches c, if the numerical difference between f(x) and L can be made as small as we like by making the positive difference between x and c small enough.

$$\lim_{x \to c} \text{fix} = L \quad \left(\begin{array}{l}\text{i.e. Change from x to c is}\\ \text{small value.}\end{array}\right)$$

1.1.7 Derivative

Let y =f(x) since y depends upon x, any change in x results in change in y. Suppose Δx represents change in Δx and Δy represents change in y.

Δx is small change is x or Say dx.

The ratio $\frac{\Delta y}{\Delta x}$ is called the incremental ratio. Now if the change in the independent variable x is very small i.e. $\Delta x \to O$, then the ratio $\frac{\Delta y}{\Delta x}$ may approach to a definite limiting value.This limiting value or average rate of change of function f(x) is known as instantaneous rate of change.

1.1.7.1 Definition

The derivative of y = f(x) is defined as

$$\frac{dy}{dx} = \underset{\Delta x \to 0}{Lt} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Note: (i) Derivative $\frac{dy}{dx}$ measures the rate of change of the variable y with respect to the variable x.

(ii) $\frac{d}{dx}$ does not mean 'd' divided by 'dx'. It is a symbol meaning the differential coefficient of the function.

(iii) The symbol $\frac{dy}{dx}$ is written in many other ways such as y^1, y_1, $\frac{d}{dx}$ [f(x)], f'(x) Dy etc.

(iv) The derivative is also called the rate measure.

(v) The increment ratio $\frac{f(x+h)-f(x)}{h}$ is not defined when h = 0.

(vi) The derivative of f(x) will exist only if $\underset{h \to 0}{lt} \frac{f(x+h)-f(x)}{h}$ exists.

1.1.7.2 Differentiation 'ab-initio'

It is process of obtaining derivative without making use of established standard theorems on differentiation.

Let y = f(x).......................... (i)

Here x is independent variable and y is dependent variable. Suppose, there is a small increment in the value of x and is denoted by Δx, and Δy be increment in y.

$\therefore$ y + Δy = f(x+ Δx)........................... (ii)

Subtracting (i) from (ii) we get,

Δy = f(x+Δx) - f(x)

Dividing both sides by Δx

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Take limits as $\Delta x \to o$

$$\underset{\Delta x \to 0}{\text{lt}} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{\text{lt}} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$\frac{dy}{dx}$ or $Lt \frac{\Delta y}{\Delta x}$ is the derivative or differential coefficient of y with respect to x.

Example 1. Let y = $\frac{1}{x^2}$

Find $\frac{dy}{dx}$ from the first principle.

Sol. y = $\frac{1}{x^2}$ .... (i)

Let Δx be an increment in the value of x and Δy be the corresponding increment in the value of y.

$\therefore$ y + $\Delta y = \frac{1}{(x + \Delta x)^2}$ ....(ii)

Subtracting (i) from (ii)

$$\Delta y = \frac{1}{(x + \Delta x)^2} - \frac{1}{x^2} = \frac{x^2 - (x + \Delta x)^2}{x^2 (x + \Delta x)^2} = \frac{x^2 - x^2 - \Delta x^2 - 2^x \Delta^x}{x^2 (x + \Delta x)^2}$$

(by taking L.C.M.)

$$= \frac{-\Delta x(\Delta x + 2x)}{x^2 (x + \Delta x)^2}$$

(Dividing both sides by Δx)

$$\frac{\Delta y}{\Delta x} = -\frac{(\Delta x + 2x)}{(x + \Delta x)^2 x^2}$$

Take limits as $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \frac{(0+2x)}{(x+0)^2 x^2} - \frac{2x}{x^4} = \frac{-2}{x^3}$$

Example 2. Let y = $\sqrt{ax+b}$. Find $\frac{dy}{dx}$ by using the first principle.

Sol. We have to differentiate $y = \sqrt{ax+b}$ (i), by the first principle

$$y + \Delta y = \sqrt{a(x+\Delta x)+b}..................(ii)$$

Subtracting (i) from (ii)

$$\Delta y = \sqrt{a(x+\Delta x)+b} - \sqrt{ax+b}$$

Rationalizing

$$\Delta y = \sqrt{ax + a\Delta x) + b} - \sqrt{ax+b}\left(\frac{\sqrt{(ax + a\Delta x) + b} + \sqrt{ax+b}}{\sqrt{(ax + a\Delta x) + b} + \sqrt{ax+b}}\right)$$

$$= \frac{(ax + a\Delta x) + b - ax - b}{\sqrt{(ax + a\Delta x) + b} + \sqrt{ax+b}}$$

Dividing both sides by Δx,

$$\frac{\Delta y}{\Delta x} = \frac{a\Delta x}{\Delta x\left[\sqrt{ax + a\Delta x + b} + \sqrt{ax+b}\right]}$$

Take limits as $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{a}{\left[\sqrt{ax + a\Delta x + b} + \sqrt{ax+b}\right]} = \frac{a}{\sqrt{ax+b} + \sqrt{ax+b}} = \frac{a}{2\sqrt{ax+b}}$$

$$\frac{dy}{dx} = \frac{a}{2\sqrt{ax+b}}$$

1.1.8 Derivatives of Some Standard Functions:

In this section, we will discuss set of rules to find derivatives of some standard functions. We assume that the functions are differentiable.

1.1.8.1 Rules for Simple Functions

I. Derivative of a constant: If y =c, where c is constant.

then $\frac{dy}{dx} = \frac{d}{dx}(c) = 0$ or derivative of any constant value is zero.

Proof : Let y = c ....(i)

y + Δy = c ....(ii)

Substracting (i) from (ii) we get, Δy = 0

Dividing by Δx, $\frac{\Delta y}{\Delta x} = 0$

Take limits as Δx →0

$$\underset{\Delta x \to 0}{\text{lt}} \frac{\Delta y}{\Delta x} = 0$$

$$or \frac{dy}{dx} = 0 \qquad or \qquad \frac{d}{dx}(c) = 0$$

For example : If y = 7, then $\frac{dy}{dx} = 0$

II. Derivative of x^n w.r.t.x.

Derivative of x^n w.r.t. x is nx^{n-1}.

Proof: Let y = x^n ....(i)

y + Δy = $(x+\Delta x)^n$..............................(ii)

Subtracting, Δy = $(x+\Delta x)^n - x^n$

Divide by Δx on both sides,

$$\therefore \frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^n - x^n}{\Delta x} ....(ii)$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{x^n}{\Delta x}\left[\left(1 + \frac{\Delta x}{x}\right)^n - 1\right]$$

Apply Binomial Theorem (for +ve index)

$[(x+a)^n = x^n + {}^nc_1\ x^{n-1}\ a + {}^nc_2\, x^{n-2}\ a^2 + {}^nc_3\, x^{n-3}\ a^3 \ldots\ldots\ldots\ldots\ldots + {}^nc_n\ a^n]$

$$\frac{\Delta y}{\Delta x} = \frac{x^n}{\Delta x}\left[(1 + n\frac{\Delta x}{x} + \frac{n(n-1)}{\underline{|2}}\left(\frac{\Delta x}{2}\right)^2 \ldots\ldots\ldots - 1\right]$$

$$= \frac{1}{\Delta x} \times x^n \left[n \frac{\Delta x}{x} + \frac{n(n-1)}{\underline{|2}} \left(\frac{\Delta x}{2} \right)^2 + ... \right]$$

$$= \frac{1}{\Delta x} x^n \frac{\Delta x}{x} \left[n + \frac{n(n-1)}{\underline{|2}} \left(\frac{\Delta x}{2} \right) + ... \right]$$

$$= x^{n-1} \left[n + \frac{n(n-1)}{\underline{|2}} \left(\frac{\Delta x}{2} \right) + ... \right]$$

Take limits as $\Delta x \to 0$

$$\therefore \frac{dy}{dx} = x^{n-1} [n + 0 + 0 + ...]$$

$$\therefore \frac{dy}{dx} = nx^{n-1}$$

$$\text{or } \frac{d}{dx} [x^n] = nx^{n-1}$$

For example : (i) If $y = x^{17}$, then $\frac{dy}{dx} = 17x^{16}$

(ii) If $y = x^8$, then $\frac{dy}{dx} = 8x^7$

III. Derivative of $y = (ax+b)^n$ is $n(ax+b)^{n-1}a$.

Proof : Let $y = (ax+b)^n$ .... (i)

Change x to $x + \Delta x$ and y to $y + \Delta y$, we have,

$y + \Delta y = [a (x + \Delta x) + b]^n$

$= [ax + a\Delta x + b]^n = [ax + b + a \Delta x]^n$

$$y + \Delta y = (ax+b)^n \left(1 + \frac{a\Delta x}{ax+b}\right)^n \text{ ....(ii)}$$

Subtract (i) from (ii), we get

$$\Delta y = (ax + b)^n \left(1 + \frac{a\Delta x}{ax + b}\right)^n - (ax + b)^n$$

$$= (ax+b)^n \left[\left(1+\frac{a\Delta x}{ax+b}\right)^n - 1\right]$$

$$= (ax+b)^n \left[\left\{1+n\frac{a\Delta x}{ax+b}+\frac{n(n-1)}{2!}\left(\frac{a\Delta x}{ax+b}\right)^2+.............\right\}-1\right]$$

$$= (ax+b)^n \left[n\times\frac{a\Delta x}{ax+b}+\frac{n(n-1)}{2!}\left(\frac{a\Delta x}{ax+b}\right)^2+.............\right]$$

$$= (ax+b)^n \frac{a\Delta x}{ax+b}\left[n+\frac{n(n-1)}{2!}\left(\frac{a\Delta x}{ax+b}\right)+.............\right]$$

$$= (ax+b)^{n-1} a\Delta x\left[n+\frac{n(n-1)}{2!}\left(\frac{a\Delta x}{ax+b}\right)+.............\right]$$

Divide both sides by Δx

$$\therefore \frac{\Delta y}{\Delta x} = \frac{(ax+b)^{n-1} a\times\Delta x}{\Delta x}\left[n+\frac{n(n-1)}{2!}\left(\frac{a\Delta x}{ax+b}\right)+.............\right]$$

Take limits as $\Delta x \to 0$

$$\underset{\Delta x\to 0}{\text{Lt}} \frac{\Delta y}{\Delta x} = a\ (ax+b)^{n-1}\ [n+0+0+0+....]$$ o

$$\frac{d}{dx}(ax+b)^n = n(ax+b)^{n-1}\times a$$

For example : (i) If $y = (7x+5)^4$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(7x+5)^4 = 4(7x+5)^3 \times 7 = 28(7x+5)^3 \quad \left[\begin{array}{l}\text{by the farmula} \\ \frac{d}{dx}(ax+b)^n = n(ax+b)^{n+1}.a\end{array}\right]$$

(ii) If y = $(2x - 4)^{-5/2}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(2x-4)^{-5/2} \quad -\frac{5}{2}(2x-4)^{\frac{-5}{2}-1} = -5(2x-4)^{-7/2}$$

$$\therefore \frac{d}{dx}(2x-4)^{-5/2} = -5(2x-4)^{-7/2}$$

Cor. Differentiation does not affect a multiplicative constant.

$\therefore \frac{d}{dx}[au] = a\frac{d}{dx}(u)$ where a is constant and u = f(x)

Let y = au

y + Δy = a (u+Δu) = au + aΔu

Subtracting Δy = aΔu

Dividing by Δx, we get

$$\frac{\Delta y}{\Delta x} = a\frac{\Delta u}{\Delta x}$$

Take limits as $\Delta x \to 0$

$$\underset{\Delta x \to 0}{\text{lt}} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = a\frac{du}{dx}$$

$$\therefore \frac{d}{dx}[au] = a\frac{d}{dx}(u)$$

For example : (i) If y = $7x^9$

then, $\frac{d}{dx}(7x^9) = 7\frac{d}{dx}(x^9) = 7 \times 9x^8 = 63x^8$

(ii) If $y = 6x^7$

$$\frac{d}{dx}(6x^7) = 6\frac{d}{dx}(x^7) = 6 \times 7x^6 = 42x^6$$

Example 3: Find derivatives of the following functions:

(a) $Y = 5x^3$

$$\frac{dy}{dx} = \frac{d\left(5 \times x^3\right)}{dx} = \frac{5d\left(x^3\right)}{dx} = 5.3x^2$$

Ans. $15x^2$.

(b) $Y = 8x^{-5/2} + 7x^{-1}+3$

$$\frac{dy}{dx} = 8\left(\frac{-5}{2}\right)x^{\frac{-5}{2}-1} + 7(-1)x^{(-1-1)} + 0$$

$$= \frac{-40}{2}x^{\frac{-7}{2}} - 7x^{-2}$$

Ans. $= -20x^{\frac{-7}{2}} - 7x^{-2}$

(c) $f(x) = (2x-7)^{-8/3}$

$$f'(x) = \left(\frac{-8}{3}\right)(2)(2x-7)^{-11/3}$$

$$= \frac{-16}{3}(2x-7)^{-11/3}$$ Ans.

1.1.8.2 Sum Rule/Difference Rule

Let u and v are two functions of x, then

$$\frac{d}{dx}(u \pm v) = \frac{d}{dx}(u) \pm \frac{d}{dx}(v)$$

The derivative of the algebraic sum/difference of two functions is equal to the corresponding algebric sum of their derivatives, provided these derivatives exist.

Let y = (u+v) ....(i)

$y + \Delta y = [(u+\Delta u) + (v + \Delta v)]$............(ii)

Subtracting (i) from (ii), $\Delta y = \Delta u + \Delta v$

Dividing both sides by Δx, we get

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$$

Take limits as $\Delta x \to 0$ we get

$$\underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{Lt} \frac{\Delta u}{\Delta x} + \underset{\Delta x \to 0}{Lt} \frac{\Delta v}{\Delta x}$$

Hence = $\frac{d}{dx}(u+v) = \frac{d}{dx}(u) + \frac{d}{dx}(v)$

Similarly, we can prove $\frac{d}{dx}(u-v) = \frac{d}{dx}(u) - \frac{d}{dx}(v)$

Generalization : Let Y = $u_1 \pm u_2 \pm u_3 \pm u_4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \pm u_n$

Where u_1, u_2………………… u_n are the derivable functions of x, then

$$\frac{dy}{dx} = \frac{du_1}{dx} \pm \frac{du_2}{dx} \pm \frac{du_3}{dx} \ldots\ldots\ldots\ldots\ldots \pm \frac{du_n}{dx}$$

Example 4 : Let y = $6x^4 + \frac{1}{2}x^3 - x^2 + 4x - 8$, find $\frac{dy}{dx}$

Sol. We have to find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{d}{dx}(6x^4 + \frac{1}{2}x^3 - x^2 + 4x - 8)$$

$$\frac{dy}{dx} = \frac{d}{dx}(6x^4) + \frac{1}{2}\frac{d}{dx}(x^3) - \frac{d}{dx}(x^2) + \frac{d}{dx}(4x) - \frac{d}{dx}(8)$$

$$= 6\frac{d}{dx}(x^4) + \frac{3}{2}x^2 - 2x + 4\frac{d}{dx}(x) - 0 \quad \left[\frac{d(x^n)}{dx} = nx^{n-1}\right]$$

$$\therefore \frac{dy}{dx} = 24x^3 + \frac{3}{2}x^2 - 2x + 4$$

Example 5: Let y = $4x^4 - 3x^2 + 2$, find $\frac{dy}{dx}$

Sol. $$\frac{dy}{dx} = \frac{d}{dx}(4x^4) - \frac{d}{dx}(3x^2) + \frac{d}{dx}(2)$$

$$= 4\frac{d}{dx}(x^4) - 3\frac{d}{dx}(x^2) + 0 = 16x^3 - 6x$$

$$\therefore \quad \frac{d}{dx}(4x^4 - 3x^2 + 2) = 16x^3 - 6x$$

1.1.8.3 Product Rule

Let u and v are two functions of x, then

$$\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$$

Product Rule says

Derivative of the product of two functions = (first function x derivative of second function) + (second function × derivative of the first function).

Let y = uv..................... (i)

Change x to x + Δx and y to y + Δy, we have

y + Δy = (u + Δu) (v + Δv), we have

y + Δy = uv + vΔu + uΔv + ΔuxΔv.............. (ii)

Subtract (i) from (ii) we get

Δy = vΔu + uΔv + Δu.Δv.

Dividing both sides by Δx

$$\frac{\Delta y}{\Delta x} = \frac{v\Delta u + u\Delta v + \Delta v \times \Delta u}{\Delta x}$$

$$= v\frac{\Delta u}{\Delta x} + u\frac{\Delta v}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}$$

Take limits as $\Delta x \to 0$

$$\underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{Lt} \left(v \frac{\Delta u}{\Delta x} + u \frac{\Delta v}{\Delta x} + \Delta u \frac{\Delta v}{\Delta x} \right)$$

$$\frac{dy}{dx} = v \underset{\Delta x \to 0}{lt} \frac{\Delta u}{\Delta x} + u \underset{\Delta x \to 0}{lt} \frac{\Delta v}{dx} + \Delta u \underset{\Delta x \to 0}{lt} \frac{\Delta v}{\Delta x}$$

$$\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx} + \Delta u \frac{dv}{dx} \qquad \left(\because Lt_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{du}{dx} \right)$$

$$= v \frac{du}{dx} + u \frac{dv}{dx} \left(Lt \frac{\Delta u}{\Delta x} = \frac{dv}{dx} \right)$$

$$\therefore \frac{dy}{dx}(uv) = v \frac{du}{dx} + u \frac{dv}{dx}$$

Generalization

Let y = uvw = uv (w)

$$\therefore \frac{dy}{dx} = v \frac{dw}{dx} + w \frac{d}{dx}(uv)$$

$$uv \frac{dw}{dx} + w \left(u \frac{dv}{dx} + v \frac{du}{dx} \right)$$

$$= uv \frac{dw}{dx} + wu \frac{dv}{dx} + wv \frac{du}{dx}$$

Example 6 : Let y = $(x-4)^2 (x^2+7)$, find $\frac{dy}{dx}$.

Sol. $\frac{dy}{dx} = \frac{d}{dx}(x-4)^2 (x^2+7)$

$$= (x-4)^2 \frac{d}{dx}(x^2+7) + (x^2+7) \frac{d}{dx}(x-4)^2$$

$$\frac{dy}{dx} = (x-4)^2(2x) + (x^2+7)2(x-4)$$

$$= (x-4)^2(2x) + 2(x-4)\ (x^2+7)$$

$$\therefore \qquad \frac{dy}{dx} = (x-4)\ \left[(x-4)\quad (2x) + 2(x^2+7)\right]$$

$$= (x-4)\ \left[(2x^2 - 8x + 2x^2 + 14\right] = (x-4)\quad (4x^2 - 8x + 14)$$

Example 7 : Let y = $(6x^4+9x)\quad (3x^2+5)$, find $\frac{dy}{dx}$

Sol. $\frac{dy}{dx} = \frac{d}{dx}\left[(6x^4+9x)\quad (3x^2+5)\right]$

$$= (6x^4+9x)\frac{d}{dx}(3x^2+5) + (3x^2+5)\frac{d}{dx}(6x^4+9x)$$

$$\frac{d}{dx} = (6x^4+9x)\quad (6x) + (3x^2+5)\quad (24x^3+9)$$

1.1.8.4 QUOTIENT RULE

If y = $\frac{u}{v}$, where u and v are derivable functions of x, then

$$\frac{dy}{dx} = \frac{d(\frac{u}{v})}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Now, y = $\frac{u}{v}$ (given)............................. (i)

y + Δy = $\frac{u+\Delta u}{v+\Delta v}$........................$(ii)$

Subtracting (i) from (ii)

$$y + \Delta y - y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v}.$$

$$\Delta y = \frac{v(u + \Delta u) - u(v + \Delta v)}{v(v + \Delta v)}$$

$$\Delta y = \frac{vu + v\Delta u - uv - u\Delta v}{v(v + \Delta v)} = \frac{v\Delta u - u\Delta v}{v(v + \Delta v)}$$

Dividng both sides by Δx

$$\frac{\Delta y}{\Delta x} = \frac{\frac{v\Delta u - u\Delta v}{v(v + \Delta v)}}{\Delta x} = \frac{v\frac{\Delta u}{\Delta x} - u\frac{\Delta v}{\Delta x}}{v^2 + v\Delta v}$$

Take limits as $\Delta x \to 0$

$$\underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{Lt} \frac{v\frac{\Delta u}{\Delta x} - u\frac{\Delta v}{\Delta x}}{v^2 + v\Delta v}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$\therefore \frac{d}{dx}$ (Quotient of two functions)

$$= \frac{\text{Denominator} \times \frac{d}{dx}(\text{Numerator}) \text{ - Numerator} \times \frac{d}{dx}(\text{Denominator})}{(\text{Denominator})^2}$$

Example 8 : Let y $= \frac{x^3 - 6x^2}{5x^2 - 1}$, find $\frac{dy}{dx}$.

