M.A. (ECONOMICS) PART-I (Semester-I) PAPER-III

BASIC QUANTITATIVE METHODS

Section - B

PUNJABI UNIVERSITY PATIALA

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SECTION - B

Lesson No :

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MATRICES

Matrix algebra enables one to solve or handle a large system of simultaneous equations. Matrices provide a compact way of writing an equation system even an extremely large one. It is applicable only in linear equation system. Infact, it has the alternate name linear algebra.

Linearity assumption frequently adopted in economics may in certain case be quite reasonable and justified. On this basis, Matrix algebra is developed.

$$4x + 3y - 5z = 0$$
$$6x - 8y + 3z = 0$$

The coefficients of the above two equations can be written as:

$$A = \begin{bmatrix} 4 & 3 & -5 \\ 6 & -8 & 3 \end{bmatrix}_{2\times 3}$$

The above rectangular array is called matrix.

The above matrix A contains 2 rows (horizontal lines) and 3 columns (vertical lines). It can be said that A is a matrix of order 2×3 and is read as 2 by 3 matrix.

Definition :

A matrix is an arrangement of '$m \times n$' ordered numbers consisting of m rows and n columns. This matrix is called a matrix of order $m \times n$ and is read as 'm by n' matrix.

Matrices are denoted by capital letters A,B, …………etc.

A matrix of order mxn is written as :

$$A = \begin{bmatrix} a_{11} & a_{12} & ..... & a_{ij} & ..... & a_{1n} \\ a_{21} & a_{22} & ..... & a_{2j} & ..... & a_{2n} \\ ..... & ..... & ..... & ..... & ..... & ..... \\ a_{m1} & a_{m2} & ..... & a_{mj} & ..... & a_{mn} \end{bmatrix}_{m\times n}$$

The a's are called elements of matrix A. The above matrix A has 'm' rows and 'n' columns.

A matrix is a rectangular array of numbers and is enclosed by brackets or with double vertical lines.

The elements in the ith row and jth column is denoted by a_{ij}. Briefly we write:

$$A = [a_{ij}]$$

Thus the matrices of order 2×3, 3×2 and 2×2 are written as :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3} \qquad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times 2} \qquad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$$

Equality of Matrices : Two matrices are equal if and only if they have the same number of rows and the same number of columns and the corresponding elements in the two are equal e.g.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and } B = \begin{bmatrix} w & s \\ t & u \end{bmatrix}$$

Now A = B if a = w b = s
c = t d = u

Symbolically if A = $[a_{ij}]$, b = $[b_{ij}]$

Then A = B, if A and B have same dimensions and aij = bij for each (i , j)

Types of Matrices :

Square Matrix and Rectangular Matrix : If m = n, i.e. if the number of rows are equal to the number of columns, we have a square matrix, e.g.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \qquad \text{and } B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}_{3\times 3}$$

A matrix which is not square matrix is called a rectangular matrix, e.g.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix}_{2\times 3} \text{ and } B = \begin{bmatrix} 3 & 2 \\ 4 & 5 \\ 1 & 3 \end{bmatrix}_{3\times 2}$$

are rectangular matrices of order 2×3 and 3×2 respectively.

Diagonal Matrix : It is a square matrix in which all the elements are zero, except those in the leading diagonal, e.g.

$$A = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \text{ it is } 3 \times 3 \text{ diagonal Matrix}$$

Scalar Matrix : A diagonal matrix in which all the diagonal elements are equal is called a scalar matrix. For example :

$$A = \begin{bmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix} \text{ is a } 3 \times 3 \text{ Scalar Matrix}$$

Unit Matrix : A scalar matrix, each of whose diagonal elements is unity, is called a unit matrix and is denoted by 1.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ is a unit matrix of order } 3 \times 3$$

Zero or Null Matrix : A Matrix with every element equal to zero is called a zero or null matrix. It may be square or rectangular :

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ or } O = [\,0\ 0\ 0\,]$$

are zero matrix of order 2×2 and 1×3 respectively.

Sub-matrix : A matrix obtained by deleting some rows and some columns of a given matrix A is called sub matrix of A, e.g.

Suppose $A = \begin{bmatrix} 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 \\ 5 & 2 & 1 & 8 \end{bmatrix}$ and

$B = \begin{bmatrix} 4 & 6 \\ 5 & 1 \end{bmatrix}$ is a 2×2 sub matrix of A and is obtained by eliminating 2nd row and 2nd and 4th columns of matrix A.

Row and Column Matrix : A row matrix has only one row and is of order $1 \times n$. Similarly a column matrix has only one column and is of order $m \times 1$.

Example 1:

(i) A = [a b c] is a row matrix of order 1×3

(ii) $B = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$ is a column matrix of order 3×1

Upper and Lower Triangular Matrices : A square matrix all of whose elements below the main diagonal are zero is called upper triangular matrix. If all elements above the main diagonal are zero then it is a lower triangular matrix. For example :

(i) $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$ is an upper triangular matrix of order 3×3

(ii) $B = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is lower triangular matrix of order 3×3

Transpose of a matrix : The transpose of a matrix is obtained by writing the ith row as the jth column and the jth column as the ith row. In other words, if we interchange the rows and columns of $m \times n$ matrix A, we get an $n \times m$ matrix A, which is called the transpose of A, symbolically.