Sol. $\frac{dy}{dx} = \frac{(5x^2 - 1)\frac{d}{dx}(x^3 - 6x^2) - (x^3 - 6x^2)\frac{d}{dx}(5x^2 - 1)}{(5x^2 - 1)^2}$

$$\therefore \frac{dy}{dx} = \frac{(5x^2 - 1)(3x^2 - 12x) - (x^3 - 6x^2)(10x)}{(5x^2 - 1)^2}$$

Example 9: Let y $= \sqrt{\dfrac{1-x}{1+x}}$, differentiate y w.r.t to x.

Sol. $$\frac{dy}{dx} = \frac{(1+x)^{1/2}\frac{d}{dx}(1-x)^{1/2} - (1-x)^{1/2}\frac{d}{dx}(1+x)^{1/2}}{(\sqrt{1+x})^2}$$

$$= \frac{(1+x)^{\frac{1}{2}}\frac{1}{2}(1-x)^{-\frac{1}{2}}(-1) - (1-x)^{\frac{1}{2}}\left(\frac{1}{2}\right)(1+x)^{-\frac{1}{2}}}{1+x} = \frac{\sqrt{1+x}\,\frac{1}{2}\frac{(-1)}{\sqrt{1-x}} - \frac{\sqrt{1-x}}{2}\sqrt{1+x}}{1+x}$$

$$\frac{dy}{dx} = \frac{\frac{-1\sqrt{1+x}}{2\sqrt{1-x}} - \frac{1}{2}\frac{\sqrt{1-x}}{\sqrt{1+x}}}{1+x}$$

$$\frac{dy}{dx} = \frac{-\frac{1(1+x)+(1-x)}{2\sqrt{1-x}\sqrt{1+x}}}{1+x} = \frac{-1}{(1+x)\sqrt{1-x^2}}$$

1.1.8.5 Chain Rule/Function of a Function Rule

If y = f(u) is derivable w.r.t. u and u = φ (x) is derivable w.l.t. x, then y is derivable w.r.t. x and

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \text{ i.e. by chain we get } \frac{dy}{dx}$$

Generalisation:

If y = θ (u), u= f(z), z = g(x)

then, $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dz} \times \dfrac{dz}{dx}$

Example 10: Let $y = 2u^2 + 3$ and $u = 5x^3$, find $\frac{dy}{dx}$.

Sol. To find $\frac{dy}{dx}$ we have to find $\frac{dy}{du}$ and $\frac{du}{dx}$

Now $y = 2u^2+3$

$\Rightarrow \therefore \frac{dy}{du} = 4u,$

$u = 5x^3$

$\therefore \frac{du}{dx} = 15x^2$

So, $\therefore \frac{dy}{du} = \frac{dy}{du} \times \frac{du}{dx} = 4u \times 15x^2 = 60x^2u$

$\therefore \frac{dy}{du} = 60(5x^3)x^2 = 300x^5$

Example 11: Let, $y = at^3$ and $x = bt^2$, where 't' is the parameter. Find $\frac{dy}{dx}$

Sol. We have to find $\frac{dy}{dx}$, where t is the common parameter.

Now $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$y = at^3$, $x = bt^2$

$\therefore \frac{dy}{dt} = 3at^2$ and $\frac{dx}{dt} = 2bt \left(\because \frac{dx^n}{dx} = nx^{n-1} \right)$

$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3at^2}{2bt} = \frac{3}{2}\frac{a}{b}t.$

1.1.9 DIFFERENTIATION OF IMPLICIT FUNCTIONS

When y is defined implicitly as a function of x, then it is called implicit function. For example $2ax^2+by+9=0$ is an implicit function. Here derivatives of y.w.r.t. x may be found by considering y as a function of x and differentiating term by term.

$$\frac{d}{dx}(x^n) = nx^{n-1} \quad \text{and} \frac{d}{dx}(y)^n = ny^{n-1}\frac{d}{dx}(y)$$

Example 12: Consider x^3-$2x^2y$+$3xy^2$-25 = 0, find $\frac{dy}{dx}$

Sol. $$\frac{d}{dx}(x^3 - 2x^2y + 3xy^2 - 25) = \frac{d}{dx}(x^3) - 2\frac{d}{dx}(x^2y) + 3\frac{d}{dx}(xy^2) - \frac{d}{dx}(25) = 0$$

$$\therefore 3x^2 - 2\left[x^2\frac{dy}{dx} + y \quad \frac{d(x)^2}{dx}\right] + 3\left[x\frac{d(y^2)}{dx} + y^2\frac{d(x)}{dx}\right] = 0$$

$$\therefore 3x^2 - 2\left[x^2\frac{dy}{dx} + y \quad 2x\right] + 3\left[x \quad 2y\frac{dy}{dx} + y^2\right] = 0$$

$$\Rightarrow 3x^2 - 2x^2\frac{dy}{dx} - 4yx + 3x.2y\frac{dy}{dx} + 3y^2 = 0$$

$$\Rightarrow (6xy - 2x^2)\frac{dy}{dx} = -3(x^2 + y^2) + 4xy \quad \text{i.e. } \frac{dy}{dx}\left(6xy - 2x^2\right) = 4xy - 3\left(x^2 + y^2\right)$$

$$\therefore \frac{dy}{dx} = \frac{4xy - 3(x^2 + y^2)}{6xy - 2x^2}$$

Example 13: Let ax^2+2hxy+by^2+2gx+2fy+c=0, find $\frac{dy}{dx}$.

Sol. $$\frac{d}{dx}[ax^2 + 2hxy + by^2 + 2gx + 2fy + c] = 0$$

$$\therefore \frac{d}{dx}(ax^2) + \frac{d}{dx}(2hxy) + \frac{d}{dx}(by^2) + \frac{d}{dx}(2gx) + \frac{d}{dx}2(fy) + \frac{d}{dx}(c) = 0$$

$$\therefore 2ax + 2h\frac{d}{dx}(xy) + b\frac{d}{dx}(y^2) + 2g\frac{d}{dx}(x) + 2f\frac{d}{dx}(y) = 0$$

$$\Rightarrow 2ax + 2h[y + x\frac{dy}{dx}] + b(2y\frac{dy}{dx}) + 2g + 2f\frac{dy}{dx} = 0$$

$$\Rightarrow 2ax + 2hy + 2hx\frac{dy}{dx} + 2by\frac{dy}{dx} + 2f\frac{dy}{dx} + 2g = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2[ax + hy + g]}{2[hx + by + f]} = -\frac{(ax + hy + g)}{hx + by + f}$$

1.1.10 DIFFERENTIATION OF PARAMETRIC EQUATIONS

If x = f(t) and y = g (t), i.e. x and y both are functions of variable t, then their equations are known as parametric equations,

and $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt}$.

Example 14: Let $x = t^4+3t^2+1$, $y = 7t^2+6t+1$. Find $\frac{dy}{dx}$

Sol. For finding $\frac{dy}{dx}$, we have to find $\frac{dx}{dt}, \frac{dy}{dt}$

Now, $\frac{dy}{dt} = 14t + 6$, $\frac{dx}{dt} = 4t^3 + 6t$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{14t + 6}{4t^3 + 6t}$$

1.1.11 Summary

In this lesson, we have discussed about the functional dependence of variable y on variable x. Various types of functions have been discussed. Further, the concept of limit of a function is explained well using simple illustrations. The definition of derivative of a function and methods for finding derivatives of functions have been elaborated in detail. To find out the derivative of the functions under certain rules have explained with the help of simple examples.

1.1.12 Key Concepts

Functions, One-one function, Onto function, Limit of a function, Derivative of a function, ab-initio principle, Rules of differentiation

1.1.13 Long Questions

1. Differentiate the following by ab-initio principle. (Try it by own)

(i) $y = \sqrt{x}$

(ii) $y = \frac{4x-3}{2x+1}$

(iii) $y = \frac{x}{x+1}$

(iv) $2x^2 - 4x + 5$

(v) $\frac{ax+b}{cx+d}$

2. Let $y = 4S^3 + 2S^2 + 4$ and $x = 8S^3 + 2S^2 + 8$

Find $\frac{dy}{dx}$

3. Let $y = u^2 + 2u + 8$ and $x = u^4 + 5u + 7$

Find $\frac{dy}{dx}$.

1.1.14 Short Questions

1. Differentiate the following functions w.r.t.x.

(i) $(x^3+2x^2)^3$

(ii) $\sqrt{\frac{x^3-2ax}{a^2-2ab}}$

(iii) $(2x^4+1)^{1/2} + (x+1)^{1/2}$

(iv) $(x^2+1)(3x^2-2x^2)$

(v) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

2. Find the derivatives of y with respect to x when

(i) $y = 2u^2-3$

$$u = \frac{1}{x^2}$$

(ii) $y = 3t^2 + 1$

$t = u^2 + u,\ u = x$

1.1.15 Suggested Readings

1. D. Bose.	:	An Introduction of Mathematical Economics
2. Mehta and Madnani	:	Mathematics for Economists
3. Aggarwal and Joshi	:	Mathematics for students of Economics
4. Bhardwaj and Sabharwal	:	Mathematics for students of Economics

LESSON NO. 1.2 **AUTHOR : DR. VIPLA CHOPRA**

DIFFERENTIATION OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS

1.2.1 Objectives

The prime objective of this lesson is to discussed about logarithmic and exponential functions and their derivatives.

1.2.2 Introduction

In the previous lesson, we have studied the derivatives of simple functions. In the present lesson, we will study the derivatives of Logarithmic and Exponential Functions. This lesson has been divided into three parts. In the first part, derivatives of standard logarithmic functions will be studied. Second part is concerned with derivatives of some standard exponential functions. Successive derivatives or higher order derivatives are discussed in the third part of this lesson.

1.2.1 Derivative of Logarithmic Functions

For understanding derivatives of Logarithmic functions, the students must have to be familiar with logs and antilogs.

1.2.1.1 Definition

$y = \text{Log}_a x$ is a logarithmic function. The base of logarithm is 'a'.

$y = \text{Log}_e x$ is known as natural logarithmic function, as 'e' is the base. It can also be written as: y = log x. It represents the standard form of logarithmic function.

The derivatives of such standard logarithmic functions lead to standard results.

1.2.1.2 Derivative of $y = \text{Log}_a x$

Given $y = \log_a x$....................................I

$\Rightarrow \quad y + \Delta y = \log_a (x + \Delta x)$............................II

Subtract I from II we get

$\therefore \Delta y = \log_a (x + \Delta x) - \log_a x.$

$$= \log_a \left[\frac{(x + \Delta x)}{x} \right] \quad \left[\because \log_a \frac{m}{n} = \log_a m - \log_a n \right]$$

$$= \log_a \left(1 + \frac{\Delta x}{x} \right)$$

Divide both sides by Δx, we get

$$\frac{\Delta y}{\Delta x} = \frac{1}{\Delta x} \log_a \left(1 + \frac{\Delta x}{x} \right)$$

$$= \log_a \left(1 + \frac{\Delta x}{x} \right)^{1/\Delta x} = \log_a \left[\left(1 + \frac{\Delta x)}{x} \right)^{\frac{x}{\Delta x}} \right]^{\frac{1}{x}}$$

$$= \frac{1}{x} \log_a \left(1 + \frac{\Delta x}{x} \right)^{\frac{x}{\Delta x}} \left[\frac{1}{x} \text{Comes at the front of log a} \right]$$

Take limits as $\Delta x \to 0$

$$\frac{dy}{dx} = \frac{1}{x} \underset{\Delta x \to 0}{lt} \log_a \left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}$$

$$= \frac{1}{x} \log_a \left(\because Lt_{\Delta x \to 0} (1 + x)^{\frac{1}{x}} = e\right)$$

Hence $\frac{d}{dx}(\log a^x) = \frac{1}{x} \log_a e$......................*III*

Example 1: Let y = $\log_5 x$, find $\frac{dy}{dx}$

Sol. $\frac{dy}{dx} = \frac{1}{x} \log_5 e$

Example 2 : If y = $\log_a x^2$, find $\frac{dy}{dx}$

Sol. y = 2 $\log_a x$ $\left(\therefore \log_a m^n = n \log_a m\right)$

$$\therefore \frac{dy}{dx} = 2 \frac{d}{dx} (\log_a x) \quad \left[\because \frac{d}{dx} \log_a x = \frac{1}{x} \log_a e\right.$$

$$= \frac{2}{x} \log_a e$$

Cor. 1: When y = $\log_e x$

In order to find its derivative put a = e in III

$$\frac{dy}{dx} = \frac{1}{x} \log_e e = \frac{1}{x} \qquad (\because \log_e e = 1)$$

or $\frac{d}{dx}(\log \ x) = \frac{1}{x}$ (When base is not mentioned, it is understood to be ‘e’)

Example 3: Let y = $\log_e x^3$, find $\frac{dy}{dx}$.

Sol. Here, y = 3 $\log_e x$ ($\log m^n = n \log m$)

$$\therefore \frac{dy}{dx} = 3.\frac{1}{x} = \frac{3}{x} \qquad [\because \log_e(1) = 1]$$

Example 4 (a): Let y = log $\sqrt{x}$, find $\frac{dy}{dx}$.

Sol. $y = \frac{1}{2}\log \ x \qquad (\because \log \ m^n = n\log m)$

$$\therefore \frac{dy}{dx} = \frac{1}{2} \times \frac{1}{x} = \frac{1}{2x} \qquad [\because \log_e(1) = 1]$$

Example 4(b): Let y = $\log_e \sqrt[3]{x}$, find $\frac{dy}{dx}$.

Sol. $y = \frac{1}{3} \ \log_e x \left[\because \log_e(x)^{\frac{1}{3}} = \frac{1}{3}\log_e x\right]$

$$= \frac{1}{3} \times \frac{1}{x} = \frac{1}{3x}$$

Example 4(c): Let y = $\log x^8$, find $\frac{dy}{dx}$.

Sol. $\frac{dy}{dx} = 8\frac{d}{dx}(\log x) = 8 \times \frac{1}{x} = \frac{8}{x}. \ \left[\because \log x^8 = 8 \log x\right]$

Example 4(d): y = $\log_8 x$, find $\frac{dy}{dx}$.

Sol. $\frac{dy}{dx} = \frac{1}{x}\log_8 e$.

1.2.1.3 Function of Function Rule

Let y = $\log_a u$, where u is derivable function of x.
By chain rule, we know

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Following this rule we get,

$$\frac{d}{dx}(y)=\frac{d}{dx}(\log_a u).\frac{du}{dx}=\frac{1}{u}\log_a e\frac{du}{dx}$$

Example 5: Let y = $\log_a (x^3+1)$, find $\frac{dy}{dx}$.

As $\frac{d}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

Put u = x^3+1 $\therefore \frac{du}{dx}=3x^2$

$\therefore$ y = $\log_a u$

$$\therefore \frac{d}{dx}(\log_a u)=\frac{1}{u}\log_a e\times\frac{du}{dx}=\frac{3x^2}{x^3+1}\log_a e$$

$$\therefore \frac{d}{dx}\log_a(x^3+1)=\frac{3x^2}{x^3+1}\log_a e$$

Col 1: When y = log u, where u is the function of x.

then $\frac{dy}{dx}=\frac{1}{u}\log_a e\times\frac{du}{dx}=\frac{1}{u}\times\frac{du}{dx}$ $(\log_e e+1)$

Example 6: Let y = log $\sqrt{1-x^3}$, find $\frac{dy}{dx}$.

Sol. We have to differentiate y w.r.t.x.

$$\therefore \frac{dy}{dx}=\frac{d}{dx}(\log(1-x^3)^{1/2}$$

$$\therefore \frac{dy}{dx}\log(\sqrt{1-x^3})1=\frac{1}{2(1-x^3)}\times(-3x^2)$$

$$=-\frac{3}{2}\frac{x^2}{(1-x^3)}$$

Example 7: Let y $= \log\sqrt{\dfrac{x+1}{x-1}}$, find $\dfrac{dy}{dx}$.

Sol. $y = \log\left(\dfrac{x+1}{x-1}\right)^{1/2} = \dfrac{1}{2}\log\left(\dfrac{x+1}{x-1}\right)$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\frac{x-1}{x+1} \times \frac{d}{dx}\left(\frac{x+1}{x-1}\right) \quad \because \frac{d}{dx}\left[\log\left(f(x)\right)\right] = \frac{1}{f(x)}\frac{d}{dx}\left(f(x)\right)$$

$$= \frac{1}{2}\left(\frac{x-1}{x-1}\right)\frac{(x-1)-(x+1)}{(x-1)^2} = \frac{-2}{2(x^2-1)} = \frac{-1}{(x^2-1)}$$

1.2.1.4 Product Rule

Some functions involving the product of two or more variables may be complicated. Logarithmic derivation becomes very helpful in solving such problems.

Let y = uv

When u and v are derivable functions of x.

Take log on both sides, we get

log y = log u + log v

$$\therefore \frac{d}{dx}\log y = \frac{d}{dx}\log \quad u + \frac{d}{dx}\log v$$

$$\therefore \frac{1}{y} \times \frac{dy}{dx} = \frac{1}{u} \times \frac{du}{dx} + \frac{1}{v} \times \frac{dv}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(uv) = uv\left[\frac{1}{u} \times \frac{du}{dx} + \frac{1}{v} \times \frac{dv}{dx}\right]$$

Example 8 : Let y = $(2x^2+1)(x+1)$, find $\dfrac{dy}{dx}$.

Sol. Taking log on both sides, log y = $\log[(2x^2+1)(x+1)] = \log(2x^2+1) + \log(x+1)$

$[\because \log \quad mn = \log m + \log n]$

Take derivatives on both sides, we get

$$\frac{1}{y}\times\frac{dy}{dx}=\frac{1}{(2x^2+1)}\times\frac{d}{dx}(2x^2)+\frac{1}{(x+1)}\frac{d}{dx}(x+1)$$

$$\frac{1}{y}\times\frac{dy}{dx}=\frac{4x}{(2x^2+1)}+\frac{1}{(x+1)}$$

$$\therefore\frac{d}{dx}\left[(2x^2+1)(x+1)\right]=(2x^2+1)(x+1)\frac{4x^2+4x+2x^2+1}{(2x^2+1)(x+1)}=\frac{6x^2+4x+1}{(2x^2+1)(x+1)}$$

Example 9 : Let y = $(4x-1)^2\sqrt{x+4}$, find $\frac{dy}{dx}$.

Sol. Take log on both sides, we get

$$\log\ y=\log[(4x-1)^2\ \sqrt{x+4}\ ]$$

$$\log\ y=\log[(4x-1)^2\ \sqrt{x+4}\ ]$$ [log mn = log m + log n]

$$\log y = 2\log\ (4x-1)\ +\frac{1}{2}\log(x+4)$$ [log m^n = n log m]

Differentiate w.r.t.x.

$$\therefore\frac{1}{y}\times\frac{dy}{dx}=2\frac{1}{4x-1}\times4+\frac{1}{2}\times\frac{1}{x+4}\times1$$

$$\frac{8}{4x-1}+\frac{1}{2(x+4)}$$

$$\therefore\frac{dy}{dx}=(4x-1)^2\sqrt{x+4}\left[\frac{8}{4x-1}+\frac{1}{2(x+4)}\right]$$

$$=(4x-1)^2\sqrt{x+4}\,\frac{(20x+63)}{(4x-1)2(x+4)}$$

$$=\frac{(4x-1)(20x+63)}{2\sqrt{x+4}}$$

1.2.1.5 Quotient Rule

Let $y = \frac{u}{v}$, where u and v are derivable functions of x.