If $A_{mn} = (a_{ij})$ then $A'_{mn} = (a_{ji})$

For example if

$$A = \begin{bmatrix} 4 & 5 & 9 \\ 6 & 7 & 8 \\ 5 & 3 & 2 \end{bmatrix} \text{ then } A' \text{ or } A^t = \begin{bmatrix} 4 & 6 & 5 \\ 5 & 7 & 3 \\ 9 & 8 & 2 \end{bmatrix}$$

The symbol ' or t denotes transpose

If the matrix is square as in the above example, then the transpose of the matrix can be thought of as the matrix with the same main diagonal, with all the other elements 'reflected' in that diagonal.

$$\text{If } A = \begin{bmatrix} 5 & 6 & 1 \\ 7 & 8 & 4 \end{bmatrix} \text{ then } A^1 = \begin{bmatrix} 5 & 7 \\ 6 & 8 \\ 1 & 4 \end{bmatrix}$$

Symmetric Matrix : It is a square matrix A such the $a_{ij} = a_{ji}$ i.e. $(ij)^{th}$ element of A is equal to $(ji)^{th}$ elment of A'. In other words A = A' where A' is the transpose of matrix A. For example :

$$A = \begin{bmatrix} a & t & k \\ t & b & n \\ k & n & c \end{bmatrix} \quad A' = \begin{bmatrix} a & t & k \\ t & b & n \\ k & n & c \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 7 & 9 \\ 7 & 5 & 11 \\ 9 & 11 & 16 \end{bmatrix}, \quad B' = \begin{bmatrix} 4 & 7 & 9 \\ 7 & 5 & 11 \\ 9 & 11 & 16 \end{bmatrix}$$

$\therefore$ A = A' $\qquad\qquad$ $\therefore$ B = B'

A symmetric matrix is a 'reflection; of itself in the main diagonal.

Skew Symmetric Matrix : It is a square matrix A in which $a_{ij} = -a_{ji}$ for all values of i and j in other words $(ij)^{th}$ element of A is equal to the negative of (ji)th element of A.

Now $a_{ij} = -a_{ji}$ is a skew symmetric matrix

$$\text{or } 2a_{ij} = O \begin{pmatrix} \therefore a_{ij} = -a_{ji} \\ Put\ i = j \end{pmatrix}$$

$\therefore a_{ij} = O$

So all the diagonal elements of a skew-symmetric matrix A are zero.

$$A = \begin{bmatrix} 0 & a & n \\ -a & o & k \\ -n & -k & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 5 & 6 \\ -5 & 0 & 4 \\ -6 & -4 & 0 \end{bmatrix}, \quad -A' = \begin{bmatrix} 0 & a & n \\ -a & o & k \\ -n & -k & 0 \end{bmatrix}, \quad -B' = \begin{bmatrix} 0 & 5 & 6 \\ -5 & 0 & 4 \\ -6 & -4 & 0 \end{bmatrix}$$

Thus A and B are two skew symmetric matrices because A' = –A and B = –B'

Idempotent matrix : A square matrix A is said to be idempotent if $\overline{A^2}$ = A.A.

(Unit matrix is an example of an idempotent matrix)

Conjugate Matrix :

If A is an $m \times n$ matrix, then the $m \times n$ matrix is obtained by replacing each element of A by its complex conjugate is called Conjugate Matrix of A and is denoted by $\overline{A}$. Thus if A = (a_{ij}) then the matrix $\overline{A}$ = $\left(\overline{a}_{ij}\right)$ is called the conjugate matrix of A where $\left(\overline{a}_{ij}\right)$ is the complex conjugate of $\left(a_{ij}\right)$. Obviously A is real if and only if $\overline{A}$ = A, i.e. if all the elements of A are real.

Hermitian Matrix : A square matrix 'A' is said to be Hermitian Matrix if $A^{\theta} = A$, e.g. the matrix

$$\begin{bmatrix} 1 & 2+i \\ 2-i & 4 \end{bmatrix}$$ is a Hermitian matrix

Note : All the diagonal elements of a Hermitian matrix are real.

Skew Hermitian Matrix : A square matrix A is said to be Skew Hermitian if $A^{\theta} = A$, e.g. the matrix

$$\begin{bmatrix} 2i & 1+i & 3-2i \\ -1+i & 4i & 4+i \\ -3-2i & -4+i & 0 \end{bmatrix}$$ is a skew Hermitian matrix.

Note : In a skew Hermitian matrix the diagonal elements are purely imaginary or zero.