Take log on both sides

$$\log y = \log\left(\frac{u}{v}\right) = \log u - \log v$$

$$\Rightarrow \quad \frac{d}{dx}\log y = \frac{d}{dx}\log u - \frac{d}{dx}\log v$$

$$\frac{1}{y}\times\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} - \frac{1}{v}\frac{dv}{dx}$$

$$or \quad \frac{dy}{dx} = y\left[\frac{1}{u}\times\frac{du}{dx} - \frac{1}{v}\times\frac{dv}{dx}\right]$$

$$\therefore \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u}{v}\left[\frac{1}{u}\times\frac{du}{dx} - \frac{1}{v}\times\frac{dv}{dx}\right]$$

Exmaple 10: Let $y = \frac{x}{(x+3)(x+4)}$, find $\frac{dy}{dx}$.

Sol. Let u = x and v = (x+3) (x +4)

Take log on both sides.

$$\therefore \log y = \log\left[\frac{x}{(x+3)(x+4)}\right] \qquad \left[\because \log\left(\frac{u}{v}\right) = \log u - \log v\right]$$

log y = log x - log (x+3) (x+4)

log y = log x - [log (x+3) + log (x+4)]

Differentiate w.r.t.x.

$$\frac{1}{y}\times\frac{dy}{dx} = \frac{1}{x} - \left[\frac{1}{x+3} + \frac{1}{x+4}\right]$$

$$\frac{1}{y}\times\frac{dy}{dx}=\frac{1}{x}-\frac{2x+7}{(x+3)(x+4)}=\frac{x^2+7x+12-(2x+7)x}{x(x+3)(x+4)}$$

$$\therefore \frac{dy}{dx}=y\left[\frac{-x^2+12}{x(x+3)(x+4)}\right]$$

$$\therefore \frac{dy}{dx}=\frac{x}{(x+3)(x+4)}\times\frac{-x^2+12}{x(x+3)(x+4)}=\frac{-1(x^2-12)}{(x+3)^2(x+4)^2}$$

$$\therefore \frac{dy}{dx}=\frac{-x^2+12}{x(x+3)^2(x+4)^2}$$

Example 11: Let y $=\frac{x^3}{(x+5)^2}$, find $\frac{dy}{dx}$.

Sol. $\log y = \log x^3 - \log (x+5)^2$

$\log y = 3 \log x - 2 \log (x+5)$

Differentiate w.r.t.x. we get,

$$\frac{1}{y}\times\frac{dy}{dx}=\frac{3}{x}-\frac{2}{x+5}\left[\because \log x^1=\frac{1}{4}\log x\right]\ \frac{dy}{dx}=\frac{x^2}{(x+5)^2}\left[\frac{x+15}{x(x+5)}\right]=\frac{x^2(x+15)}{(x+5)^3}$$

1.2.4 Derivative of Exponential Functions:

In this section we will study the derivatives of exponential functions. Logarithms will be used to solve these functions.

1.2.4.1 Definition

Consider $y = a^x$. It is an exponential function. Similarly, $y = e^x$ is known as natural exponential function. $y = e^x$ is a single valued continuous and increasing function.

1.2.4.2 Differential Coefficient of a^x

Let $y = a^x$ ....(1)

Take logs on both sides, we get

Log y = x log a, where x is variable.

Differentiate w.r.t.x.

$$\therefore \frac{d}{dx}(\log y) = \frac{d}{dx}(x.\log \quad a) = \log \quad a\frac{d}{dx}(x)$$

$$\therefore \frac{1}{y} \quad \frac{dy}{dx} = \log a.1$$

$$\Rightarrow \qquad \frac{dy}{dx} = y\log \quad a = a^x \log \quad a \qquad [\text{from } (1)]$$

$$\therefore \ \frac{d}{dx}(a^x) = a^x.\log \quad a$$

Cor.1 Put a = e, we get

y = e^x

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(e^x) = e^x \quad \log \quad e = e^x \qquad (\log \quad e = 1)$$

Thus $\frac{d}{dx}(e^x) = e^x$

Cor. 2 If y = a^u where u is function of x

By Chain rule, we know

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

Now y = a^u

$$\therefore \frac{dy}{du} = \frac{d}{du}(a^u) = a^u \log a \qquad \left(\because \frac{d(a)^x}{dx} = a^n \log a\right)$$

Put in (i)

$$\therefore \frac{dy}{dx} = a^u \log a \frac{du}{dx}$$

Cor. 3 If y = e^u where u is the function of x.

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^u\right) = e^u \log_e \frac{du}{dx} = e^u \frac{du}{dx} \quad [\because \log_e = 1]$$

$$\therefore \frac{dy}{dx}(e^u) = e^u \frac{du}{dx}$$

1.2.4.3 Function of Function rule:

Let y = e^u where u is the derivable function of x.

Take logarithms of both sides

$$\therefore \log y = \log e^u = u \log e = u$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{du}{dx}e^u \qquad \Rightarrow \frac{d}{dx} = (e^u)\frac{du}{dx}$$

Example 12: Let y = 3^{x+2}, find $\frac{dy}{dx}$.

Sol. $\frac{dy}{dx} = (x+2)\log 3$

Example 13: Let y =ex^2+2x, find $\frac{dy}{dx}$.

Sol. Put x^2+ 2x = u

$$\therefore \frac{dy}{dx} = e^u \frac{du}{dx}$$

$$\Rightarrow \frac{dy}{dx} = e^{x^2+2x}\frac{d}{dx}(x^2 + 2x) = (2x+2)e^{x^2+2x}$$

Example 14: Consider y = $a^{\sqrt{x^2+1}}$, find $\frac{dy}{dx}$.

Sol. Put $\sqrt{x^2+1} = u$

Now, y = a^u

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\Rightarrow \frac{dy}{dx} = a^u \log a \times \frac{1}{2}(x^2+1)2x$$

$$= a^{\sqrt{x^2+1}} \log a \times \frac{x}{\sqrt{x^2+1}}$$

Example 15: Consider y = e^{x^x} , find $\frac{dy}{dx}$.

Sol. Take logarithms on both sides, we get

log y = $\log e^{x^x} = x^x \log e = x^x$

Take logarithms again, we get

log (log y) = x log x

Differentiate, $\frac{1}{\log y}\frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{x} + \log x.1$

$$\therefore \frac{dy}{dx} = y \log \ y[1 + \log \ x]$$

$$= \ = e^{x^x} \log \ e^{x^x} \ [1 + \log \ x]$$

$$= \ = e^{x^x} \log \ e^{x^x} \ \times \log \ ex$$

Example 16 : If x = y (1 + log x), show that

$$\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$$

Sol. $y = \frac{x}{(1 + \log x)}$

$$\frac{dy}{dx} = \frac{1(1 + \log x) - x\left(0 + \frac{1}{x}\right)}{(1 + \log x)^2}$$

$$= \frac{1 + \log x - \frac{x}{x}}{(1 + \log x)^2}$$

$$= \frac{\log x}{(1+\log x)^2}$$

1.2.5 Summary

In this lesson, we have learnt in detail about the derivatives of logarithmic and exponential functions. Moreover, we have discussed certain rules for finding derivatives using logarithms and exponents.

1.2.6 Key Concepts

Derivatives of logarithmic functions, Derivatives of exponential functions.

1.2.7 Long Questions

(1) Differentiate the following to obtain $\frac{dy}{dx}$

(i) $y = \log\sqrt{a^2+x^2-x}$

(ii) $y = \log (x+2) (x^3-x)$

(2) If x = y (1 + log x), show that

$$\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$$

(3) If $y = \sqrt{\frac{1-x}{1+x}}$, show that

$$(1-x)^2 \quad \frac{dy}{dx} + y = 0$$

(4) Find differential coefficient of x^x.

(5) Find $\frac{dy}{dx}$ when $y = a^x + x^x$

(6) Find $\frac{dy}{dx}$ when $y = (\log x)^{\log x}$

1.2.8 Short Questions

(1) Find the derivative of the following functions w.r.t.x.

(i) $\log_a (x^3+1)$ (ii) $\log (4-2x^3)$ (iii) $\log \frac{1+x^2}{1-x}$

(2) If y = $= \frac{(x+2)+(x+3)}{(x-2)(x-3)}$, find $\frac{dy}{dx}$

(3) If $y = \log_a (3x^2 + 4x + 5)$, find derivatives of y with respect to x.

(4) Find the differential coefficient of the following function:

$$\log(x+\sqrt{x^2+a^2})$$

1.2.9 Suggested Readings

1. D. Bose.	:	An Introduction of Mathematical Economics
2. Mehta and Madnani	:	Mathematics for Economists
3. Aggarwal and Joshi	:	Mathematics for students of Economics
4. Bhardwaj and Sabharwal	:	Mathematics for students of Economics

LESSON NO. 1.3 **AUTHOR : Mrs. AMANPREET KAUR**

MAXIMA AND MINIMA

1.3.1 Objectives

In this lesson, we are going to study about the determination of maximum and minimum values of a function using derivatives, which will further help us to allocate the maxima and minima of that particular function.

1.3.2 Introduction

We shall first define the concepts of increasing (↑) and decreasing (↓) functions.

1.3.2.1 INCREASING AND DECREASING FUNCTIONS

If y = f(x), then y is said to be increasing (↑) function of x at the point x = a if.

$\frac{dy}{dx}$ at x = a is always positive i.e. $\left(\frac{dy}{dx}\right)_{x=a} > 0$.

Further, y is said to be a decreasing (↓) function of x at the point x = a if

$\frac{dy}{dx}$ at x = a is always negative i.e. $\left(\frac{dy}{dx}\right)_{x=a} < 0$.

Example 1 : Test the cost function $y = 40-6x+x^2$ for ↓ or ↑ function at the point

(i) x = 0

(ii) x = 2 (ii) x = 4

Solution: Since y = 40-6x+x²

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(y) = \frac{d}{dx}(40 - 6x + x^2)$$

$$= \frac{d}{dy}(40) - 6\frac{d}{dx}(x) + \frac{d}{dx}(x^2)$$

= 0-6 + 2x

= 2x - 6

$$\therefore At\, x = 0, \frac{dy}{dx} = 2.0 - 6 = -4 < 0$$

The function or the curve, at the point x = 0 is decreasing

$$\therefore At\, x = 2, \frac{dy}{dx} = 2.2 - 6 \quad (2x - 6)$$

= 4 - 6 = –2<0

The function or the curve is decreasing at the point x = 2

$$\therefore At\, x = 4, \frac{dy}{dx} = 2.4 - 6$$

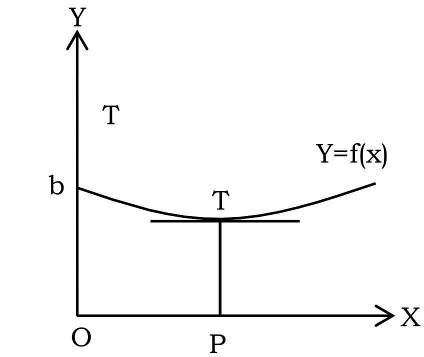
= 8 - 6 2 > 0

The function or the curve is increasing at x = 4.

1.3.3 CONCAVE UPWARDS & CONCAVE DOWNWARDS CURVE

If y = f(x) $\frac{dy}{dx} > 0$ at x = a, then y = f(x) has been defined as an ↓ function of x.

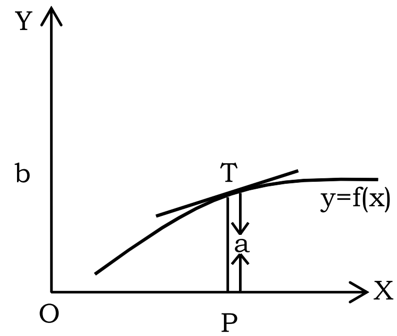
But if f(x) or $\frac{dy}{dx} > 0$, then we say that the function is increasing at an increasing rate, i.e. the rate of change of y is↑ . The curve y = f(x) lies above the tangent and we say that the curve is concave upward (or convex downward), if f(x) = 0, we get a straight line. These cases are illustrated diagramatically as follows:



$f'(a)>0, f'(a)>0$

(i)

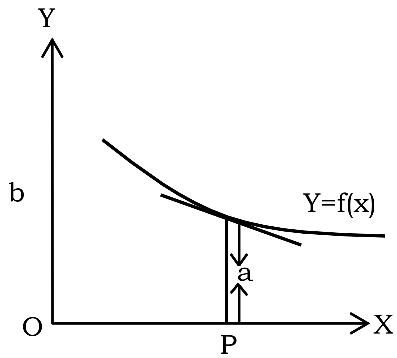
for increasing



$f'(a)>0, f'(a)<0$

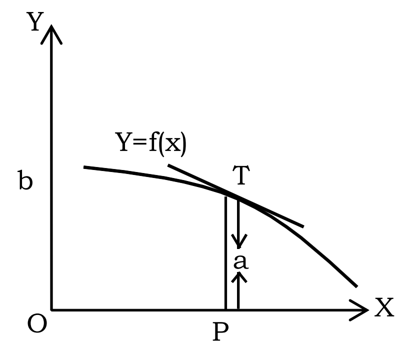
(ii)

for decreasing



$f'(a)<0, f'(a)>0$

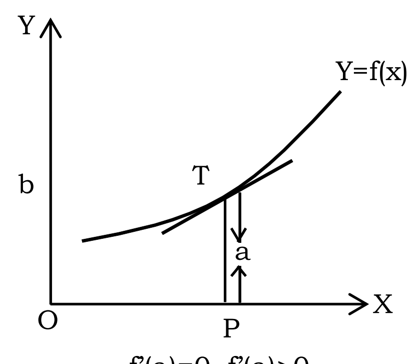
(iii)



$f'(a)<0, f'(a)<0$

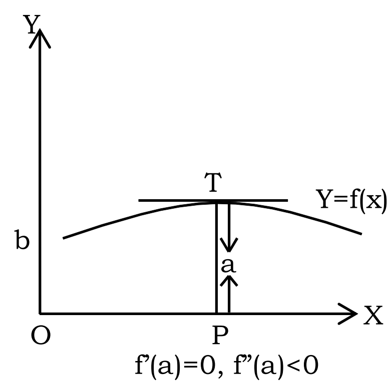
(iv)

f' (a) represents derivative.



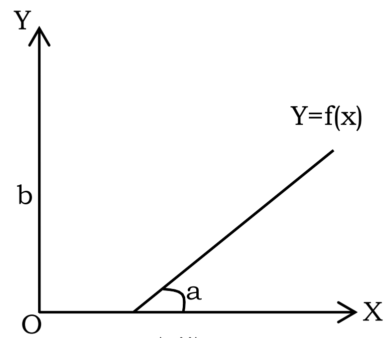
$f'(a)=0, f'(a)>0$

(v)



$f'(a)=0, f''(a)<0$

(vi)



(vii)

f' (a) represents second derivative.

We shall explain these cases below:

Case I : If f (x) > 0 and f^n (x) > 0

The curve will have shape as given in (i) above. It is concave upwards or Convex downwards.

Since f″ (x)>0, the slope of the curve is the positive and since f" (x)>0 , the slope of the curve tends to become steeper as x increases. (it depends upon the first derivative & Seem & derivative.

Case II: f′ (x)> 0 and f″ (x) <0

The curve will have shape as given in (ii) above. It is Concave downwards or Convex upwards.

Since f'(x)> 0, the slope of the curve is positive and since f" (x) <0, the slope of the curve goes on decreasing as x increases.

Case III: If f′ (x) <0 and f″ (x) >0

The curve will have shape as given in (iii) above. It is Concave upward or Convex downwards.

If f' (x)<0, the slope of the curve is negative and since f" (x)>0, the (negative) slope of the curve goes on decreasing as x increases.

Case IV : f′(x) < 0 and f″ (x) <0

The curve will have shape as given in (iv) above. It is Concave downwards or Convex upwards.

Since f'(x)< 0 the slope of the curve is negative and since f" (x) <0, the (negative) slope of the curve goes on decreasing as x increases.

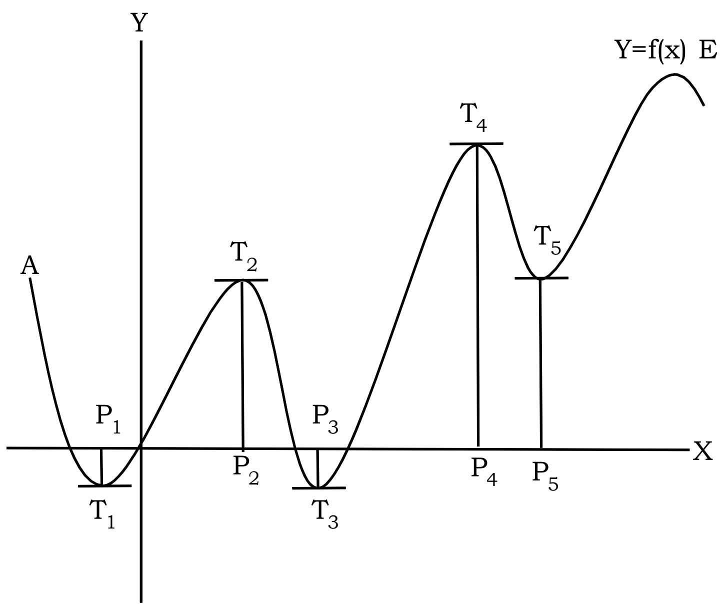
Thus with the help of second derivatives, we have been in a position to decide about the rising and falling nature of the curve.

The only problem which now remains is about maximum or minimum point of the curve when $\frac{dy}{dx} = 0$ [as in c(vi) above]. Before we deal with the problem, we shall define the maximum and minimum points of the function.

1.3.3.1 Maximum and Minimum Values of Function:

Let us take y = f (x). Let it take the following form when plotted:

1. The curve is falling from A to T_1 from T_2 to T_3 and T_4 to T_5 and the corresponding function is decreasing.
2. The curve is rising from T_1 to T_2, T_3, to T_4, T_5 to E and the corresponding function is increasing.
3. If the curve rises to a certain position and then falls. Such a position is called a *Maximum Point* of the curve. T2 and T4 are such points in the above curve. The ordinate, that the value of the function at such a point is called the *Maximum Value* of the function.



4. If the curve falls to a certain position and then rises, such a position is called Minimum Points of the curve. T_1, T_3 and T_5 are such points in the above curve. The ordinate, the value of the function at such a point is called the Minimum Value of the function.
5. The curve is concave upwards (or convex downward) between A and B, C and D, T_4 and E. It is Concave downwards (or Convex upwards) between B

and C, D and T^5.

Now we give a formal definition of extreme value of a function at a point.

A function Y = f(x) is said to have maximum value at x = c

if f(c) > f(x) for all x's(x # c) lying in the interval ($c-\partial c, c+\partial c$)

Similarly a function y = f(x) is said to have a minimum value at x=c if f(c) <f(x) for all x (x#c) lying in the interval ($c-\partial c, c+\partial c$)

i.e. if the function (or the curve) is plotted on the graph the curve is rising at nearyby points on the left of a maximum point and falling for nearby points on the left and rising for nearby points on the right.

1.3.4 Second Order Conditions for Minimum Values:

We have seen that a given function y = f(x) will have a maximum or minimum value at a point where the first derivative viz $\frac{dy}{dx}$ is zero and the second derivative at the points is less than zero, greater than zero respectively. Thus we observe that the given function should satisfy two conditions in order to decide about maximum or minimum value at a particular point. These conditions are known as *Second Order Conditions*.

(a) Conditions for Maximum Value

(1) First-order condition (or Necessary condition) $f'(x) = 0$

(2) Second order condition (or sufficient condition) $f''(x) < 0$

(b) Conditions for minimum Value

(1) First order condition (or necessary condition) $f'(x) = 0$

(2) Second order condition (or sufficient condition) $f''(x) > 0$

(3) In certain cases when $f''(x) = 0$, the above conditions fail to give the maximum values of the given function.

If $f''(x) = 0$, it means there may be a *Point of Inflexion.*

Criteria for Points of Inflextion

(1) If f(x) = 0 but f" (x) # 0, there is a point of inflextion.

(2) If f"(x) = 0, f" (x), but f" (x#0, then f" (x)<0 gives maxima and f" (x)>0 gives minima.