Example 2: If A is any square matrix, show that A+ A^{θ} is Hermitian

Sol. We know that a square matrix A is Hermitian if A^{0} =A

Now $(A + A^{\theta})^{\theta} = A^{\theta} + (A^{\theta})^{\theta} = A^{\theta} + A$ $\qquad ((A^{\theta})^{\theta} = A$

$= A + A^{\theta}$

Hence A = A^{θ} is Hermitian

Orthogonal Matrix : A square matrix A is said to the orthogonal if

A'A = 1 = A'A

Nilpotent Matrix : A square matrix A is said to be nilpotent of index n if A^{n} =0

Example 3:

$$A = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$ is nilpotent because $$A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Idempotent Matrix :

A matrix reproduces itself when multiplied by itself is called an idempotent matrix. That is if A.A = A then A is called an idempotent matrix.

For example, $$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ is an idempotent matrix

$$\therefore\ A.A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A$$

Example 4 : Show that A = $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is an idempotent matrix.

Solution : Here A.A. = $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 4+2-4 & -4-6+8 & -8-8+12 \\ -2-3+4 & 2+9-8 & 4+12-12 \\ 2+2-3 & -2-6+6 & -4-8+9 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

Since A.A. = A

Hence A is an idempotent matrix.

Example 5 : If A = $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ Prove that B $= I_3 - A(A'.A)^{-1}A^1$.

Show that B is an idempotent matrix

Solution: Since A = $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ A' = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$$\therefore \; A'.A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

and $|A'A|$ = (42-36) = 6

adj (A'A) = $\begin{bmatrix} +14 & -6 \\ -6 & +3 \end{bmatrix}$

$$\therefore (A'A)^{-1} = \frac{1}{|A'A|} \text{ adj } (A'A)$$

$$(AA)^{-1} = \frac{\begin{bmatrix} 14 & -6 \\ -6 & +3 \end{bmatrix}}{6} = \begin{bmatrix} \frac{14}{6} & -1 \\ -1 & \frac{3}{6} \end{bmatrix}$$

and $A(A'.A)^{-1}A' = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \frac{14}{6} & -1 \\ -1 & \frac{3}{6} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \frac{14}{6}-1 & \frac{14}{6}-2 & \frac{14}{6}-3 \\ -1+\frac{3}{6} & -1+1 & -1+\frac{9}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \frac{8}{6} & \frac{2}{6} & \frac{-4}{6} \\ \frac{-3}{6} & 0 & \frac{3}{6} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{6} & \frac{2}{6} & \frac{-1}{6} \\ \frac{2}{6} & \frac{2}{6} & \frac{2}{6} \\ \frac{-1}{6} & \frac{2}{6} & \frac{5}{6} \end{bmatrix}$$

Now $B = I_3 - A\,(A'.A)^{-1}A'$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{5}{6} & \frac{2}{6} & \frac{-1}{6} \\ \frac{2}{6} & \frac{2}{6} & \frac{2}{6} \\ \frac{-1}{6} & \frac{2}{6} & \frac{5}{6} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{6} & \frac{-2}{6} & \frac{1}{6} \\ \frac{-2}{6} & \frac{4}{6} & \frac{-2}{6} \\ \frac{1}{6} & \frac{-2}{6} & \frac{1}{6} \end{bmatrix}$$

$$\text{Now } B.B = \begin{bmatrix} \frac{1}{6} & \frac{-2}{6} & \frac{1}{6} \\ \frac{-2}{6} & \frac{4}{6} & \frac{-2}{6} \\ \frac{1}{6} & \frac{-2}{6} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} \frac{1}{6} & \frac{-2}{6} & \frac{1}{6} \\ \frac{-2}{6} & \frac{4}{6} & \frac{-2}{6} \\ \frac{1}{6} & \frac{-2}{6} & \frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{6}{36} & \frac{-12}{36} & \frac{6}{36} \\ \frac{-12}{36} & \frac{24}{36} & \frac{-12}{36} \\ \frac{6}{36} & \frac{-12}{36} & \frac{6}{36} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{6} & \frac{-2}{6} & \frac{1}{6} \\ \frac{-2}{6} & \frac{4}{6} & \frac{-2}{6} \\ \frac{1}{6} & \frac{-2}{6} & \frac{1}{6} \end{bmatrix} = B$$

Since B.B = B

$\therefore$ B is an idempotent matrix

Theorems on Idempotent

1. If AB = A and BA = B then A and B are idempotent

Let us consider the matrix ABA

Since ABA = (AB)A = AA = A^2

Since ABA = (A) BA = AB = A

Which gives A^2 = A

Hence A is idempotent

Again BAB = (BA) B = B.B = B^2 $\quad\therefore\quad$ BA =B

Also BAB = B (AB) = BA = B $\quad$ ($\therefore$ $\quad$ AB = A, BA = B)

Which gives B^2 = B

Hence B is idempotent

2. If B is idempotent show that A=1-B, is also idempotent and that AB=BA = o

Since B is idempotent $\quad\therefore B^2 = B$

Now $A^2 = (1\text{-}B)^2 = (1\text{-}B) \,.\, (1 - B)$ $\quad$ ($\therefore$ A = 1 – B)

$= 1^2 - 21B + B^2$

$= 1 - 2B + B$ $\quad$ ($\therefore$ 1B = B and B^2 =B)

$= 1 - B$

$= A$

Since $A^2 = A$ $\therefore$ A is idempotent

Again AB = (1 – B) B

= 1.B – B.B $\therefore$ 1.B = B and B.B = B^2

= B – B

= 0

Similarly BA = B (1-B) = B1 – B.B = B – B = 0

Trace of a matrix

In a square matrix, all those elements a_{ij} for which i = j (i.e. a_{ij}) are called the diagonal elements and the line along which they lie is called the principal diagonal. The sum of the elements in the principal diagonal of a square matrix is called the trace of the matrix :

e.g. if A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$ then

trace of A = tr. A = Sum of the diagonal elements

= $a_{11} + a_{22} + a_{33} = \sum_{i=1}^{3} a_{ij}$

In general if A = $(a_{ij})_{n\times m} = \begin{bmatrix} a_{11} & a_{12} & ..... & a_{1n} \\ a_{21} & a_{22} & ..... & a_{2n} \\ ..... & ..... & ..... & ....._{nxn} \\ a_{n1} & a_{n2} & ..... & a_{nxn} \end{bmatrix}$

Trace of A = tr. A = $a_{11} + a_{22} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots a_{nn}$

$\sum_{i=1}^{n} a_{ij}$

To take a numerical example

Let A = $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 3 \\ 1 & 3 & -1 \end{bmatrix}$

Then trace of A = tr. A = 1 + 0 + (-1) = 0

Note : It may be noted that trace of a null matrix of any order is zero and trace of an identity matrix of nth order is n

For example :

Let $O_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

then tr. $O_3 = 0 + 0 + 0 = 0$

Similarly let

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then tr. $I_3 = 1 + 1 + 1 = 3$

if we take $I_n = \begin{bmatrix} 1 & 0 & 0 & ..... & ..... & 0 \\ 0 & 1 & 0 & ..... & ..... & ..... \\ ..... & ..... & ..... & ..... & ..... & ..... \\ 0 & 0 & 0 & ..... & ..... & 1 \end{bmatrix}_{n\times n}$

then tr. $I_n = 1 + 1 + \ldots\ldots\ldots\ldots + 1 = n$ (upto n)

Theorems on Trace

1. tr. (A') = tr. A, A. being a square matrix $(a_{ij})_{n\times n}$ of order n.
2. tr. (AA') = tr. (A'A)
3. tr. (kA) = K tr. (A), k being scalar
4. tr. (A+B) = tr. (A) + tr. (B) if A and B are square matrices of the same order
5. tr. (AB) = tr. (BA) if AB and BA are both defined.
6. tr. (ABC) = tr. (BCA) = tr. (CAB), if ABC, BCA and CAB are all defined.

1. Verification : tr. A' = tr. A

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

then $A' = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

$\therefore$ tr. $A' = a_{11} + a_{22} + a_{33}$ = tr. A

2. tr. (A.A') = tr. (A'A)

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

so that A' = $\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

and AA' = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{2} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

$$= \begin{bmatrix} a^2_{11} + a^2_{12} + a^2_{13} & a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} & a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} \\ a_{21}a_{11} + a_{22}a_{12} + a_{23}a_{13} & a^2_{21} + a^2_{22} + a^2_{23} & a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} \\ a_{31}a_{11} + a_{32}a_{12} + a_{33}a_{13} & a_{31}a_{21} + a_{32}a_{22} + a_{33}a_{23} & a^2_{31} + a^2_{32} + a^2_{33} \end{bmatrix}$$

$\therefore$ tr. (AA') = $a^2_{11} + a^2_{12} + a^2_{13} + a^2_{21} + a^2_{22} + a^2_{23} + a^2_{31} + a^2_{32} + a^2_{33}$

Similarly

A'A = $\begin{bmatrix} a_{11}+a_{21}+a_{31} & & \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $\quad a^2_{13}+a^2_{23}+a^2_{33}$

$$= \begin{bmatrix} a^2_{11} + a^2_{21} + a^2_{31} & \ldots\ldots\ldots & \ldots\ldots\ldots \\ \ldots\ldots\ldots & a^2_{12} + a^2_{22} + a^2_{32} & \ldots\ldots\ldots \\ \ldots\ldots\ldots & \ldots\ldots\ldots & a^2_{13} + a^2_{23} + a^2_{33} \end{bmatrix}$$

$\therefore$ tr. (A'A) = $a^2_{11} + a^2_{12} + a^2_{13} + a^2_{21} + a^2_{22} + a^2_{23} + a^2_{31} + a^2_{32} + a^2_{33}$

$\therefore$ tr. (AA') = tr. (A'A)

3. tr. (kA) = k tr. (A)

Let A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then kA = $\begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix}$

$\therefore$ tr. (kA) = $ka_{11} + ka_{22} + ka_{33}$

= K $(a_{11} + a_{22} + a_{33})$ = k tr. A

4. tr. (A+B) = tr. (A) + tr. (B)

Let A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ B = $\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$

tr A = $a_{11} + a_{22} + a_{33}$

tr B = $b_{11} + b_{22} + b_{33}$

$$A + B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix}$$

$\therefore$ tr. (A+B) = $(a_{11} + b_{11}) + (a_{22} + b_{22}) + (a_{33} + b_{33})$

= $(a_{11} + a_{22} + a_{33}) + (b_{11} + b_{22} + b_{33})$

= tr. (A) + tr. (B)

5. tr. (AB) = tr. (BA).