1.3.5 Practical Method for Maxima and Minima

If follows from the above discussion that the maximum or minimum, points can be determined from the functions y as follows:-

(a) 1. Equate $\frac{dy}{dx}$ to zero and solve for x.

(e.g. $y = 8x^3 \ \frac{dy}{dx} = \frac{d(8x^3)}{dx} = 0$ & Solve for x)

2. Determine whether $\frac{dy}{dx}$ is positive or negative for nearby points on the left and right.

3. A point where $\frac{dy}{dx} = 0$ is a maximum point if,

$\frac{dy}{dx} > 0$ for nearby points on left and

$\frac{dy}{dx} < 0$ for nearby points on the right.

4. A point nearby $\frac{dy}{dx} = 0$ for a minimum point if,

$\frac{dy}{dx} < 0$ for nearby points on the left and

$\frac{dy}{dx} > 0$ for nearby points on the right.

Remarks

Since the curve is *Concave downwards (or Convex upward)* at ordinary maximum point, the second derivative is negative if is not zero. Similarly since the curve is *Concave upward (or Convex downwards)* at an ordinary minimum point, the second derivative is positive it is not zero.

This gives the following method move frequently used for ordinary maximum or minmum points of a given function y = f(x).

(B) 1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

2. Put $\frac{dy}{dx}$ = 0 and solve for x to get x = x_1, x_2, x_3, etc.

3. Substitute each of these values of x into $\frac{d^2y}{dx^2}$ and find its value.

4. A value of x for which $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ is negative is maximum.

5. A value of x for which $\frac{dy}{dx}$ = 0 and $\frac{d^2y}{dx^2}$ is positive and is minimum .

6. If $\frac{d^2y}{dx^2}$ =0 for some value of x, then we have examined for points of inflextion.'

   (i) If f' (x) = 0 but f" (x) # 0, there is a point of inflextion.

   (ii) f" (x) <0 gives maxima and

   f" (x) > 0 gives minima

Properties of Extreme Values

1. A necessary but not sufficient condition for f(c) to be an extreme value of the function f(x) is that f (c) = 0.
2. If (i) f(x) is continuous at x=c and (ii) f' (x) exists in a certain neighbourhood of c excluding c; then f(c) is extreme vlaue of (x) and if f(x) changes sign as x passes through the value of c: f being maximum if the sign changes from minus to plus.
3. If f(x) possesses continuous derivatives upto the second order in a certain neighbourhood of the point c. then

   (i) f(c) is a maximum value of (x) if

   f(c) = 0 and f" (c) < 0

   (ii) f(c) = is a minimum value of(x) if

   f" (c) = 0 and f" (c) > 0

Example 1 : Find the extreme values, if any, of the function y = $2x^2-x^3$

Solution: First Method

Let y = $2x^2–x^3$

$$\frac{dy}{dx} = \frac{d}{dx}(2x^2-x^3)$$

$= 4x - 3x^2$

$= x(4-3x)$

For maxima and minima put $\frac{dy}{dx} = 0$

Here $\frac{dy}{dx} = 0$ gives x (4-x) = 0

Either x = 0 or 4 - 3x=0

i.e. x =4/3

$\therefore$ The two points at which, maxima can exist are x = 0 and x=4/3.

Now $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(4x-3x^2\right)$ =4-6x

Now we test the Second derivative at the two points.

At x = 0

$\frac{d^2y}{dx^2}$ = 4-6(0) = 4-0=4>0

$\therefore y$ is minimum at x=0 and Min. value is = 2x0-3=0

At $x=\frac{4}{3}$

$$\frac{d^2y}{dx^2} = 4 - 6\text{x}\frac{4}{3} = -4 < 0$$

$\therefore$ y is maximum at x =4/3 and the maximum value of the function at x=4/3 is given by

$f(4/3) = 2(4/3)^2-(4/3)^3$

$$= 2\frac{16}{9} - \frac{64}{27} - \frac{32}{27} = \frac{32}{9} - \frac{64}{27}$$

Second Method

Since $y = 2x^2 - x^3$

$$\frac{dy}{dx} = 4x - 3x^2 = x(4-3x)$$

$$\therefore \frac{dy}{dx} = 0 \text{ gives } x = 0 \text{ and } x = 4/3$$

Firstly we discuss maxima or minima at the point x = 0

When x is slightly less than 0, $\frac{dy}{dx}$ = (-) (+) = -

When x is slightly greater than 0, $\frac{dy}{dx}$ = (+) (+) = +

So $\frac{dy}{dx}$ changes sign from (-) to (+), as x passes through the point O. Hence it gives a minimum value.

Now we discuss maxima or minima at the point x = 4/3

When x is slightly less than 4/3, $\frac{dy}{dx}$ = (+) (+) = +

When x is slightly greater than 4/3, $\frac{dy}{dx}$ = (+) (-) = -

So $\frac{dy}{dx}$ changes from (+) to (-) as x passes through the point 4/3.

Hence it gives a maximum value at x = 4/3.

Maximum value at x = 4/3 and minimum value at x = 0 are found be calculating f(4/3) and f(0) respectively as in the first method.

Example 2: Find the extreme values of the function .

$y = 3x^4 - 10x^3 + 6x^2 + 5$

Solution

$y = 3x^4 - 10x^3 + 6x^2 + 5$

$\frac{dy}{dx}$ = $12x^3$-$30x^2$+12

$\frac{d^2y}{dx^2}$ = $36x^2$-60x+12

First Order Condition

For maxima and minima, $\frac{dy}{dx}$ = 0

$\therefore$ $12x^3$-$30x^2$+12x = 0

or 3x($4x^2$-10x+4) = 0

or 6x(x-2) (2x-1) = 0

$\therefore$ x=0 or x-2=0 or 2x–2=0

i.e. $\therefore$ x = 0 or x=2 or x=1/2

Second Order Condition

(i) $\frac{d^2y}{dx^2}$ = 36 $(0)^2$- 60. (0) + 12 = 12>0

at x = 0

Hence the function has a minimum value at x = 0 and the minimum vlaue is:

f(O) = $3(0)^4$ - $10(0)^3$ + $6(0)^2$ + 5 = 5

(ii) $\frac{d^2y}{dx^2}$ at x = 1/2 = $36(1/2)^2$ - 60. $\frac{1}{2}$ +12

= 9-30+ 12 = -9<0

Hence the function has a maximum value at = 1/2 and maximum value is

f $\left(\frac{1}{2}\right)$= 3 $\left(\frac{1}{2}\right)^4$- 10 $\left(\frac{1}{2}\right)^3$ + 6 $\left(\frac{1}{2}\right)^2$ + 5

$$= \frac{3}{16} + \frac{3}{2} + 5 - \frac{5}{4} = \frac{87}{16}$$

(iii) $\frac{d^2y}{dx^2}$ at x = 2

$= 36(2)^2 - 60(2) + 12$

$= 36x4-120+12$

$= 144-120+12 = 36$

Hence the function has +a minimum value at x=2 and the minimum value is f(2)

Example 3: Show that y = $x \times \frac{1}{x}$ has extreme vlaues and the minimum value is not four more than the maximum value.

Solution:

$$y = x \times \frac{1}{x}$$

$$\therefore \frac{dy}{dx} = 1 \; \frac{1}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{2}{x^3}$$

First Order Condition

For maxima or minima $\frac{dy}{dx} = 0$

$\therefore 1 - \frac{1}{x^2} = 0$ or $x^2 = 1$ or $x = \pm 1$

Second Order Condition

$\frac{d^2y}{dx^2}$ at $x=1 = \frac{2}{(1)^3} = \frac{2}{1} = 2 > 0$

Hence the function f(x) has a minimum value at x=1 and the minimum value is given by

$$f(1) = 1 + \frac{1}{1} = 2$$

$$\frac{d^2y}{dx^2} \; at \; x = -1 = \frac{2}{(-1)} = \frac{2}{-1} = -2 < 0$$

Clearly f (-2) =-2+ $\frac{1}{-2} = -4$

i.e. minimum value viz. f(+1) is greater than the maximum value vix, f(-1) by 4.

Example 4: Show that the minimum value of $\left(\frac{1}{x}\right)^x$ is $(e)^{1/e}$.

Solution:

$$y = \left(\frac{1}{x}\right)^x$$

$$\log \quad y = \log \left(\frac{1}{2}\right)^x = x \log \left(\frac{1}{x}\right) = -x \log x$$

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(= x \log x)$$

or $$\frac{1}{y} \times \frac{dy}{dx} (= \left(-x\frac{1}{x}\right) + (\log x.1)$$

= -(1+log x)

$$\therefore \frac{dy}{dx} = y(1+\log x) = -\left(\frac{1}{x}\right)^x (1+\log x) \quad \left(\because y = \left(\frac{1}{x}\right)^x\right)$$

For maxima, or minima $\frac{dy}{dx}$ =0

$$\therefore \left(\frac{1}{x}\right)^x (1+\log x) = 0$$

which gives 1+log x = 0

or log x -1 = -log e= log $\frac{1}{e}$

$\therefore x = \frac{1}{e}$

Now we discuss maxima or minima at x= $\frac{1}{e}$

When x is slightly less than $\frac{1}{e}$, $\frac{dy}{dx}$ = (+) (+) = +

Since $\frac{dy}{dx}$ changes sign from (-) to (+)

as x passes through the point x = 1/e.

We get a minimum value x =1/e and the minimum value is given by $\left(\frac{1}{e}\right)^{1/e}$

Example 5: Show that the curve y = 2x-3+ $\frac{1}{x}$

is convex from the below (or concave from the above) for all possible value of x.

Solution:

Here y = 2x-3+ $\frac{1}{x}$

$$\frac{dy}{dx} = 2 - \frac{1}{x^2} \qquad \left[\because \frac{d(2x)}{dx} = 2 \quad \frac{d\left(\frac{1}{x}\right)}{dx} = \frac{-1}{x^2}\right]$$

$$\frac{d^2y}{dx^2} = \frac{2}{x^3}$$

If this curve is to be Convex downwards (or Convave upwards, then it must possess minimum value for all positive values of x.

Hence $\frac{d^2y}{dx^2} > 0$

First Order Condition

$$\frac{d^1y}{dx^2} = 2 - \frac{1}{x^2} = 0$$

i.e. $x^2 = \frac{1}{x}$ or $x = \pm\frac{1}{\sqrt{2}}$

Second Order Condition

$$\frac{d^2y}{dx^2} \text{ at } x = \pm\frac{1}{\sqrt{2}}$$

$$\frac{d^2y}{dx^2} = \frac{2}{x^3} = \frac{2}{\left(+\frac{1}{\sqrt{2}}\right)^3} = \left(\frac{2}{\frac{1}{\sqrt{2}}}\right)^3 = +4\sqrt{2} > 0$$

Hence condition for minimum value is satisfied

$$\frac{d^2y}{dx^2} \, at \, x = -\frac{1}{\sqrt{2}} = -2\left(\frac{1}{\sqrt{2}}\right)^3 = -4\sqrt{2 < 0}$$

Hence conditon for minimum value is not satisfied.

Thus the curve $y = 2x - 3 + \frac{1}{x}$ is Convex from below (or downwards for all positive values of x.)

1.3.6 Summary

In this lesson, we have discussed about the increasing and decreasing functions along with maximum and minimum values of the functions. Further, rules to allocate maxima and minima of a function have been described.

1.3.7 Key Concepts

Increasing function, Decreasing function, Maximum value, Minimum value, Concave upwards (convex downwards), Convex upwards (concave downwards), Points of inflexion, Maxima, Minima, Extreme values.

1.3.8 Long Questions

1. Show that $y=40-6x+x^2$ is decreasing at x=2, stationary at x=3, and increasing at x=5.
2. Show that the maximum value of the function $f(x)=x^3-27x+108$ is 108 more than the minimum value.
3. Find the extreme values of the following functions :-

   (i) $y = x3+4x^2-3x+1$ (ii) $y = (x-3)^3 (x+1)^2$
4. Show that $y=x^5-5x^4+5x^3-10$ has maximum value at x=1, a minimum value when x=3 and neither maximum nor minimum value when x=0.

1.3.9 Short Questions

1. Define increasing and decreasing functions.
2. Define maximum and minimum values of a function.
3. Discuss about maxima and minima of a function.

1.3.10 Suggested Readings

1. D. Bose. : An Introduction of Mathematical Economics
2. Mehta and Madnani : Mathematics for Economists
3. Aggarwal and Joshi : Mathematics for students of Economics
4. Bhardwaj and Sabharwal : Mathematics for students of Economics

LESSON NO. 1.4 AUTHOR : Mrs. AMANPREET KAUR

ECONOMIC APPLICATIONS OF DERIVATIVES

1.4.1 Objectives

In the previous lesson, we have studied the concept of derivatives. In the present lesson we shall study some simple applications of derivatives in Economics.

1.4.2 Introduction

We know that the relation between economic variables can be expressed by means of functions and curves. Our aim is to interpret the derivatives with reference to some economic relations. The derivatives obtained from these functions give us the marginals in economics. All decision processes in economics make use of the concept of the marginal utility, marginal cost, marginal revenue etc. The concept of derivatives is used to find the rate of change in one variable with respect to some other variables. For example, the rate of change of consumption with respect to income, or rate of change of demand with respect to price, or the rate of change in output with respect

* **Note:** The lesson is basically economic application of derivatives in economics which also covers their application in maxima and minima.

to some input such as labour or capital etc. Now we turn to economic applications of derivatives.

1.4.3 Slope of the Demand Curve at a Point

Let q be the quantity demanded and p be the price. The demand curve is defined as q = f(p). The slope of the demand curve at a point is defined as $\frac{dq}{dp}$ i.e. rate of change of q.w.r. to p. Since the demand curve is monotonically decreasing, so the slope of demand curve is $\frac{dq}{dp}$ but this is negative.

Example 1: Let q = a-b p be the demand curve.

The slope of demand curve at a particular point = $\frac{dq}{dp}$

Now $\frac{dq}{dp} = -b$

$\therefore$ Slope of demand curve = -b.

The negative sign with b indicates that other things being equal, p and q are inversely related.

1.4.4 Price Elasticity of Demand

Price elasticity of demand is defined as the value of the ratio of the relative (or proportionate) change in the demand to the relative (or proportionate) change in the price.

Precisely, if we suppose that the demand changes from (x) to (x+ Δ x) when the price changes from (p) to (p+ Δ p), then elasticity of demand as per definition is given by:

$$\eta_d = \frac{\text{Proportionate change in quantity demanded}}{\text{Proportionate change in price}}$$

$$= \underset{\Delta p \to 0}{\text{Lt}} \frac{\Delta x / x}{\Delta p} \qquad \begin{pmatrix} \frac{\Delta x}{x} \text{ is change in quantity} \\ \frac{\Delta P}{P} \text{ is change in price} \end{pmatrix}$$

$$\underset{\Delta p \to 0}{Lt} \frac{p}{x} \times \frac{\Delta x}{\Delta p}$$

$$= \frac{p}{x} \underset{\Delta p \to 0}{Lt} \frac{\Delta x}{\Delta p} = \frac{p}{x} \frac{dx}{dp}$$

Since the normal demand curve is a monotonic decreasing function, the elasticity of demand will be negative at all prices.

Hence, the elasticity of demand i.e $\eta_d = \frac{-p}{x}, \frac{dx}{dp}$

Note: Slope of the demand curve may remain the same but elasticity of demand at different points on the curve may differ.

Example 2: Let the demand curve is $x = \frac{q}{\sqrt{p}}$, show that the price elasticity of demand is constant and equal to $\frac{-1}{2}$.

Sol. Given $x = \frac{9}{\sqrt{p}} = 9p^{-1/2}$, Differentiating w.r. to p.

$$\therefore \frac{dx}{dp} = \frac{d}{dp}\left(9p^{-1/2}\right) = -\frac{9}{2}p^{-3/2}$$

$$\therefore |\eta_d| = \left|\frac{p}{x}.\frac{dx}{dp}\right| = \left|\frac{p}{9p^{\frac{-1}{2}}}\left(-\frac{9p^{-\frac{3}{2}}}{2}\right)\right| = -\frac{1}{2}$$

Since, η_d is negative, $\therefore$ price elasticity of demand is constant and equal to $\frac{-1}{2}$.

Example 3: Given the demand function $q=100-2p-2p^2$. Calculate the price elasticity of demand when p = 10.

Given $q = 100-2p-2p^2$

$$\therefore \frac{dx}{dp} = -2-4p$$

Now, $|\eta_d| = \left|\frac{p}{q}.\frac{dq}{dp}\right| = \left|\frac{10}{100-2p-2p^2} \times \{2+4\times 10\}\right|$

$$= \left|\frac{10}{[100-(2\times 10)-(2\times 100)]} \times (-42)\right| = \left|\frac{10\times 7}{20}\right| = \frac{7}{2}$$

Since η_d is negative, $\therefore$ price elasticity of demand is constant and equal to = $-\frac{7}{2}$

1.4.5 Concept of Marginal Revenue

Marginal revenue is defined as the rate of change of total revenue with respect to output.

i.e. MR = $\frac{d}{dq}$ (TR)

We know that, total revenue = Price $\times$ Number of units sold (or output)

$$\therefore MR = \frac{d}{dq}(pq) = p\frac{d}{dq}(q) + q.\frac{dp}{dq} = p + q\frac{dp}{dq}$$

Example 4: The revenue from the sale of units of a good is given by:

$$R = 5q + \frac{12}{3q+1}$$

Find marginal revenue when (i) q = 8(ii) q = 16

Sol. Given R = $5q + \frac{2}{3q+1}$

$$\therefore MR = \frac{d}{dq}(R) = 5 + \frac{-12\times 3}{(3q+1)^2} = \frac{5(3q+1)^2 - 36}{(3q+1)^2}$$

When q = 8, MR = $\frac{5(25)^2 - 36}{(25)^2} = \frac{3089}{625} = 4.924$

When q = 16, MR = $\frac{5(49)^2 - 36}{(49)^2} = 4.985$

Example 5: The demand curve for a monopolist is q=100-4p. Find total revenue, average and marginal revenue. At what value of q, MR=0

Sol. Given q = 100-4p or p=25-$\frac{q}{4}$

TR = $p \times q = (25 - \frac{q}{4})q = 25q - \frac{1}{4}q^2$

Average revenue= $\frac{TR}{q} = \frac{pq}{q} = \frac{(25q - 1/4q^2)}{q}$

$$AR = = \frac{25q - 1/4q^2}{q} = 25 - \frac{q}{4}$$

Marginal revenue, $MR = \frac{d}{dq}(pq) = \frac{d}{dq}(TR) = 25 - \frac{1}{2}q$

Put, MR $= 0 \Rightarrow 25 = \frac{q}{2}$

Or q = 50

Example 6: If the demand function is p=$\sqrt{9-q}$; find at what level of output q, the TR will be maximum and what will it be?

Sol. Since we know that TR is maximum when MR =0. Thus by equating MR function to zero, we find the output level q and then substituting this value of q in the given demand function we estimate price p.

Given p = $\sqrt{9-q}$

Suppose the level of output that maximises the TR is q. with price = P in the market.

TR = q.p.

Or TR = $q\sqrt{9-q}$ = q $(9\text{-}q)^{1/2}$

Now, MR = $q.\frac{1}{2}(9-q)^{-\frac{1}{2}}(-1)+(9-q)^{1/2} = \frac{2(9-q)-q}{2\sqrt{9-q}}$

TR is maximum when MR = 0, thus making MR = 0

$$MR = \frac{18-3q}{2\sqrt{9-q}} = 0$$

$\Rightarrow \quad \therefore 6-q=0 \quad \therefore q=6$

Hence TR will be maximum when q = 6.

Substituting value of q in p = $\sqrt{9-q} = \sqrt{3}$

$\therefore TR = pq = 6\sqrt{3}$

Example 7: The demand curve of a monopolist is given by $p = \frac{50-q}{5}$

1. Find the marginal revenue for any output q.
2. What is MR when
   (a) q=0
   (b) q=25

Solution: The demand curve is $p = \frac{50-q}{5}$ or $p = \frac{50}{10} - \frac{q}{5}$

Total Revenue R is given by

$$R = pq = 10q - \frac{1}{5}q^2$$

$\therefore$ **Marginal Revenue =** $\frac{dR}{dq} = 10 - \frac{2q}{5}$ .... (i)

(a) Value of MR when q = 0

Putting q = 0 in (i), we get

$$\therefore 10 - \frac{2(0)}{5} = 10 \quad \text{(b)} \quad MR \; at \; q = 25 \quad \therefore \; 10 - \frac{2}{5}(25) = 0$$

Thus MR at q=0 is 10 and at q = 25 is zero.

1.4.6 Marginal Cost

Suppose that total cost function on $\bar{\wedge}$ of output q is given by = f(q) then Marginal cost is defined as the rate of change in total cost with respect to output.

$$\text{Thus MC} = \frac{d\bar{\wedge}}{dq}$$

Example 8 : Let the total cost function be $\bar{\wedge}$ = $3+2x+5x^2$

$$\text{then MC} = \frac{d\bar{\wedge}}{dq} = \frac{d}{dx}(3+2x+5x^2) = 2+10x$$

Example 9: Given $\bar{\wedge}$ = aq^2+bq+c

Find AC and MC and hence show that the slope of AC Curve = $\frac{1}{q}(MC - AC)$

Solution : Given $\bar{\wedge}$ = aq^2+bq+c

$$AC = \frac{\bar{\wedge}}{q} = aq + b + \frac{c}{q}$$

$$\text{Slope of the AC Curve} = \frac{d}{dq}\left(\frac{\bar{\wedge}}{q}\right)$$

$$= \frac{d}{dq}\left(aq + b + \frac{c}{q}\right) = a + 0 + \left(\frac{-c}{q^2}\right)$$

$$= a - \frac{c}{q^2} \;\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Now AC = aq+b+ $\frac{c}{q}$

MC = 2aq+b

$$\therefore \frac{1}{q}(MC-AC)=\frac{1}{q}\left[(2aq+b)-(ab+b+\frac{c}{q})\right]-$$

$$= =\frac{1}{q}\left[aq-\frac{c}{q}\right]=a-\frac{c}{q^2}..............(ii)$$

From (i) and (ii) we see that

Slope of AC curve $=\frac{1}{q}\left[MC-AC\right]=a-\frac{c}{q^2}$

Example 10 : The total function for producting a commodity in x quantity is TC = 60-12x+2x^2

(a) Find the average cost function. (b) the level of output at which this function is minimum. (c) Verify that AC =MC at the minimum point of AC curve.

Solution: (a) The total cost function is given by TC = 60 -12x+2x^2

Average cost function = $\frac{TC}{x}=\frac{60-12x+2x^2}{x}=\frac{60-12x+2x}{x}$

Marginal cost function $=\frac{d}{dx}(TC)=\frac{d}{dx}(60-12x+2x^2)$

= -12+4x

Slope of AC curve = $\frac{1}{x}(MC-AC)$

$$=\frac{1}{x}\left[-12+4x-\frac{60+12-2x}{x}\right]=2-\frac{60}{x^2}=\frac{1}{\mathrm{x}}\left[\frac{-12\mathrm{x}+4\mathrm{x}^2-60+12-2\mathrm{x}}{\mathrm{x}}\right]$$

(b) When AC function is minimum the slope of AC function is zero.

$$\therefore i.e.\frac{1}{x}(MC-AC)=0$$

Thus

$$2-\frac{60}{x^2}=0 \quad or\, x^2=30\, x=\sqrt{30}=5.477.$$

(c) Now at the minimum point of AC curve x = $\sqrt{30}$

When

$$x=\sqrt{30},\, AC\,\frac{60}{30}-12+2\sqrt{30}$$

$$=2\sqrt{30}-12+2\sqrt{30}=4\sqrt{30}-12$$

When

x = $\sqrt{30}$, MC = -12+4 $\sqrt{30}$

$\therefore$ At the minimum point of AC curve, AC = MC.

1.4.7 Marginal Utility: Marginal utility is defined as the additional utility made to the total utility by increasing or adding one more unit of the commodity.

Mathematically, MU= $\frac{dU}{dx}$

Where, x is the quantity of a commodity consumed and U is tital utility.

Example 11: For the total utility function U= $20x^4+7x^3+13x^2+12x+9$, Compute Marginal Utility

Given U = $20x^4+7x^3+13x^2+12x+9$

MU = $\dfrac{dU}{dx} = \dfrac{d(u)}{dx}$

MU= $\dfrac{d}{dx}$ $(20x^4+7x^3+13x^2+12x+9)$

MU= $80x^3+21x^2+26x+12$

Example 12: Given the average revenue function AR= $\dfrac{20}{Q}-10$

Find the total and marginal revenue functions.

Solution: AR= $\dfrac{20}{Q}-10$

Therefore TR = AR x Q

TR = $\left(\dfrac{20}{Q}-10\right)$ (20-10Q)=-10

= 20-10Q.

$$MR = \frac{dR}{dQ} = \frac{dR}{dQ} = \frac{dR}{dQ} = (20-10Q) = -10$$

Example 13: Find out the TR, AR and MR for the demand function p = $\dfrac{1}{q^4}+32,$ where P is the price and q is the quantity demanded.

Solution: TR = Pq

$$= \left[\frac{1}{q^4}+32\right]q$$

$= q^{-3} + 32q$

AR= $\frac{TR}{q} = \frac{q^{-3} + 32q}{q}$

$= \frac{1}{q^4} + 32$

MR= $\frac{d}{dq}(TR) = \frac{d}{dq}(TR)$

$= \frac{d}{dq}(q^{-3} + 32q)$

$=-3q^{-4}+32$

Example 14: Given the total cost function C= $\frac{1}{3}Q^3 - 3Q^2 + 9Q$, find Average Cost and also find the marginal cost at the level Q = $\frac{9}{2}$

Solution: C= $\frac{1}{3}Q^3 - 3Q^2 + 9Q$

Therefore, AC= $\frac{C}{Q} = \frac{\frac{1Q^3 - 3Q^2 + 9Q}{3}}{Q}$

$AC = \frac{Q^2}{3} - 3Q + 9 \;\left(\frac{1}{3}Q^3 - 3Q^2 + 9Q\right)$

MC= $= \frac{d(c)}{dQ} = \frac{d}{dQ}\frac{(1 - Q^3 - 3Q^2 + 9Q)}{3}$

MC = Q^2-6Q +9

At Q $= \frac{9}{2}\, MC\, is = \left(\frac{9}{2}\right)^2 - 6 \times \frac{9}{2} + 9$

$$=\frac{81-108+36}{4}=\frac{9}{4}=2.25$$

1.4.8 Application of Derivatives for Finding Maxima and Minima

The technique of derivation is also used for finding out the maximum and minimum values of a function. The maximum and minimum values are also known as extreme vlaues of the function.

Necessary and sufficient Conditions:

Let y = f(x) be a function of x.

(a) The necessary condition for all extreme values is that $\frac{dy}{dx}=0$

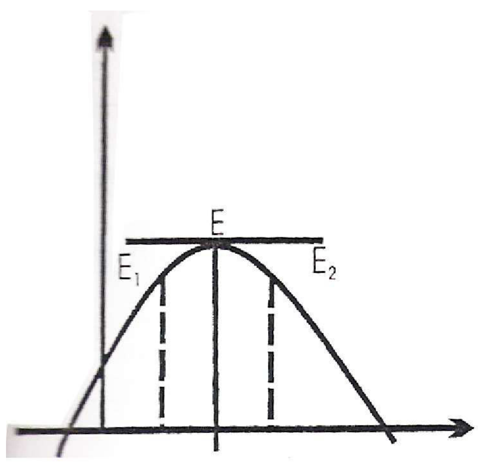


Fig.1 Maximum

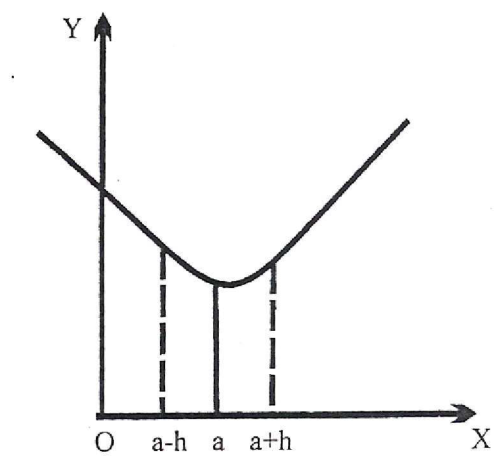


Fig.2 Minimum

Fig. 1 and Fig. 2 show that maximum or minimum occurs at x = a where $\frac{dy}{dx}=0$. In Fig. 3, $\frac{dy}{dx}=0$ at point D on the curve y = f(x).

Here the value is neither maximum nor minimum. Therefore $\frac{dy}{dx}=0$ is a necessary condition but not a sufficient condition.

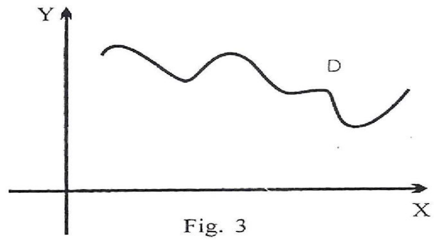


Fig. 3

(b) (i) For a function to have a maximum values at x = a, the sufficient condition is:

$$\frac{d^2y}{dx^2} < 0 \, or \, f''(x)$$ is negative. (Second derivative is negative)

(ii) For a function to have a minimum values at x =a, the sufficient condition is:

$$\frac{d^2y}{dx^2} > 0 \, or \, f''(x)$$ is positive. (Second derivative is positive)

Note:

At maximum point, f”(x) = 0 and f" (x) is -ve. (When second derivative is negative)

At a minimum point, f” (x) = 0 and f” (x) is +ve.

In Economics when the revenue function is given, we can find the maximum revenue by the technique of Maxima and Minima. Similarly minimum cost can be found out from total cost function by the procedure of Maximum and Minimum.

Example 15: The cost C of manufacturing a certain article is given by formula

$\frac{5+48+3x^2}{x}$. Where x is the number of article manufactured.

Find minimum value of C.

Solution: Given C = $\frac{5+48+3x^2}{x}$ diff. w.r. to x.

$$\frac{dc}{dx} = \frac{d}{dx}(6x^{-1} + 48)$$

$$\frac{dc}{dx} = \frac{-48}{x^2} + 6x$$

diff. again w.r. to x.

$$\frac{d^2c}{dx^2} = \frac{d}{dx}\left(\frac{-48}{x^2} + 6x\right) = = \frac{d}{dx}\left(-48^{-2} + 6x\right)$$

$$= = \frac{96}{x^3} + 6$$

For extreme values $= \frac{dc}{dx} = 0 \frac{-48}{x^2} + 6x = 0$

i.e. $6x^3=48$ or $x^3= 8$

$\therefore$ x = 2

$$\frac{d^2c}{dx^2} = \frac{96+6}{(2)^2} = \frac{96+6}{8} = \frac{144}{8} = 18 > 0$$

Min. cost = $5 + \frac{48}{2} + (2)^2$

$$= \frac{10+48+24}{2} = \frac{82}{2} = 41$$

Min. cost = 41, x -=2.

Example 16 : If $p = \frac{121-1}{q+4}$ find output level at which total revenue is maximum, also find maximum revenue.

Solution : As $p = \frac{121-1}{q+4}$

R = Total Revenue $pq = \frac{(121-1)q}{q+4} = \frac{121}{q-4}q - q$

diff. w.r. to q

= $\frac{dR}{dq} = \frac{d}{dq}\frac{(121)}{q+4}q - \frac{d(q)}{dq}$

$= \frac{(q+4)(121)-121q}{(q+4)^2} - 1$

= $= \frac{121 \times 4}{(q+4)^2} - 1$

diff. again w.r. to q.

$\frac{d^2R}{dq^2} = \frac{d}{dq}\left[484(q+4)^{-2} - 1\right]$

= 484(-2)(q+4)$^{-3}$-O= $\frac{-968}{(q+4)^3}$

For Max or Mini $\frac{dR}{dq} = 0$

$\Rightarrow \frac{484}{(q+4)^2} - 1 = 0$ i.e. $(q+4)^2 = 484$

-q+4=22 Or q = 18 i.e. -q = 22-4 = q = -18

At q $\Rightarrow 18, \frac{d^2R}{dq^2} = \frac{-968}{(18+4)^3} < 0$

q=18 gives maximum revenue.

Max. Revenue = $\dfrac{121\times18}{18\frac{1}{4}}-18=81$

1.4.9 Summary

Thus derivatives have a wide range of application in Economics. For finding various concepts like marginal utility, marginal revenue, marginal cost, marginal propensity to consume, marginal propensity to save, marginal productivity etc. derivatives are used. They are also used for finding maximum and minimum values of a function.

1.4.10 Key Concepts

Demand curve, Elasticity of demand, Marginal cost, Marginal revenue, Marginal utility, Maxima and Minima.

1.4.11 Long Questions

1. A demand function is given by $P=\dfrac{10}{(3+q)}$, $0 \le q \le 9$, where p is the price per unit of quantity demanded and q is the number of units demanded. Find the elasticity at this end points (i.e. 0 and 9) of the given interval.

   For this function show that $\dfrac{dq}{dp}=\dfrac{1}{\frac{dp}{dq}}$

2. Let the revenue function be given by $R = 14x-x^2$ and cost function $T = x(x^2-2)$. Find AC, MC, AR and MR.
3. The consumption function for an economy is given by C = 50 +4Y, where C is aggregate consumption and Y is the aggregate income. Find the marginal propensity to consume.
4. Assuming labour as the only variable factor of production, a particular production function is given by:
   $q = 50+0.5L^2-0.001L^3$
   and wage rate = 90. Find MC when L = 300.
5. The total cost function of a firm is given by = $0.4q^3 -0.9q^2 + 10q + 10$.
   Find (i) AC (ii) MC (iii) Slope of MC.

1.4.12 Short Questions

1. Given the demand curve $P = 16-D^2$, find the total revenue curve and marginal revenue curve when D =1.
2. Let the demand function for a commodity be P = 10-2D, where P is the price and D is the quantity demanded. Find AR and MR.
3. Find the elasticity of demand of the demand function:

   $P = 12-4^x$ at P = 2.

4. Find the slope of the demand curve:

   $$V = \frac{40}{P+1} \, at \; P = 4$$

5. Given $\frac{-}{\wedge}$ = aq^2+bq+c

   Find AC and MC and hence show that the slope of AC curve = $\frac{1}{q}(MC - AC)$

1.4.13 Suggested Readings

1. D. Bose. : An Introduction of Mathematical Economics
2. Mehta and Madnani : Mathematics for Economists
3. Aggarwal and Joshi : Mathematics for students of Economics
4. Bhardwaj and Sabharwal : Mathematics for students of Economics

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BBA PART- I **PAPER : BBA-103**

BUSINESS MATHEMATICS

LESSON NO. 1.5 **AUTHOR : Mrs. AMANPREET KAUR**

PARTIAL DIFFERENTIATION

1.5.1 Objectives

In this lesson, our prime objectives are :

(i) to study partial derivatives and total derivative

(ii) to introduce homogeneous function

(iii) to find the derivatives of implicit functions

1.5.2 Partial Derivatives:

Let Z=f(x,y) be a function of two variables x and y.

If we keep y as constant and vary x alone then, z is a function of x only. The derivative of z with respect to x, treating y as constant, is called the partial derivative of z with respect to x and is denoted by one of the following symbols :

$$\frac{\partial z}{\partial x}, \frac{\partial f}{\partial x}, f_x(x,y), D_x f.$$

Thus $$\frac{\partial z}{\partial x} = \underset{\Delta x \to o}{Lt} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

Similarly, partial derivative of z with respect to y is denoted by one of the following symbols :

$$\frac{\partial z}{\partial y}, \frac{\partial f}{\partial y}, f_y(x, y), D_y f.$$

$$\text{Thus, } \frac{\partial z}{\partial y} = \Delta y \overset{Lt}{\rightarrow} o \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

Similarly, If z is a function of three or more variables x_1, x_2, x_3 _____ then partial derivative of z with respect to x_1, is obtained by differentiating z with respect to x_1, keeping all other variables constant and is written as $\partial z / \partial x_1$.

In general f_x and f_y are also functions of x and y and so these can be differentiated further partially with respect to x and y as :

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \text{ or } \frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \text{ or } \frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \text{ or } \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy,}$$

and

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial z^2}{\partial y^2} \text{ or } \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy.}$$

It can easily be verified that, in all ordinary cases,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

Sometimes we use the following notation.

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x \partial y} = s, \frac{\partial^2 z}{\partial y^2} = t$$

Example 1 : Find the first and second order partial derivatives of $z=x^3 + y^3-3axy$.

Sol. We have $Z = x^3+y^3-3axy$.

$$\frac{\partial z}{\partial x} = 3x^2 + O - 3ay(1)$$

$$= 3x^2 - 3ay$$

and $$\frac{\partial z}{\partial y} = o + 3y^2 - 3ax(1) = 3y^2 - 3ax$$

Also, $$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(3x^2 - 3ay\right) = 6x$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(3x^2 - 3ay\right) = -3a$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(3y^2 - 3ax\right) = 6y$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(3y^2 - 3ax\right) = -3a$$ and

We observe that, $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$

1.5.3 Homogeneous Function

An expression of the form $a_o x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \ldots\ldots.. + a_n y^n$ in which every term is of the n^{th} degree is called a homogeneous function of degree n. This can be rewritten as

$$x^n\left[a_o + a_1(y/x) + a_2(y/x)^2 + \ldots\ldots\ldots\ldots + a_n(y/x)^n\right]$$

Thus, any function f(x,y) which can be expressed in the form $x^n \phi(y/x)$, is called homogenous function of degree n in x and y.

For instance, $x^3 Cos(y/x)$ is a homogenous function of degree 3, in x and y.

In general, a function f(x,y,z,t.......) is said to be a homogeneous function of degree n in x, y,z,t......., if it can be expressed in the form $x^n \phi(y/x, z/x, t/x)$.

Euler's theorem on homogenous functions :

Statement : If u be a homogenous function of degree n in x and y, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

Proof : Since u is a homogeneous function of degree n in x and y, therefore, $u=x^n f(y/x)$.

$$\therefore \frac{\partial u}{\partial x} = nx^{n-1}f\left(\frac{y}{x}\right) + x^n f^1\left(\frac{y}{x}\right).y\left(-\frac{1}{x^2}\right) = nx^{n-1}f\left(\frac{y}{x}\right) - yx^{n-2}f^1\left(\frac{y}{x}\right)$$

$$\text{and, } \frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right).\frac{1}{x} = x^{n-1}f^1\left(\frac{y}{x}\right).$$

$$\text{Hence } x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f\left(\frac{y}{x}\right) = nu$$

In general, If u be a homogeneous function of degree n in x, y, z, t,......, then.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + t\frac{\partial u}{\partial t} + ... = nu.$$

Example 2 : Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u \log u$

Where $\log u = (x^3 + y^3)/(3x + 4y)$

Sol. Since $Z = \log u = \frac{x^3 + y^3}{3x + 4y} = \frac{x^2.\left[1 + (y/x)^3\right]}{3 + 4(y/x)} = x^2\phi\left(\frac{y}{x}\right)$

$\therefore$ Z is a homogeneous function of degree 2 in x and y.