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

then $A\,B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

tr. (AB) = $a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22}$

$BA = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$= \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{21} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{bmatrix}$$

tr. (BA) = $(a_{11}b_{11} + b_{12}a_{21} + b_{21}a_{12} + b_{22}a_{22})$

tr. (AB) = $(a_{11}b_{11} + a_{22}b_{22} + a_{12}b_{21} + a_{21}b_{12})$

= $(a_{11}b_{11} + a_{21}b_{12} + a_{12}b_{21} + a_{22}b_{22})$ = tr. (BA)

Operations of Matrices

Operations of Matrices include addition, subtraction and multiplication of matrices. They are discussed below one by one.

Addition of Matrices : Two matrices can be added if and only if they have the same dimentions (order). When this dimensional requirement is met, the matrices are said to be comfortmable for addition. In such a situation, the sum of matrix A = (a_{ij}) and matrix B = (b_{ij}) is the addition of each pair of corresponding elements.

In other words, the sum of the m × n matrix A and the (m × n) matrix B is said to be a matrix C such that $c_{ij} = a_{ij} + b_{ij}$ and we call the process of forming C the addition of B to A.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} r & s \\ t & k \end{bmatrix}$$

Then the sum of these

$$A + B = \begin{bmatrix} a+r & b+s \\ c+t & d+k \end{bmatrix}$$

Suppose that A denotes the quantities bought by Mr. Alien in each of the three weeks of four different goods. Let B denote the quantities bought of the same goods by Mr. Allen in three weeks. The total purchases will be :

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 5 & 4 & 3 \\ 2 & 1 & 3 \\ 0 & 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 & 0 \\ 5 & 3 & 2 \\ 5 & 3 & 1 \\ 4 & 0 & 0 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1+2 & 4+3 & 5+0 \\ 5+5 & 4+3 & 3+2 \\ 2+5 & 1+3 & 3+1 \\ 0+4 & 2+0 & 3+0 \end{bmatrix} = \begin{bmatrix} 3 & 7 & 5 \\ 10 & 7 & 5 \\ 7 & 4 & 4 \\ 4 & 2 & 3 \end{bmatrix}$$

It two matrices have different dimensions/order, their addition is not defined, e.g.

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 3 & 2 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 5 \\ 2 & 1 \end{bmatrix}$$

Since A is of 2 × 3 order and B = 2 × 2 order hence their sum is not possible.

Subtraction of Matrices : The difference between two matrices (subtraction) can be known if and only if the two matrices have the same dimensions. In such a situation the difference between the matrices i.e. A-B can be obtained by subtracing the elements of B from the corresponding elements of A. in other words.

$(d_{ij}) = (a_{ij}) - (b_{ij})$

it can also be stated that

if $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{m\times n}$

then $A - B = (a_{ij})_{m\times n} - (b_{ij})_{m\times n}$

for example :

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} r & s \\ t & u \end{bmatrix}$$

Then

$$A - B = \begin{bmatrix} a-r & b-s \\ c-t & d-u \end{bmatrix}$$

or, if the following information is given to us

A = $\begin{bmatrix} 20 & 4 \\ 15 & 7 \end{bmatrix}$ and B = $\begin{bmatrix} 8 & 3 \\ 7 & 9 \end{bmatrix}$

Then the difference between two matrices

A – B = $\begin{bmatrix} 20-8 & 4-3 \\ 15-7 & 7-9 \end{bmatrix} = \begin{bmatrix} 12 & 1 \\ 8 & -2 \end{bmatrix}$

Thus the subtraction operation may be taken alternatively as an addition operation involving a matrix A and another matrix (-1) B, or A-B = A + (-B).

Multiplication of Matrices : The multiplication of two matrices is contingent upon the satisfaction of another dimensional requirement.

The conformability, conditions for the multiplication is that the column dimension of matrix A (the 'lead matrix in the expression AB) must be equal to the row dimension of B (the 'lag') matrix) for example :

Let A = $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}_{2\times3}$, B = $\begin{bmatrix} p \\ q \\ r \end{bmatrix}_{3\times1}$

Matrix A has 3 columns and B has 3 rows.

The procedure of Multiplication is that each element of a row from the first matrix is to be multiplied by the corresponding element of a column of the second matrix and the sum of these product constitutes an element of the matrix AB.

Thus the product of matrix AB, in the above mentioned matrices A and B will be:

AB = $\begin{bmatrix} a_1 p + b_1 q + c_1 r \\ a_2 p + b_2 q + c_2 r \end{bmatrix}$ and it is 2 × 1 matrix.

The procedure will be similar for higher dimension matrices. For example

A = $\begin{bmatrix} 1 & -2 & 3 & 4 \\ 2 & 1 & 0 & 3 \\ 2 & 1 & -2 & 4 \end{bmatrix}_{3\times4}$ and B = $\begin{bmatrix} 2 & 1 \\ 0 & 2 \\ -2 & 3 \\ 1 & 2 \end{bmatrix}_{4\times2}$

Now A is 3 × 4 matrix and B is a 4 ×2 matrix, hence the condition for multiplying A by B is satisfied

Hence the product will be

$$AB = \begin{bmatrix} 1\times2+(-2)0+3(-2)+4\times1 & 1\times1+(-2)2+3\times3+4\times2 \\ 2\times2+1\times0+0\times(-2)+3\times1 & 2\times1+1\times2+0\times3+3\times2 \\ 2\times2+1\times0+(-2)(-2)+4\times1 & 2\times1+1\times2+(-2)\times3+4\times2 \end{bmatrix}$$

or $AB = \begin{bmatrix} 0 & 14 \\ 7 & 10 \\ 12 & 6 \end{bmatrix}$ and it is 3 × 2 matrix

It should be noted that if AB is defined, BA need not be. In the definition, AB would be defined only if number of column of B is equal to number or rows of A. This would allow the operation, AB would still be a different product from BA. Because in the first case A's rows are multiplied by B's column, whereas in the second case B's rows are multiplied by A's column, so that BA and AB are completely different.

In the example mentioned above BA is not possible, because B is a matrix of order 4 × 2 and A or order 3 × 4. The number of columns of B is not equal to number or rows of A.

If the product of AB of the matrices A and B is possible and A is of order m × n and matrix B of order n × p, then product AB is of the order m × p.

If A and B are both square matrices of order n × n each then the product AB and BA are both possible and each of them is square matrix of order n and they are different.

Laws of Matrices :

(1) Matrix addition is Commutative as well as Associative

Commutation tells the fact that matrix addition calls only for the addition of the corresponding elements of two matrices and that the order in which each pair of corresponding elements is added is immaterial.

Commutative law :

Commutative law holds good when

$A + B = B + A$

Proof : $A + B = (a_{ij}) + (b_{ij}) = (b_{ij}) + (a_{ij}) = B + A$

So $A + B = B + A$

Suppose $A = \begin{bmatrix} 3 & 5 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

$A + B = B + A = \begin{bmatrix} 7 & 8 \\ 6 & 4 \end{bmatrix}$

Associative law of addition

Associative law holds good when $(A+B) + C = A + (B + C)$

Proof : $(A + B) + C = [(a_{ij}) + (b_{ij})] + (c_{ij})$

$= (a_{ij} + b_{ij} + c_{ij}) = a_{ij} + [(b_{ij}) + (c_{ij})]$

$= A + (B + C)$

This can be extended to the sum of any finite number of matrices of the same order.

The subtraction operation A-B can simply be regarded as the addition operation $A + (-B)$

(2) In general Matrix Multiplication is not commutative:

(3) Matrix Multiplications is Associative and also follows Distributive law.

Associative Law of Multiplication :

(AB) C = A (BC) = ABC

In forming the product ABC the conformability condition must be satisfied by the matrices. If A is m × n, B is n × p and C is p × q.

Then

$A_{(m\times n)} \times B_{(n\times p)} \times C_{(p\times q)}$ are conformable.

Example 6: show that the product of

$$[x\ y\ z]\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

= $(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy)$

Solution :

$$[x\ y\ z]\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$=[x\ y\ z]\begin{bmatrix} ax+hy+gz \\ hx+by+fz \\ gx+fy+cz \end{bmatrix}$$

= [x (ax + hy + gz) +y (hx +by +fz) + z (gx + fy + cz)]

= $[ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy]$ hence provded.

Distributive Law :

Distributive law holds good when

A (B+C) = AB + AC. (Pre-multiplied by A)

Or (B+C) A = BA + CA (post-multiplied by A)

In the product AB, the matrix A is said to be pre-multiplied by A, and the matrix B is post multiplied by B. In BA the matrix A is said to be pre-multiplied by B and B to be post-multiplied by A.

In each case, the conformability condition for addition as well as multiplication must of course be observed.

Example 7:

Show that :

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$$ is a nilpotent matrix of index 2.

Solution :

$$A^2 = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix} \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3-4 & -3-9+12 & -4-12+16 \\ -1-3+4 & 3+9-12 & 4+12-16 \\ 1+3-4 & -3-9+12 & -4-12+16 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ thus } A^2 = O$$

Hence A is a nilpotent matrix

Example 8:

If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ show that

$A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ where n is any +ve integar.

Proof : We will prove by the method of induction

Now

$$A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$$

if we put n = 1, we get

$$A^1 = \begin{bmatrix} 1+2.1 & -4.1 \\ 1 & 1-2.1 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

Which is the same as given value of A.

Hence the value of A^n is true if n = 1.