By Euler's theorem, we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z \qquad ....(i)$$

But $\frac{\partial z}{\partial x} + \frac{1}{u}\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y} = \frac{1}{u}\frac{\partial u}{\partial y}$

Hence (i) becomes

$$x.\frac{1}{u}\frac{\partial u}{\partial x} + y.\frac{1}{u}\frac{\partial u}{\partial y} = 2\log u$$

or $$x.\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u$$

Example 3 : If $u = \sin^{-1}\left(\frac{x+2y+3z}{x^8+y^8+z^8}\right)$,

find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

Sol. Here u is not a homogeneous function.
we therefore, write

$$w = \text{Sin } u = \frac{x+2y+3z}{x^8+y^8+z^8} = \frac{x^{-7}.\left[1+2(y/x)+3(z/x)\right]}{1+(y/x)^8+(z/x)^8}$$

Thus w is a homogeneous function of degree –7 in x,y,z. Hence by Euler's theorem :

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = -7(w) \qquad \text{.................. (1)}$$

But $\frac{\partial w}{\partial x} = \text{Cos}u\frac{\partial u}{\partial x}$,

$$\frac{\partial w}{\partial y} = \text{Cos}u\frac{\partial u}{\partial y},$$

$$\frac{\partial w}{\partial z} = \text{Cos}u\frac{\partial u}{\partial z}.$$

$\therefore$ (1) becomes, $x \ \text{Cosu}\frac{\partial u}{\partial x} + y \ \text{Cosu}\frac{\partial y}{\partial y} + z \ \text{Cosu}\frac{\partial u}{\partial z} = -7\sin u$

or $x\frac{\partial y}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -7\tan u$

Example 4 : If z is a homogeneous function of degree n in x and y, show that

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z.$$

Sol. By Euler's theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Differeentiating (i) partially w.r.t x, we get

$$x\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} + y\frac{\partial^2 z}{\partial x\partial y} = n\frac{\partial z}{\partial x}$$

i.e. $$x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x\partial y} = (n-1)\frac{\partial z}{\partial x} \ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

Again differentiating (i) partially w.r.t y, we get

$$x\frac{\partial^2 z}{\partial y\partial x} + \frac{\partial z}{\partial y} + y\frac{\partial^2 z}{\partial y^2} = n\frac{\partial z}{\partial y}$$

i.e. $$x\frac{\partial^2 z}{\partial x\partial y} + y\frac{\partial^2 z}{\partial y^2} = (n-1)\frac{\partial z}{\partial y} \ldots\ldots\ldots\ldots (iii)$$

Mutiplying (ii) by x and (iii) by y and adding, we get

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) = n(n-1)z$$

1.5.4 Total Derivative

If u = f(x,y), where $x = \phi(t)$ and y = r(t) then we can express u as a function of t alone by substituting the values of x and y in f(x,y). Thus we can find the ordinary derivative du/dt which is called the total derivative of u to distinguish it from the partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$.

Now to find $\partial u/\partial t$ without actually substituting the values of x and y in f(x,y), we establish the following Chain Rule:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

Proof : we have u = f(x,y) .... (i)

Giving increment Δt to t, let the corresponding increments of x,y and u be Δx, Δy and Δu respectively.

Then, $u + \Delta u = f(x + \Delta x, y + \Delta y)$ .... (ii)

On Subtracting, $\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y)$

(i) from (ii), $= \{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)\} + \{f(x, y + \Delta y) - f(x, y)\}$

$$\Rightarrow \quad \frac{\Delta u}{\Delta t} = \frac{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)}{\Delta x}.\frac{\Delta x}{\Delta t} + \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}.\frac{\Delta y}{\Delta t}$$

Taking limits as $\Delta t \to o$, Δx and Δy also $\to o$, we have

$$\frac{du}{dt} = \underset{\Delta x \to o}{Lt}\left\{\frac{f(x + dx, y + \Delta y) - f(x, y + \Delta y)}{\Delta x}\right\}\frac{dx}{dt}$$

$$+ \underset{\Delta y \to o}{Lt}\left\{\frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}\right\}\frac{dy}{dt}$$

$$= \underset{\Delta y \to o}{Lt}\left\{\frac{\partial f(x, y + \Delta y)}{\partial y}\right\}\frac{\partial x}{dt} + \frac{\partial f(x, y)}{\partial y}.\frac{dy}{dt}$$

[Supposing $\partial f(x,y)/\partial x$ to be a Continuous function of y]

$$= \frac{\partial f(x,y)}{\partial x}.\frac{dx}{dt} + \frac{\partial f(x,y)}{\partial y}.\frac{dy}{dt}$$

Which is the desired formula.

Cor. Taking t = x, (i) becomes, $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial y}{\partial y}.\frac{dy}{dx}$ .... (iii)

Obs. If u = f(x,y,z), where x, y, z are all functions of a variable t, then chain rule is

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} + \frac{\partial u}{\partial z}.\frac{dz}{dt}$$

1.5.5 Differentiation of Implicit Functions.

If f(x,y) = c be an implicit relation between x and y which defines as a differentiable function of x, then (iii) becomes

$$o = \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{dy}{dx}$$

This gives the important formula

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y}$$

for the first differential coefficient of an implicit function.

Example 5 : Given u = Sin (x/y), $x=e^t$ and $y=t^2$. find du/dt as a function of t. Verify your result by direct subsitution.

Sol. We have $\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$

$$= \left(\text{Cos}\,\frac{x}{y}\right)\frac{1}{y}.e^t + \left(\text{Cos}\,\frac{x}{y}\right)\left(-\frac{x}{y^2}\right)2t$$

$$= \text{Cos}\left(e^t/t^2\right).e^t/t^2 - 2\text{Cos}\left(e^t/t^2\right).\,e^t/t^3$$

$$\Rightarrow \frac{du}{dt} = \left\{(t-2)/t^3\right\} e^t \text{Cos}\left(e^t/t^2\right) \quad \text{.................. (i)}$$

Also $u = \text{Sin}(x/y) = \text{Sin}\left(e^t/t^2\right)$

$$\frac{du}{dt} = \text{Cos}\left(\frac{e^t}{t^2}\right) . \frac{t^2e^t - e^t.2t}{t^4}$$

$$= \frac{t-2}{t^3} e^t \text{ Cos}\left(\frac{e^t}{t^2}\right) \quad \text{................ (ii)}$$

from (1) and (2), it is clear that both are same.

Formula for the Second differential coefficient of implicit function :

If f (x,y) = 0, Show that

$$\frac{d^2y}{dx^2} = -\frac{q^2r - 2\,pqs + p^2t}{q^3}$$

Sol. We have, $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y} = \dfrac{-p}{q}$

$$\therefore \frac{d^2y}{dx^2} = \frac{-d}{dx}\left(\frac{dy}{dx}\right) = \frac{-d}{dx}\left(\frac{p}{q}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-\left[q(dp/dx) - p(dq/dx)\right]}{q^2} \quad \text{........... (i)}$$

Using the notations:

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial p}{\partial x},\ s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial q}{\partial x},\ t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial q}{\partial y},$$

We have, $\dfrac{dp}{dx} = \dfrac{\partial p}{\partial x} + \dfrac{\partial p}{\partial y} . \dfrac{dy}{dx} = r + s(-p/q)$

$$= \frac{qr - sp}{q}$$

and $$\frac{dq}{dx} = \frac{\partial q}{dx} + \frac{\partial q}{\partial y} . \frac{dy}{dx} = S + t\left(-p/q\right)$$

$$= \frac{qs - pt}{q}$$

Substituting the values of dp/dx and dq/dx in (ii), we get

$$\frac{d^2y}{dx^2} = \frac{-1}{q^2}\left[q\left(\frac{qr - ps}{q}\right) - p\left(\frac{qs - pt}{q}\right)\right]$$

$$= -\frac{q^2 r - zpqs \times p^2 t}{q^3}$$

1.5.6 Standard Limits:

(i) $\underset{x \to a}{\text{Lt}} \frac{x^n - a^n}{x - a} = na^{n-1},$ n any rational number.

(ii) $\underset{x \to 0}{\text{Lt}} \frac{\text{Sin } x}{x} = 1$

(iii) $\underset{x \to 0}{\text{Lt}} \left(1 + x\right)^{\frac{1}{x}} = e$

(iv) $\underset{x \to 0}{\text{Lt}} x^{\frac{1}{x}} = 1$

(v) $\underset{x \to 0}{\text{Lt}} \frac{a^x - 1}{x} = \log_a e$

1.5.7 Summary

In this lesson, we have learnt about the derivatives of functions of two or more variables i.e. parital derivatives. We have introduced homogeneous function and discussed Euler's theorem based upon it. Further, we have discussed the concept of total derivative and derivative of implicit functions.

1.5.8 Key Concepts

Limits, Partial Derivatives, Homogeneous functions, Euler's theorem, Total derivatives, Differentiation of implict function.

1.5.9 Long Questions

1. If $z = \frac{xy}{x+y}$, prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$

2. State and prove Euler's theorem.

1.5.10 Short Questions

1. If $z=e^{x2+y2}$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
2. If $u = \log (x^2+y^2)$, find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.
3. Define Homogeneous functions.
4. Discuss the technique of finding derivative of implicit functions.

1.5.11 Suggested Readings

1. D. Bose.	:	An Introduction of Mathematical Economics
2. Mehta and Madnani	:	Mathematics for Economists
3. Aggarwal and Joshi	:	Mathematics for students of Economics
4. Bhardwaj and Sabharwal	:	Mathematics for students of Economics

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LESSON NO. 1.6 **AUTHOR : Mrs. AMANPREET KAUR**

MATRICES

Structure

1.6.1 Objectives

In this lesson, our prime objectives are :

(i) to study matrices and its types

(ii) to understand the operations on matrices

1.6.2 Introduction

In this lesson, we will study the meaning, type and different operations of matrices. A matrix is a mathematical technique which is quite useful in solving simultaneous equations and for studying linear programming.

Suppose we have two equations :

$3x_1 + 4x_2 - 5x_3 = 3$ (i) or $3x_1 + 5x_2 = 5$ ....(i)

and $6x_1 + 7x_2 + 2x_3 = 0$ (ii) $7x_1 + 4x_2 = 7$ ....(ii)

The coefficient matrix of the above two equations can be written as

$$A = \begin{bmatrix} 3 & 4 & -5 \\ 6 & 7 & 2 \end{bmatrix} B = \begin{bmatrix} 3 & 5 \\ 7 & 4 \end{bmatrix}$$

The above rectangular array is called a matrix. The matrix A contains two rows (horizontal lines) and three columns (vertical lines). It is a matrix of order 2×3.

1.6.3 Meaning/Definition

A Matrix is an arrangement of mn numbers in a rectangular array consisting of m rows and n columns. It is called matrix of order m × n. A matrix A of order m×n is written as

$$A = \begin{bmatrix} a_{11} \; a_{12} \;............a_{1j}...........a_{1n} \\ a_{21} \; a_{22} \;............a_{2j}...........a_{2n} \\ .... \\ a_{i1} \; a_{i2} \;............a_{ij}...........a_{in} \\ .... \\ a_{m1} \; a_{m2} \;............a_{mj}...........a_{mn} \end{bmatrix}_{m \times n} \quad ....(i)$$

The a's are called elements of the matrix A.

Matrices are denoted by capital letters, A, B, C .... etc.

It is enclosed by any one of [ ], ( ), | |, the brackets.

In (i) we have used double subscripts to represent elements of the matrix A. The element have been written as a_{ij} where i = 1, 2, ...m and j = 1, 2, ...n.

The first subscript attached to the element identifies the row and the second subscript identifies the column.

For example, element a_{ij} means element of i^{th} row and j^{th} column.

We can also writes $A = (a_{ij})_{m \times n}$

$$A_1 = \begin{bmatrix} 5 & 6 & 1 \\ 7 & 3 & 1 \\ 3 & 9 & 4 \end{bmatrix}_{3 \times 3}$$ is a matrix of the order of 3 × 3

$A_{23} = 1$, $A_{32} = 9$

1.6.4 Types of Matrices

1. Row and Column Matrices

A matrix of order 1xn is called a Row matrix, i.e. a row matrix has only one row.

The matrix of order mx1 is called a column matrix i.e. a column matrix has only one column.

For example A = $[a_{11}\ a_{12}\ a_{13}\ \ldots\ldots\ldots\ldots a_{in}]$ is a row matrix.

$$B = \begin{bmatrix} b_{11} \\ b_{21} \\ 1 \\ 1 \\ 1 \\ b_{m1} \end{bmatrix}_{m\times 1} \text{ is a column matrix. } A = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

Row matrix B = 5 6 7

2. Square and Rectangular Matrices

A Matrix having the same number of rows and columns i.e. if m = n is called a square matrix.

For example $\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}_{2\times 2}$ $\begin{bmatrix} 1 & 4 & 3 \\ 2 & 0 & 3 \\ 4 & 3 & 1 \end{bmatrix}_{3\times 3}$ are square matrices. resembelse

A matrix which is not a square matrix is called a rectangular matrix

$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2\times 3}$, $\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 1 \end{bmatrix}_{3\times 2}$ are examples of rectangular matrices.

3. Diagonal Matrix

Before defining diagonal matrix we will first define diagonal elements and principal Diagonal of a matrix.

The element a_{ij} of the matrix A = $(a_{ij})_{mxn}$ for i = j are called the diagonal elements of the matrix A and the line along which they lie is called the Principal diagonal of the matrix A.

Thus if A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$ is a matrix of order 4.

Then the elements a_{11}, a_{22}, a_{33}, a_{44} are diagonal elements and the line along which they lie is called Principal diagonal.
Diagonal matrix is a square matrix in which all the elements excepting the diagonal elements are zero. For example : A = diag [d_1, d_2, d_3] is a diagonal matrix of order 3.

4. Scalar Matrix

A diagonal matrix whose all diagonal elements are equal, is called a scalar matrix.

For example, A = $\begin{bmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix}$ is a scalar matrix of order 3.

5. Identity Matrix or Unit matrix

Identity matrix or a unit matrix is special case of scalar matrix where the scalar quantity is equal to unity.

For example I_3 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3 and is denoted by 1_3.

$I_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

6. Zero Matrix or Null Matrix

If all elements in a matrix are equal to zero, the matrix is called a null matrix or zero matrix. A null matrix is generally denoted by O.

O = $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is zero matrix of order 2 x 3.

7. Upper Triangular and Lower Triangular Matrices.

A square matrix A = [aij] is called upper triangular matrix, if aij = 0 for i > j and lower triangular matrix if a_{ij} = 0, for i < j.

For example $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$ and $\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$

8. Sub-Matrix

A matrix obtained by deleting some rows or some columns both of a given matrix. A is called a sub matrix of A.

Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$

then $\begin{bmatrix} 1 & 3 \\ 9 & 11 \end{bmatrix}$

obtained by deleting the second row second and fourth column of A is a submatrix of A.

1.6.5 Matrix Operations

1.6.5.1 Addition and Subtraction of Matrices

If two matrices A = $[a_{ij}]$ and B = $[b_{ij}]$ are of the same order mxn, then A + B or A-B is defined as a new matrix C = $[c_{ij}]$ or $[c'_{ij}]$ of the same order (i.e. mxn)

in which cij = $a_{ij} + b_{ij}$ (or $c_{ij} = a_{ij} - b_{ij}$)

Thus if $A = \begin{bmatrix} a_{11} & a_{12} & ..... & a_{1n} \\ a_{21} & a_{22} & ..... & a_{2n} \\ ..... & ..... & ..... & ..... \\ a_{m1} & a_{m2} & ..... & a_{mn} \end{bmatrix}_{mxn}$ and $B = \begin{bmatrix} b_{11} & b_{12} & ..... & b_{1n} \\ b_{21} & b_{22} & ..... & b_{2n} \\ ..... & ..... & ..... & ..... \\ b_{m1} & b_{m2} & ..... & b_{mn} \end{bmatrix}_{mxn}$

Then $C = A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & ..... & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & ..... & a_{2n}+b_{2n} \\ ..... & ..... & ..... & ..... \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & ..... & a_{mn}+b_{mn} \end{bmatrix}_{mxn}$

$$C = A - B = \begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} & \ldots\ldots & a_{1n}-b_{1n} \\ a_{21}-b_{21} & a_{22}-b_{22} & \ldots\ldots & a_{2n}-b_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{m1}-b_{m1} & a_{m2}-b_{m2} & \ldots\ldots & a_{mn}-b_{mn} \end{bmatrix}_{mxn}$$

Example 1 : If $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & 5 \\ 4 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 5 & 4 \\ 2 & 3 & 7 \end{bmatrix}$

Find A + B, A - B.

$$A + B = \begin{bmatrix} 1+2 & 3+0 & 2+1 \\ 2+3 & 0+5 & 5+4 \\ 4+2 & 1+3 & 2+7 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 5 & 5 & 9 \\ 6 & 4 & 9 \end{bmatrix}$$

Now

$$A - B = \begin{bmatrix} 1-2 & 3-0 & 2-1 \\ 2-3 & 0-5 & 5-4 \\ 4-2 & 1-3 & 2-7 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 1 \\ -1 & -5 & 1 \\ 2 & -2 & -5 \end{bmatrix}$$

1.6.5.2 Properties of Addition of Matrices

(i) Addition of two common matrices is possible only if they are of same order.

(ii) The addition of matrices is commutative. It means that A + B = B + A.

(iii) The addition of matrices is associative. It means that

$$A + (B + C) = (A + B) + C = A + B + C$$

1.6.5.3 Scalar Multiple of a Matrix

The Scalar multiple of the matrix A = $(a_{ij})_{mxn}$ by a scalar k is the matix C = $[cij]_{mxn}$ where $c_{ij} = ka_{ij}$ for all values of i and j.

For example if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then $kA = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}$

1.6.5.4 Properties of Scalar Multiplication

(i) If A and B are matrices of same order and 'λ (read as lembda) is a scalar quantity, then λ (A + B) = λA + λB

(ii) If A is any matrix and λ_1 and λ_2 are any two vectors, then

(a) $(\lambda_1 + \lambda_2) A = \lambda_1 A + \lambda_2 A$

(b) $\lambda_1 (\lambda_2 A) = \lambda_1 \lambda_2 (A)$

(c) If A is any matrix and λ is a scalar than $\lambda A = A\lambda$

$$\text{If } A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 9 \\ 7 & 6 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A\text{-}2B = \begin{bmatrix} 1 & (3-2) & 2 \\ (2-2) & 6 & 9 \\ 7 & 6 & (2-2) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 6 & 9 \\ 7 & 6 & 0 \end{bmatrix}$$

1.6.5.5 Multiplication of Matrices

If A = $[a_{ij}]$ and B = $[b_{jk}]$ be two matrices of the order mxn and nxp respectively where i varies from 1 to m, j varies from 1 to n and k from 1 to p, then the product C = AB is defined as a matrix C of the order mxp, whose $(c_{ij})^{th}$ element is given by

$$C_{ij} = a_{i1} b_{1k} + a_{i2} b_{2k} + a_{i3} b_{3k} + \dots\dots\dots + a_{in} b_{nk}$$

$$= \sum_{j=1}^{n} a_{ij} b_{jk}$$

Two matrices A and B are conformable for the product AB when the number of columns of A is equal to the number of rows in B.

In product AB, A is called pre-factor and B is called post factor.

For Example :

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2x3}, B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}_{3x3}$$

The product AB is possible if AB = C_{2x3}

$$\text{Let } C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix} \dots\dots\dots\dots\dots\dots\dots (I)$$

C_{11} = Sum of products of corresponding elements of first row of prefactor

A and first column of post factor B which are

$$[a_{11}\ a_{12}\ a_{13}]\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$

C_{12} = Sum of products of corresponding elements of first row of A and second column of B which are

$$[a_{11}\ a_{12}\ a_{13}]\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}$$

Similarly $C_{13} = a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33}$

$C_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}$

$C_{22} = a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}$

$C_{23} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33}$

Substituting C_{11}, C_{12} etc. in (1), we have

C = AB

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \end{bmatrix}$$

1.6.6 Equality of Matrices

Two matrices are said to be equal only if

(i) the matrices are of the same order.

(ii) each element belonging to one is equal to the corresponding element belonging to the other.

Thus, the matrices $A = (a_{ij})_{mxn}$ and $B = (b_{ij})$ are equal if $a_{ij} = b_{ij}$ f all values of i and j.

For example, the matrices

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 & (2)^2 \\ (2)^2 & 5 & \left(\sqrt{6}\right)^2 \end{bmatrix}$$

are equal as A and B are of the same order and $a_{ij} = b_{ij}$ for all values of i and j.