Where n =2

$$A^2 = \begin{bmatrix} 1+2.2 & -4.2 \\ 2 & 1-2.2 \end{bmatrix} = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}$$

Now $A^2 = A.A$

$$= \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 9-4 & -12+4 \\ 3-1 & -4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}$$

Hence A^n is true when n = 2

Assume that value of A^n is true when n = m

$$A^m = \begin{bmatrix} 1+2m & -4m \\ m & 1-2m \end{bmatrix}$$

$$A^mA = \begin{bmatrix} 1+2m & -4m \\ m & 1-2m \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3+6m-4m & -4-8m+4m \\ 3m+1-2m & -4m-1+2m \end{bmatrix} = \begin{bmatrix} 3+2m & -4-4m \\ m+1 & -2m-1 \end{bmatrix}$$

= A^{m+1} (Value obtained from A on putting n = m+1)

Hence Value of A^n is true fro n = m + 1

Hence A^n is true whenever n is any +ve interger.

Exercise : 1. Explain with illustration :

(i) Conjugate Matrix

(ii) Symmetric and Skew – Symmetric matrix

(iii) Orthogonal Matrix

2. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

so that $(ABC)^1 = C^1 B^1 A^1$

3. Show that the matrix

$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ satisfies the equation $A^3 -6A^2 + 9A -4I + 0$

LESSON NO. 2.2 **AUTHOR : SH. C.S. AGGARWAL**

DETERMINANTS AND THEIR PROPERTIES

1. Meaning
2. Properties of Determinants
3. Questions

There is some debate amongst economists about how much one needs to know about determinants. It is true that evaluating determinants is rather a tedious way of solving numerical problems and that theoretical result can be obtained without them. However, they are extremely useful and are widely used in the existing economic literature.

A determinant is a number associated to a square matrix.

If A = (a_{11}) be a matrix then determinant of A = a_{11}.

The determinant of A is written as |A| or det A and is read as determinant 'A'.

In other words, if A = (a_{ij}) i = 1, 2, ...n

j = 1, 2, ...n

denotes a square matrix of order n then determinant of A = $|a_{ij}|$

Important Note :

A determinant is reducible to a number and matrix is whole block of numbers.

A determinant is defined only for a square matrix whereas a matrix as such may not be square.

Order of Determinants

1. Order One : If A = (a_{11}) be a matrix, then det. A = $|a_{11}|$

2. Order Two :

If A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a matrix of order 2 × 2 then determinant of A is defined as follows :

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

a scalar which is obtained by multiplying the two elements in the principle diagonal of A and then subtracting the product of the two remaining elements.

If A = $\begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix}$

Then |A| = 2 × 4 – 1 × 2 = 6

3. Order Three :

If A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

is a matrix of order 3 3 then determinant A has value :

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$= a_{11} (a_{22} a_{33} - a_{32} a_{23}) - a_{12} (a_{21} a_{33} - a_{31} a_{23}) + a_{13} (a_{21} a_{32} - a_{31} a_{22})$

$= a_{11} a_{22} a_{33} - a_{11} a_{32} a_{23} - a_{12} a_{21} a_{33} + a_{12} a_{31} a_{23} + a_{13} a_{21} a_{32} - a_{13} a_{31} a_{22}$

Direct Method :

There is another way of finding the determinant of a matrix which is called the direct method or Sarrus Diagram. This method is as follows :

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

$= a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{31} a_{22} a_{13} - a_{32} a_{23} a_{11} - a_{33} a_{21} a_{12}$

This method is quite simple – Write the given columns of matrix and also write 1st and 2nd column again. Then calculate as shown above :

Multiply elements :

The sum of all the six products will be the value of the determinant.

For Example :

If A = $\begin{bmatrix} 3 & 4 & 5 \\ 2 & 1 & 5 \\ 6 & 4 & 3 \end{bmatrix}$

$$\begin{array}{ccccc} 3 & 4 & 5 & 3 & 4 \\ 2 & 1 & 5 & 2 & 1 \\ 6 & 4 & 3 & 6 & 4 \end{array}$$

then $|A|$ = (3) (1) (3) + (4) (5) (6) + (5) (2) (4) – (6) (1) (5) – (4) (5) (3) – (3) (2) (4)

= 169 -114 = 55

For higher order determinants we use Laplace's Expansion Method which is based on co factors. Thus we find that determinant has a definite value.

The above example can also be solved by adopting the other method :

$|A|$ = 3 [1x3 – 5 (4)] – 4 (2 × 3 – 6 × 5) + 5 (2 × 4 – 6 × 1)

= 3 (3 – 20) – 4 (6 – 30) + 5 (8 – 6) = 3 (-17) – 4 (-24) + 5 (2)

= -15 + 96 + 10 = 55

Thus we find that the same result is obtained.

Minors

The determinants formed by taking equal number of rows and columns out of the elements of a matrix are called minors of a matrix. The order of a minor is the order of the determinant. Minor is itself a determinant and has a value. In general, (M_{ij}) can be used to represent the minor obtained by deleting the ith row and jth column.

Minor of First order of matrix : A is formed by each element individually taken.

Minor of Second order of Matrix : A is formed by taking 2 elements of two rows and two columns.

Minor of Third order of Matrix : A is formed by taking 3 elements of 3 rows and 3 columns.