Example 2 : $A = \begin{pmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & +1 \end{pmatrix}$

Compute AB and BA, Show that AB ≠ BA

Sol. $AB = \begin{bmatrix} 1 \times 2 + (-2) \times 4 + 3 \times 2 & 1 \times 3 + (-2) \times 5 + 3 \times 1 \\ -4 \times 2 + 2 \times 4 + 5 \times 2 & -4 \times 3 + 2 \times 5 + 5 \times 1 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & -10 \\ 10 & 3 \end{bmatrix}$

BA w $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & +1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$

BA

$AB = \begin{bmatrix} 2 \times 1 + 3(-4) & 2(-2) + 3(2) & 2 \times 3 + 3 \times 5 \\ 4 \times 1 + 5 \times (-4) & 4(-2) + 5(2) & 4 \times 3 + 5 \times 5 \\ 2 \times 1 + 1(-4) & 2(-2) + 1(2) & 2 \times 3 + 1 \times 5 \end{bmatrix}$

$= \begin{bmatrix} 2 - 12 & -4 + 6 & 6 + 15 \\ 4 - 20 & -8 + 10 & 12 + 25 \\ 2 - 4 & -4 + 2 & 6 + 5 \end{bmatrix} = \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$

Hence AB ≠ BA

1.6.7 Trace of a Matrix

The sum of all the diagonal elements of a square matrix A is called the trace of A and is written as tr (A)

If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{bmatrix}$

then tr. (A) = $\sum_{i=1}^{n} a_{ij}$

For Example : If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$,

tr: (A) = 1 + 5 + 9 = 15

For Example : If A = $\begin{vmatrix} 5 & 7 \\ 8 & 2 \end{vmatrix}$

trace (A) = 5 + 2 = 7

1.6.8 Summary

This lesson provided a detailed study about matrices, its types, properties and operations on matrices. Further, we have discussed the equality of matrices and trace of a matrix.

1.6.9 Key Concepts

Matrix, Diagonal matrix, Square matrix, Identity matrix, Addition of matrices, Subtraction of matrices, Scalar multiplication, Multiplication of matrices, Equality of matrices, Trace of a matrix.

1.6.10 Long Questions

1. Show that the product of two matrices

$$\begin{bmatrix} 7 & -11 & 16 \\ -3 & 5 & -7 \\ 1 & -2 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & -3 \\ 2 & 5 & 1 \\ 1 & 3 & 2 \end{bmatrix} \text{ is } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Verify that (i) A (BC) = (AB) C and A (B + C) = AB + AC

Where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

(b) (3 + 2) A = 3A + 2A

(c) (3A) B = 3 (AB)

3. If $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Prove that AB $\neq$ BA

4. Evaluate x.y.z and t if $\begin{bmatrix} x-2y & 3z-2t \\ x+2y & z+t \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 8 & 9 \end{bmatrix}$

5. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 3 \\ 2 & 0 & 1 \end{bmatrix}$, find A^2-A-I.

1.6.11 Short Questions

1. If $A = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 0 & -1 \end{bmatrix}$

Find A + B, A - B.

2. Whether the following summation is possible, if not, why ?

$$\begin{bmatrix} 1 & 2 & 3 \\ 8 & 2 & 4 \\ 5 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 7 & 8 \\ 3 & 2 & 0 \end{bmatrix}$$

3. Find out X, if

$$X + \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$$

4. $A = \begin{bmatrix} 4 & 7 & 8 \\ 3 & 2 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 6 \\ 4 & 1 & 5 \end{bmatrix}$

Show that (a) 4 (A + B) = 4A + 4B

5. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 1 & 2 \end{bmatrix}$

Can you find AB ? Justify your answer.

6. If $A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ 3 & 5 \end{bmatrix}$,Find a and b such that AB = BA.

Compute. 3A+5B.

1.6.12 Suggested Readings

1. D. Bose.	:	An Introduction of Mathematical Economics
2. Mehta and Madnani	:	Mathematics for Economists
3. Aggarwal and Joshi	:	Mathematics for students of Economics
4. Bhardwaj and Sabharwal	:	Mathematics for students of Economics

LESSON NO. 1.7

ADJOINT AND INVERSE OF A MATRIX

1.7.1 Objectives

In this lesson, we focus on the evaluation of determinant of a matrix and inverse of a matrix. We will also discussed about the properties of determinants.

1.7.2 Introduction

For finding inverse of a matrix we need to define two concepts namely determinant and adjoint of a matrix.

A determinant is a number associated with a Square Matrix.

If A = (a_{11}), be a one by one matrix then determinant of A = a_{11}.

The determinant of A is written as $|A|$ or det. A and is read as determinant of A.

In other words, If A = (a_{ij})

$i = 1, 2 \ldots\ldots\ldots n$

$j = 1, 2 \ldots\ldots\ldots n$

denotes square matrix of order n, then determinant of A or det A = $|a_{ij}|$

Important Notes

A determinant is reducible to a number or whole number.

A determinant is defined only for a square matrix where as a matrix as such may or may not be square.

1.7.3 Evaluation of Determinant

1. Order one

If A = (a_{11}) be a one by one matrix then det A = $|a_{11}|$. i.e. A = [2]

2. Order Two

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a matrix of order 2 × 2 then determinant of A is defined as follows : i.e.

$$A = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$$

i.e. $|A| = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} a_{11} a_{22} - a_{21} a_{12}$

|A| is obtained by multiplying the two elements in the principal diagonal of A and then subtracting the product of the two remaining elements.

For example :

If $A = \begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix}$ then |A| = (2 × 4 - 1 × 2) = 6

3. Order Three

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

then determinant of A has the value

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11} (a_{22} a_{33} - a_{23} a_{32}) - a_{12} (a_{21} a_{33} - a_{23} a_{31}) + a_{13} (a_{21} a_{32} - a_{22} a_{31})$$

$$= a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}$$

The sum of all the six products will be the value of determinant.

For example :

$$\text{If } A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & 1 & 5 \\ 6 & 4 & 3 \end{bmatrix}$$

We can find the det. A by using Laplace's Expansion Method.

For this method we write

$$A \quad = \quad \begin{matrix} 3 & 4 & 5 & 3 & 4 \\ 2 & 1 & 5 & 2 & 1 \\ 6 & 4 & 3 & 6 & 4 \end{matrix}$$

$|A| =$ (3) (1) (3) + 4 (5) (6) + (5) (2) (4)

-(6) (1) (5) – (4) (5) (3) – (3) (2) (4)

= 169 – 114 = 55

Thus for higher order determinants we use Laplace's Expansion Method and it is based on cofactors. Thus we find the determinant has a definite value.

The above example can be solved by adopting the order method.

$|A| = 3\,(1 \times 3 - 5 \times 4) - 4\,(2\times3 - 5\times6) + 5\,(2\times4 - 1\times6)$

$= 3\times(-17) - 4\,(-24) + 5(2)$

$= -51 + 96 + 10$

$= 55$

Thus we find that the same result is obtained.

1.7.4 Minors

The determinants formed by taking equal number of rows and columns out of the elements of a matrix are called minors of a matrix. The order or minor is the order of the determinant. Minor is itself a determinant and has a value. In general M_{ij} can be used to represent the minor obtained by deleting the i^{th} row and j^{th} column.

Minor of first order of matrix A is formed by each element of the matrix individually taken. Minor of Second order of matrix A is formed by taking 2 elements of two rows and two columns.

Minor of third order of matrix A is formed by taking of 3 elements of 3 rows and 3 columns.

Consider a matrix of order 3 × 3

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

When we delete any one row and column which contains the elements of A, we get a 2 × 2 sub matrix of A. The determinant of sub-matrix is called a minor of set A.

Thus

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

is minor of a_{33} of det. A.

Similarly

Minors of a_{22}, a_{13}, a_{31} of det. A are

$$\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \text{ and } \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \text{respectively} ;$$

For Example :

In the determinant

$$\begin{vmatrix} 8 & 5 & 4 \\ 3 & 9 & 2 \\ 7 & 1 & 6 \end{vmatrix}$$

the minor of element 5 is M_{12}, which is (for find in) a minor leave +n at row or coloumn fall which we are taking minor)

$$M_{12} = \begin{vmatrix} 3 & 2 \\ 7 & 6 \end{vmatrix} = 18 - 14 = 4$$

Similarly, minor of element 6 is M_{33} which is

$$M_{33} = \begin{vmatrix} 8 & 5 \\ 3 & 9 \end{vmatrix} = 72 - 15 = 57$$

1.7.5 Cofactors

Another concept closely connected to minor is known as cofactor. A cofactor is a minor with a prescribed sign attached with it.

The Cofactor of C_{ij} of a_{ij} is $(-1)^{i+j}$ times the determinants of the submatrix obtained by deleting row i and column j from matrix A (called the minor of a_{ij}) or $|C_{ij}| = (-1)^{i+j} |M_{ij}|$.

If the sum of the two subscripts i and j in the minor $|M_{ij}|$ is even then cofactor is of the same sign as the minor, that is $|C_{ij}| = |M_{ij}|$

If it is odd, then the cofactor takes the opposite sign to the minor, that is $|C_{ij} = (-) M_{ij}|$

The $(-1)^{i+j}$ follows the chessboard rule i.e.

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

for Example :

for a_{53} $(-1)^{5+3}$

$= (-1)^8$

$= +1$

And for a_{12} $(-1)^{1+2} = (-1)^3 = -1$

Thus it is obvious that the expression $(-1)^{i+j}$ can be positive only if 'i + j' is even otherwise it will be of negative sign.

It should be noted that it is possible to expand determinants by a cofactor of any row or for that matter, of any column. For instance, if the first column of a third order det. A consist of elements a_{11}, a_{21} and a_{31}, expansion by cofactors of these elements will also yield the value of $|A|$.

$$|A| = a_{11} (C_{11}) + a_{21} (C_{21}) + a_{31} (C_{31})$$

$$= 4 (-16) - 5 (-2) + 7 (45)$$

$$= -64 + 10 + 315 = 261$$

$$\text{Let } A = \begin{bmatrix} 4 & 5 & 7 \\ 6 & 3 & 2 \\ 1 & 8 & 0 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 4 & 5 & 7 \\ 6 & 3 & 2 \\ 1 & 8 & 0 \end{vmatrix}$$

If we expand it with the help of Ist column.

$|C_{11}|$ = Cofactor of 4 = $(-1)^{1+1} \begin{vmatrix} 3 & 2 \\ 8 & 0 \end{vmatrix}$ = (0- 16) = -16

$|C_{21}|$ = Cofactor of 6 = $(-1)^{2+1} \begin{vmatrix} 5 & 7 \\ 8 & 0 \end{vmatrix}$ = (0 - 56) = 56

$|C_{31}|$ = Cofactor of 1 = $(-1)^{3+1} \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix}$ = (10 - 21) = -11

$|A|$ = 4 × (-16) + 6 × 56 + 1 × -11

$|A|$ = - 64 + 336 - 11 = 261

Similarly, if we expand it with the help of first row.

$|A|$ = - 64 + 10 + 315 = 261

1.7.6 Properties of Determinants

Determinant posses certain basic properties which are common to determinants of all order. These properties are ;

1. The interchange of rows and columns does not affect the value of the determinant.

   It means determinant of a matrix A has same value as that of its transpose Á.

   For Example :

$$\begin{vmatrix} p & q \\ r & s \end{vmatrix} = (ps - qr) = \begin{vmatrix} p & r \\ q & s \end{vmatrix}$$

or $$\begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 5 & 3 \end{vmatrix} = 2$$

2. The interchange of rows (or two columns) will alter the sign, but not the numerical value of the determinant.

   For Example :

$$\begin{vmatrix} p & r \\ q & s \end{vmatrix} = (ps - qr) = \text{with the interchange or rows, we get}$$

$$\begin{vmatrix} q & s \\ p & r \end{vmatrix} = (qr - ps) = - (ps - qr)$$

or $$\begin{vmatrix} 5 & 2 \\ 3 & 4 \end{vmatrix}$$

$|A|$ = (20-6) = 14, with the interchange of two columns, we get

$$\begin{vmatrix} 2 & 5 \\ 4 & 3 \end{vmatrix} = (6 - 20) = - 14$$

3. A matrix with a row (or column) or zeros has a zero determinant.
For Example

$$\begin{vmatrix} p & q \\ 0 & 0 \end{vmatrix} = (p0 - q0) = 0$$

or $$\begin{vmatrix} 4 & 5 & 0 \\ 3 & 2 & 0 \\ 6 & 2 & 0 \end{vmatrix} = 4\ 2(0)-2(0)-5[(3)(0)-(6)(0)]+0\{3(2)-6(2)]$$

(Student should expand themselves to verify the result).

4. If any single row (or column) of a matrix is multiplied by a scalar 'k' then the determinant is also multiplied by 'k'.

For Example :

$$\begin{vmatrix} kp & kq \\ r & s \end{vmatrix} = (kps - krq) = k\ (ps-qr) = k \begin{vmatrix} p & q \\ r & s \end{vmatrix}$$

or $= \begin{vmatrix} 4 & 3 \\ 2 & 3 \end{vmatrix}$ 12 - 6 = 6

Suppose k = 2 and multiplying Ist column by 2.

$$\begin{vmatrix} 4 \times 2 & 3 \\ 2 \times 2 & 3 \end{vmatrix} = (8 \times 3 - 4 \times 3) = 12 = 2\ (6)$$

5. If one row (or column is multiple of another row) the value of the determinant will remain same i.e. zero.

For Example :

$$\begin{vmatrix} 2p & q \\ 2p & q \end{vmatrix} = 2pq - 2pq = 0$$

or $\begin{vmatrix} 4 & 20 \\ 5 & 25 \end{vmatrix} = 0$

(Student should expand themselves to verify the result).

6. If two columns (or rows) are identical then determinant will be zero.

For Example :

$$\begin{vmatrix} p & q \\ p & q \end{vmatrix} = (pq - pq) = 0$$

or $\begin{vmatrix} 13 & 11 \\ 13 & 11 \end{vmatrix} = 0,$ or $\begin{vmatrix} 12 & 12 \\ 4 & 4 \end{vmatrix} = 0$

7. The addition (or subtraction of a multiple of any row to (from) another row will leave the value of the determinant unaltered.

Similar is true for column.

$$\begin{vmatrix} p & q \\ r & s \end{vmatrix} = (ps - qr)$$

and $\begin{vmatrix} p & q \\ r + p & s + q \end{vmatrix} = p\ (s + q) - (r + p)\ q$

$= (ps - qr)$

Similarly

$$\begin{vmatrix} 4 & 5 \\ 4+3 & 3+2 \end{vmatrix} = 4\ (3 + 2) - 5\ (4 + 3) = -15$$

Student should verify the result by subtraction.

Thus these basic properties or determinants help in simplifying and evaluating them. Subtraction and interchange or rows (or column) further help us in reducing our work.

1.7.7 Adjoint of a Matrix

Let A = $\{a_{ij}\}_{nxn}$ = $\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nj} & \ldots & a_{nn} \end{bmatrix}_{n \times n}$

be a square matrix or order n, so that A is the determinant of the nth order.

i.e. $|A| = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$

Let the co-factor of a_{11}, a_{12} etc. of the matrix be denoted by the corresponding capital letters A_{11}, A_{12} etc.

Thus A_{ij} denotes the cofactor of a_{ij} in $|A|$.

Let us form the matrix of cofactors of the corresponding small letters a_{ij} in $|A|$ and denote it by C of (A) and C(A)

Thus C(A) = $\begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1n} \\ A_{21} & A_{22} & \ldots & A_{2n} \\ A_{n1} & A_{n2} & \ldots & A_{nm} \end{bmatrix}_{n \times n_1}$

C(A) is called a co-factor matrix of A.

Let us take the transpose of the matrix C(A) so that

$$C(A) = \begin{bmatrix} A_{11} & A_{12} & \dots & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & \dots & A_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & \dots & A_{nn} \end{bmatrix}_{n \times n}$$

$$C'(A) = \begin{bmatrix} B_{11} & B_{12} & \dots & \dots & B_{1n} \\ B_{21} & B_{22} & \dots & \dots & B_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ B_{n1} & B_{n2} & \dots & \dots & B_{nn} \end{bmatrix} \text{(Say)}$$

So that C' (A) = $[A_{ij}]_{n \times n} = [B_{ij}]$, where B_{ij} (or A_{ij}) is cofactor of a_{ij} in $|A|$.

C' (A) is called the adjoint (or adjugate) of the matrix A and is generally written as Adj. A.

Let A = $[a_{ij}]_{n \times n}$ be a square matrix of order n then the adjoint of A is the transpose of the cofactor matrix (matrix obtained by replacing a_{ij} of A by their corresponding cofactors).

We shall illustrate the concept with examples.

Example 1: Let A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2 \times 2}$ Find adj A

Solution :

Here $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Cofactors of a = + d

Cofactors of b = - c

Cofactors of c = - b

Cofactors of d = + a

$\therefore$ C (A) = Cofactors matrix = $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

$\therefore$ adj A = C' (A) = $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example 2 : Find adjoint of A = $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Solution :

Here A = $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Cofactor of the elements of the first row of A are

$$+\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}, +\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$

i.e. -1 +8 -5

Cofactor of the elements of the second row of A are

$$-\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}, +\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix}, -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix}$$

i.e. 1 -6 + 3

Cofactor of the elements of the third row of (A) are

$$+\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}, -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix}, +\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}$$

i.e. -1 +2 -1

Now C (A) = Cofactor matrix

$$C\ (A) = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

$\therefore$ adj A = C' (A) = transpose of the Cofactor matrix

$$= \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

1.7.8 The Inverse of Matrix

Let A be any square matrix; B if it exists such that AB = BA = I is called the inverse of A, I being the unit matrix and we write B = A^{-1} to be read as inverse of A.

Remarks :

Inverse of a matrix A exist only if the matrix is non singular i.e. $|A| \neq 0$.

In other words

(i) Every Matrix needn't have an inverse

(ii) every square matrix needn't have an inverse.

(iii) Every square non-singular matrix has an inverse.

How to find Inverse of Matrix A ?

We shall explain the method to find inverse of a matrix with the help of adjoint.

Let A have an inverse B so that

A is, by definition, non-singular i.e. $|A| = 0$

Now B will be the inverse of A only if it satisfies.

AB = BA = 1

Let us choose $B = \frac{\text{adj } A}{|A|}$

Since $|A| \neq 0$ our choosing B above is justified.

Now $AB = \frac{A \text{ adj } A}{|A|}$ $(\therefore A \text{ adj } A = |A|)$

$= \frac{|A|}{|A|} = 1$ (prove it)

Similarly BA = I

Hence AB = BA = I (I = identity matrix)

which shows that B is the inverse of A

or $|A|$ adj A is the inverse of A

or $A^{-1} = |A|$. adj A

Thus the necessary and sufficient condition for matrix A to possess an inverse is that it is non-singular.

i.e. $|A| = 0$

Remarks :

For finding the inverse of a matrix A, we shall find the determinant of A if $|A| = 0$, inverse doesn't exist. $|A| \neq 0$, we shall find the adjoint matrix and divide it by $|A|$ to get the inverse matrix.

Example 3 : Find the inverse of A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Solution

We have already calculated adj A in example 1 above

i.e. adj A = $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Also |A| = ad - bc

Assuming A to be non-zero i.e. ad ≠ bc,

A^{-1} = adj. A/|A| = $\dfrac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad - bc}$

verification : AA^{-1} should be equal to I

Here $A.A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \Big/ (ad - bc)$

$$= 1/(ad - bc) \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= 1/(ad - bc) \begin{bmatrix} ad - bc & -ab + ab \\ cd - cd & -bc + ad \end{bmatrix}$$

$$= 1/(ad - bc) \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = \frac{1}{(ab - bc)} (ab - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

Hence $A^{-1} = 1/(ad - bc). \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example 4 : Find the inverse of A = $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Solution :

We have already calculated adj A in example 2

i.e. adj A = $\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

$\therefore$ Also |A| = 0C (0) - 1C (1) + 2C (2) (from the first row)

= 0 (-1) - 1 (-8) + 2 (-5)

= 0 + 8 -10)

= -2 ($\neq$ 0)

$\therefore A^{-1} = \text{adj } A/|A| = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix} \Big/ -2$

or = $\begin{vmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{vmatrix}$

Verification :

A A^{-1} should be I

Here A. A^{-1} = $\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} \begin{vmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{vmatrix}$

= $\begin{vmatrix} 0-4+5 & 0+3-3 & 0-1+1 \\ 1/2-8+15/2 & -1/2+6-9/2 & 1/2-2+3/2 \\ 3/2-4+5/2 & -3/2+3-3/2 & 3/2-1+1/2 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\text{Hence } A^{-1} = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{vmatrix}$$

Example 5 : Find the adjoint of the following matrix. Hence or otherwise find the inverse.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$$

Solution :

$$\text{Here } A = \begin{vmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{vmatrix}$$

|A| = 1 C (1) + 0C (0) + (-1) C (-1)
= 1 (-28 + 30) + 0 (-21 +0) + (-1) (-18 + 0)
= 1(2) + 0(21) + (-1) (-18)
= 2 + 0 + 18
= 20

We shall first find the cofactor of the elements of A

Cofactors of the elements of first row of A

$$+\begin{vmatrix} 4 & 5 \\ -6 & -7 \end{vmatrix}, -\begin{vmatrix} 3 & 5 \\ 0 & -7 \end{vmatrix}, +\begin{vmatrix} 3 & 4 \\ 0 & -6 \end{vmatrix}$$ i.e (-28+30), -(-21-0), + (-18-0)

i.e. 2, + 21, - 18

Cofactor of the elements of the second row of A

$$-\begin{vmatrix} 0 & -1 \\ -6 & -7 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 0 & -7 \end{vmatrix}, -\begin{vmatrix} 1 & 0 \\ 0 & -6 \end{vmatrix}$$

i.e. 6, - 7, + 6

Cofactor of the elements of the third row of A

$$+\begin{vmatrix}0 & -1\\4 & 5\end{vmatrix}, -\begin{vmatrix}1 & -1\\3 & 5\end{vmatrix} + \begin{vmatrix}1 & 0\\3 & 4\end{vmatrix}$$

i.e. 4, -8, 4

Hence C(A) = Cofactor matrix of A

$$= \begin{vmatrix}2 & 21 & -18\\6 & -7 & 6\\4 & -8 & 4\end{vmatrix}$$

and adj A = C' (A) = Transpose of the Cofactor matrix

$$= \begin{vmatrix}2 & 6 & 4\\21 & -7 & -8\\-18 & 6 & 4\end{vmatrix}$$

Since $|A| = 20\ (\neq 0)$

$$\therefore A^{-1} = 1/|A|.\ \text{adj}A = \tfrac{1}{20}\begin{vmatrix}2 & 6 & 4\\21 & -7 & -8\\-18 & 6 & 4\end{vmatrix}$$

$$= \begin{vmatrix}\tfrac{2}{20} & \tfrac{6}{20} & \tfrac{4}{20}\\ \tfrac{21}{20} & -\tfrac{7}{20} & -\tfrac{8}{20}\\ -\tfrac{18}{20} & \tfrac{6}{20} & \tfrac{4}{20}\end{vmatrix}$$

Verification :

AA^{-1} should be I

$$\text{Here } AA^{-1} = \begin{bmatrix}1 & 0 & -1\\3 & 4 & 5\\0 & -6 & -7\end{bmatrix}\begin{bmatrix}\tfrac{1}{10} & \tfrac{3}{10} & \tfrac{2}{10}\\ \tfrac{21}{20} & -\tfrac{7}{20} & -\tfrac{4}{10}\\ -\tfrac{9}{10} & \tfrac{3}{10} & \tfrac{1}{5}\end{bmatrix}$$

$$= \tfrac{1}{20}\begin{vmatrix}1 & 0 & -1\\3 & 4 & 5\\0 & -6 & -7\end{vmatrix}\begin{vmatrix}2 & 6 & 4\\21 & -7 & -8\\-18 & 6 & 4\end{vmatrix}$$

$$= \frac{1}{20}\begin{vmatrix} 2+0+18 & 6-0-6 & 4-0-4 \\ 6+84-90 & 18-28+30 & 12-32+20 \\ 0-126+126 & 0+42-42 & 0+48-28 \end{vmatrix}$$

$$= \frac{1}{20}\begin{vmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

Hence $A^{-1} = \frac{1}{20}\begin{vmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{vmatrix} = \begin{vmatrix} 2/20 & 6/20 & 4/20 \\ 21/20 & -7/20 & -8/20 \\ -18/20 & 6/20 & 4/20 \end{vmatrix}$

$$= \begin{vmatrix} 1/10 & 3/10 & -1/5 \\ 21/20 & -7/20 & -2/5 \\ -9/10 & 3/10 & 1/5 \end{vmatrix}$$

Example 6 : Find the inverse of A **=** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

Solution : Here $|A|$ **=** $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$

= 1 C(1) + 2 C(2) + 3C (3)
= 1[(15-16)] -2 [+ (10-12)] + 3[8-9]
= (-1) + 2(2) -3
= -1 + 4 -3
= 0

Since $|A| = 0$ therefore A^{-1} does not exist.

Remarks : Here, we needn't to find adj A. It should be noted that we shall always find $|A|$ first.

If $|A| \neq 0$ only then we should proceed further to find adj and hence A^{-1}.

Example 7 : Find the inverse of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution : Here $|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$

$$C(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{and, adj } A = C(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{ X adj } A = \frac{1}{1} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = A$$

In this example, A has its own inverse. So every identity matrix is inverse of itself.

1.7.9 Summary

In this lesson, we have learnt to find out the determinant of a matrix and inverse of a matrix. For this purpose, we introduced many terms such as minors, co-factors and adjoint of a matrix.

1.7.10 Key Concepts

Determinant, Minor, Co-factor, Adjoint, Inverse, Properties of determinants.

1.7.11 Long Questions

1. Show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^3 \end{vmatrix}$ = (a - b) (b - c) (c - a)

2. Show that $\begin{vmatrix} [b+c]^2 & a^2 & a^2 \\ b^2 & [c+a]^2 & b^2 \\ c^2 & c^2 & [a+b]^2 \end{vmatrix}$ = 2abc $[a + b + c]^2$

3. Show that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$ = $[a + b + c]^2$

4. Explain the meaning and properties of Determinants.

1.7.12 Short Questions

1. Define and illustrate Minor.
2. Explain with the help of suitable examples :
   (i) interchange of rows or columns will alter the sign but not numerical value.
   (ii) Cofactor

3. If A = $\begin{Bmatrix} 1 & 2 & 3 \\ 2 & 3 & 6 \\ 4 & 5 & 5 \end{Bmatrix}$ find the cofactor of the elements a_{11}, a_{12}.

1.7.13 Suggested Readings

1. D. Bose.	:	An Introduction of Mathematical Economics
2. Mehta and Madnani	:	Mathematics for Economists
3. Aggarwal and Joshi	:	Mathematics for students of Economics
4. Bhardwaj and Sabharwal	:	Mathematics for students of Economics

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BBA PART- I **PAPER : BBA-103**

BUSINESS MATHEMATICS

LESSON NO. 1.8 **AUTHOR : Mrs. AMANPREET KAUR**

SOLUTION OF SIMULTANEOUS EQUATIONS

1.8.1 Objectives

With the help of this lesson, the students would be able to attain knowledge about the applications of matrices for solving system of linear equations.

1.8.2 Introduction

In the previous lesson we have studied how to find inverse of a matrix. In the present lesson we will study the application of matrices in solving a system of Linear equations. We will consider two cases :

1) Two simultaneous equations in two unknowns.
2) Three simultaneous Linear equations in three unknowns.

1.8.3 Two simultaneous Equations in Two Unknowns :

Let AX = B be the given system of equations.

Premultiplying both sides by A^{-1}

$A^{-1}AX = A^{-1}B$ (AX = B)

$(A^{-1}A)X = A^{-1}B$ ----------- (Since Associative Law)

$IX = A^{-1}B$ ---------------- (Since $A^{-1}A = I$)

$X = A^{-1}B$

Steps to solve the Linear Equations

1) Write the equations in Matrix form.
2) Find the determinant of the Coefficient Matrix. If $|A| \neq 0$, go to step 3, and if $|A| = 0$, we cannot find inverse.
3) Compute Minors (M_{ij})
4) Find Cofactor matrix (A_{ij})
5) Compute Adjoint Matrix i.e., Transpose of the Co-factor Matrix (Adj A).
6) Find Inverse (A^{-1}).
7) Multiply the Inverse Matrix (A^{-1}) by the Constant Vector (B) i.e., $A^{-1}B$.

Example 1 : Solve the following set of Linear Simultaneous Equations :

$2x_1+3x_2 = 5$ --------------- (i)

$11x_1-5x_2 = 6$ -------------- (ii)

Solution :

Step 1 : Write the above equations in Matrix form

$$\begin{bmatrix} 2 & 3 \\ 11 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

A X B

A - Coefficient Matrix

X - Variable Vector

B - Constant Vector

Step 2 : Find the determinant of the Coefficient Matrix.

$$A = \begin{bmatrix} 2 & 3 \\ 11 & -5 \end{bmatrix}$$

$|A| = -10-33 = -43 \neq 0$

Since $A \neq 0$, go to step (3).

Step 3 : Find Minors

$M_{11} = -5,\quad M_{12} = 11,\quad M_{21} = 3,\quad M_{22} = 2$

$$M_{ij} = \begin{bmatrix} -5 & 11 \\ 3 & 2 \end{bmatrix}$$

Step 4 : Find Cofactor Matrix

$A_{11} = -5$

$A_{12} = -11$

$A_{21} = -3$

$A_{22} = 2$

$$A_{ij} = \begin{bmatrix} -5 & -11 \\ -3 & 2 \end{bmatrix}$$

Step 5 : Find Adjoint Matrix (Adj A)

i.e. Transpose of the Cofactor Matrix

$$\begin{bmatrix} -5 & -3 \\ -11 & 2 \end{bmatrix}$$

Step 6 : Find Inverse (A^{-1})

$$A^{-1} = \frac{\text{Adj } A}{|A|}$$

$$= \frac{1}{-43}\begin{bmatrix} -5 & -3 \\ -11 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-5}{-43} & \frac{-3}{-43} \\ \frac{-11}{-43} & \frac{2}{-43} \end{bmatrix}$$

Step 7 : Multiplying the Inverse Matrix (A^{-1}) by the constant Vector (B).

i.e. $X = A^{-1}B$

$$= \begin{bmatrix} \frac{-5}{-43} & \frac{-3}{-43} \\ \frac{-11}{-43} & \frac{2}{-43} \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-5}{-43} \times 5 & \frac{-3}{-43} \times 6 \\ \frac{-11}{-43} \times 5 & \frac{2}{-43} \times 6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{25+18}{43} \\ \frac{55-12}{43} \end{bmatrix} = \begin{bmatrix} \frac{43}{43} \\ \frac{43}{43} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

But $X = A^{-1}B$

i.e. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, Therefore, $x_1 = 1$ and $x_2 = 1$

By Cramer's Rule we find

$$|A| = \begin{vmatrix} 2 & 3 \\ 11 & -5 \end{vmatrix} = -43$$

$$|A_1| = \begin{vmatrix} 5 & 3 \\ 6 & -5 \end{vmatrix} = -43$$

$$|A_2| = \begin{vmatrix} 2 & 5 \\ 11 & 6 \end{vmatrix} = -43$$

$$\therefore x = \frac{|A_1|}{|A|} = 1$$

$$y = \frac{|A_2|}{|A|} = 1$$

1.8.4 Three Simultaneous Linear Equations in Three Unknowns :

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = h_1$
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = h_2$
$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = h_3$

Let us change the given equations in matrix form :

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

But $\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Therefore, the given set of equations can be put as :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

$$\downarrow \quad \downarrow \quad \downarrow$$
$$A \qquad X \qquad H$$

$$\begin{bmatrix} \text{Where A is a } 3 \times 3 \text{ matrix} \\ \text{X is a } 3 \times 1 \text{ matrix} \\ \text{and H is a } 3 \times 1 \text{ matrix} \end{bmatrix}$$

This way, the three simultaneous equations are written in the form of a single matrix equation as AX = H.

Example 2 : Solve for x, y and z from the following set of equations.

$x - 2y + 3z = 1$

$3x - y + 4z = 3$

$2x + y - 2z = -1$

Put the given equations in matrix form

$$\begin{bmatrix} 1 & -2 & 3 \\ 3 & -1 & 4 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

$$\downarrow \quad \downarrow \quad \downarrow$$
$$A \qquad X \qquad H$$

$$A^{-1} = \frac{\text{Adj } A}{|A|}$$

$$\text{Now } A = \begin{bmatrix} 1 & -2 & 3 \\ 3 & -1 & 4 \\ 2 & 1 & -2 \end{bmatrix}$$

$$|A| = 1 \begin{vmatrix} -1 & 4 \\ 1 & -2 \end{vmatrix} + 2 \begin{vmatrix} 3 & 4 \\ 2 & -2 \end{vmatrix} + 3 \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix}$$

$= 1(2-4) + 2(-6-8) + 3(3+2)$

$= -2-28+15 = -15$

$$\text{Adj } A = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}^T$$

$$c_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 4 \\ 1 & -2 \end{vmatrix} = (2-4) = -2 \qquad (i) = (-1)^{i+j} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$c_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 4 \\ 2 & -2 \end{vmatrix} = -(-6-8) = 14$$

$$c_{13} = (-1)^{1+3} \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} = (3+2) = 5$$

$$c_{21} = (-1)^{2+1} \begin{vmatrix} -2 & 3 \\ 1 & -2 \end{vmatrix} = -(4-3) = -1$$

$$c_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 2 & -2 \end{vmatrix} = (-2-6) = -8$$

$$c_{23} = (-1)^{2+3} \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = -(1+4) = -5$$

$$c_{31} = (-1)^{3+1} \begin{vmatrix} -2 & 3 \\ -1 & 4 \end{vmatrix} = (-8+3) = -5$$

$$c_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} = -(4-9) = 5$$

$$c_{33} = (-1)^{3+3} \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} = (-1+6) = 5$$

$$\text{Adj } A = \begin{bmatrix} -2 & 14 & 5 \\ -1 & -8 & -5 \\ -5 & 5 & 5 \end{bmatrix}^T = \begin{bmatrix} -2 & -1 & -5 \\ 14 & -8 & 5 \\ 5 & -5 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{\begin{bmatrix} -2 & -1 & -5 \\ 14 & -8 & 5 \\ 5 & -5 & 5 \end{bmatrix}}{-15} = \begin{bmatrix} \frac{2}{15} & \frac{1}{15} & \frac{1}{3} \\ \frac{-14}{15} & \frac{8}{15} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{1}{3} & \frac{-1}{3} \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{2}{15} & \frac{1}{15} & \frac{1}{3} \\ \frac{-14}{15} & \frac{8}{15} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{1}{3} & \frac{-1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{2}{15} & +\frac{1}{5} & -\frac{1}{3} \\ \frac{-14}{15} & +\frac{8}{15} & -1 \\ \frac{-1}{3} & +1 & \frac{+1}{3} \end{bmatrix} = \begin{bmatrix} 0 \\ +1 \\ 1 \end{bmatrix}$$

$\therefore$ x = 0, y = +1, z = 1

By Cramer's Rule, we find

$$|A| = \begin{vmatrix} 1 & -2 & 3 \\ 3 & -1 & 4 \\ 2 & 1 & -2 \end{vmatrix} = -15$$

$$|A_1| = \begin{vmatrix} 1 & -2 & 3 \\ 3 & -1 & 4 \\ -1 & 1 & -2 \end{vmatrix} = 0$$

$$|A_2| = \begin{vmatrix} 1 & 1 & 3 \\ 3 & 3 & 4 \\ 2 & -1 & -2 \end{vmatrix} = -15$$

$$|A_3| = \begin{vmatrix} 1 & -2 & 1 \\ 3 & -1 & 3 \\ 2 & 1 & -1 \end{vmatrix} = -15$$

$$\therefore x = \frac{|A_1|}{|A|} = 0, \; y = \frac{|A_2|}{|A|} = 1, \; z = \frac{|A_3|}{|A|} = 1$$

Example 3 Use matrix method to solve system of equations

$x + y - z = 3$

$2x + 3y + z = 10$

$3x - y - 7z = 1$

Solution Writing the given equations in matrix form, we have

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 1 \end{bmatrix}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$A \qquad X \qquad B$$

i.e. AX = B

$$\text{Where } A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 3 \\ 10 \\ 1 \end{bmatrix}$$

$$\text{Now} \quad C_{11} = \begin{vmatrix} 3 & 1 \\ -1 & -7 \end{vmatrix} = -21 + 1 = -20$$

$$C_{12} = \begin{vmatrix} 2 & 1 \\ 3 & -7 \end{vmatrix} = 17$$

$$C_{13} = \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = -11$$

$$C_{21} = \begin{vmatrix} 1 & -1 \\ -1 & -7 \end{vmatrix} = 8$$

$$C_{22} = \begin{vmatrix} 1 & -1 \\ 3 & -7 \end{vmatrix} = -4$$

$$C_{23} = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = 4$$

$$C_{31} = \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} = 1 + 3 = 4$$

$$C_{32} = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -3$$

$$C_{33} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1$$

$$|A| = a_{11}c_{11}+a_{12}c_{12}+a_{13}c_{13} = 1(-20)\ +1(17)+(-1)(-11)$$
$$= -20+17+11 = 8$$

$$\text{Adj } A = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}^T$$

$$\text{Adj } A = \begin{bmatrix} -20 & 17 & -11 \\ 8 & -4 & 4 \\ 4 & -3 & 1 \end{bmatrix}^T$$

$$\text{Adj } A = \begin{bmatrix} -20 & 8 & 4 \\ 17 & -4 & -3 \\ -11 & 4 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{Adj}\, A = \frac{1}{8}\begin{bmatrix} -20 & 8 & 4 \\ 17 & -4 & -3 \\ -11 & 4 & 1 \end{bmatrix}$$ (A = 8) i.e. determinant of A

Now X = A^{-1} B i.e.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -20 & 8 & 4 \\ 17 & -4 & -3 \\ -11 & 4 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 10 \\ 1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 24 \\ 8 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ x = 3, y = 1, z = 1

Example 4 : Solve for x and y (using matrix inverse method)

x + 3y = 7 ; 4x - y = 2

Putting the given equations in matrix form, we have

$$\begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

↓ ↓ ↓

A X H

X = A^{-1} H

We have A = $\begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 3 \\ 4 & -1 \end{vmatrix}$ = (-1-12) = -13

Adj A = $\begin{bmatrix} -1 & -4 \\ -3 & 1 \end{bmatrix}^T = \begin{bmatrix} -1 & -3 \\ -4 & 1 \end{bmatrix}$

$$A^{-1} = \frac{\text{Adj}\, A}{|A|} = \begin{bmatrix} -1 & -3 \\ -4 & 1 \end{bmatrix} \Big/ -13$$

$$= \begin{bmatrix} \frac{1}{13} & \frac{3}{13} \\ \frac{4}{13} & \frac{-1}{13} \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{1}{13} & \frac{3}{13} \\ \frac{4}{13} & \frac{-1}{13} \end{bmatrix} \begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{7}{13} & \frac{6}{13} \\ \frac{28}{13} & \frac{-2}{13} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\therefore$ x = 1, y = 2

1.8.5 Summary

In this lesson, we have discussed an important application of matrices i.e. solving system of linear equations involving two or more than two variables.

1.8.6 Key Concepts

Applications of matrices, Solving system of linear equations.

1.8.7 Long Questions

1. Solve the following system of equations using matrix method :

$$3x + 4y + 2z = 8$$
$$2y - 3z = 3$$
$$x - 2y + 6z = -2$$

2. Find A^{-1}, where $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}$

and hence solve $x+2y-3z = -4$

$2x+3y+2z = 2$

and $3x-3y-4z = 11$

3. Solve the following set of equations by matrix method :

$3x_1+x_2+x_3 = 1$

$2x_1+2x_3 = 0$

$5x_1+x_2+2x_3 = 2$

4. Solve the following set of equations by matrix method :

$2x_1 + 3x_2 = 7$

$3x_2 + 2x_3 = 5$

$2x_1 + 7x_3 = 10$

1.8.8 Short Questions

5. Solve the following equations using matrix method :

$2x_1+ 3x_2 = 13$

$x_1 + 7x_2 = 23$

1.8.9 Suggested Readings

1. D. Bose. : An Introduction of Mathematical Economics
2. Mehta and Madnani : Mathematics for Economists
3. Aggarwal and Joshi : Mathematics for students of Economics
4. Bhardwaj and Sabharwal : Mathematics for students of Economics

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