Consider a matrix of order (3 × 3)

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

When we delete any one row and column, which contains the elements of A, we get a (2 × 2) sub-matrix of A. The determinant of sub-matrix is called a minor of det. A. Thus.

$$\begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}$$

is minor of a_{32} of det. A

Similarly minors of a_{12}, a_{13} a_{32} of det. A are as follows :

$$\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}, \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}, \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix} \text{respectively}$$

For example, in the determinant $\begin{vmatrix} 1 & 5 & 4 \\ 3 & 9 & 2 \\ 7 & 1 & 6 \end{vmatrix}$ the minor of element of 5 in M_{12} which

is $M_{12} = \begin{vmatrix} 3 & 2 \\ 7 & 6 \end{vmatrix} = 18 - 14 = 4$

Similarly , minor of element 6 of M_{33} which is

$M_{33} = \begin{vmatrix} 1 & 5 \\ 3 & 9 \end{vmatrix} = 9 - 15 = -6$

Cofactor : Another concept closely connected to Minor is known as Cofactor.

A Cofactor is a minor with a prescribed sign attached with it :

The Cofactor C_{ij} is $(-1)^{1+j}$ times the determinant of the sub-matrix obtained by deleting row i and column j from matrix A (called the minor of (a_{ij}) or $C_{ij} = (-1)^{i+j}|M_{ij}|$

If the sum of the two subscripts i and j in the minor (M_{ij}) is even, then the cofactor is of the same sign as the minor : that is $(C_{ij}) = (M_{ij})$

if it is odd, then the cofactor takes the opposite sign to the minor , that is

$C_{ij} = (-1)^{i+j} (M_{ij})$

$(-1)^{i+j}$ follows the Chessboard Rule i.e.

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

For example, for a_{53}, $(-1)^{5+3} = (-1)^8 = +1$

And for a_{12}, $(-1)^{1+2} = (-1)^3 = -1$. Thus it is obvious that the expression $(-1)^{i+j}$ can be positive only if 'i + j' is even, otherwise it will be of negative sign.

It should be noted that it is possible to expand a determinant by the cofactor of any row or for that matter, of any column. For instance, if the first column of third order det. A consist of elements a_{11}, a_{21}, a_{31} expansion by cofactors of these elements will also yield that value of :

$|A| = a_{11} [C_{11}] + a_{21} [C_{21}] + a_{31} [C_{31}]$

For Example :

Evaluate the determinant of the matrix

$$\begin{bmatrix} 4 & 5 & 7 \\ 6 & 3 & 2 \\ 1 & 8 & 0 \end{bmatrix}$$

Let A = $\begin{bmatrix} 4 & 5 & 7 \\ 6 & 3 & 2 \\ 1 & 8 & 0 \end{bmatrix}$

Then |A| = $\begin{bmatrix} 4 & 5 & 7 \\ 6 & 3 & 2 \\ 1 & 8 & 0 \end{bmatrix}$

If we expand it with the help of Ist column

$[C_{11}]$ = Cofactor of 4 = $(-1)^{1+1} \begin{vmatrix} 3 & 2 \\ 8 & 0 \end{vmatrix}$ = (0-16) =-16

$[C_{21}]$ = Cofactor of 6 = $(-1)^{2+1} \begin{vmatrix} 5 & 7 \\ 8 & 0 \end{vmatrix}$ = - (0-56) = 56

$[C_{11}]$ = Cofactor of 1 = $(-1)^{3+1} \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix}$ = + (10-21) =-11

|A| = 4 (-16) + 6 (56) +1 (-11)

|A| = -64 + 336 – 11 = 261

Similarly if we expand it with the help of first row,

= 4 (0-16) -5 (0-2) + 7 (48-3)

|A| = -64 + 10 +315 = -64+325=-261

Properties of Determinants

Determinants posseses certain properties that are common to determinants of all orders. These properties are :

1. The interchange of rows and columns (i.e. transpose of a matrix), does not effect the value of the determinant

It means the determinant of a matrix A has the same value as that of its transpose A'.

For example $\begin{vmatrix} p & q \\ r & s \end{vmatrix} = \begin{vmatrix} p & r \\ q & s \end{vmatrix}$ = (ps – qr)

or $\begin{vmatrix} 4 & 5 \\ 3 & 2 \end{vmatrix} = \begin{vmatrix} 4 & 3 \\ 5 & 2 \end{vmatrix}$ = - 7

2. The interchange of two rows (or two columns) will alter the sign, but not the numerical value of the determinant :

For example $\begin{vmatrix} p & q \\ r & s \end{vmatrix}$ = (ps – qr) but with interchange of two rows.

we get $\begin{vmatrix} r & s \\ p & q \end{vmatrix}$ = (qr – ps) = - (ps – qr)

or $\begin{vmatrix} 5 & 2 \\ 3 & 4 \end{vmatrix}$ = (20 – 6) = 14 with the interchange of two columns,

we get $\begin{vmatrix} 2 & 5 \\ 4 & 3 \end{vmatrix}$ = (6 – 20) = - 14

3. A matrix with a row (or column) of zeros has a zero determinant.

$\begin{vmatrix} p & q \\ 0 & 0 \end{vmatrix}$ = (p0 – q0) = 0

or $\begin{vmatrix} 4 & 5 & 0 \\ 3 & 2 & 0 \\ 5 & 2 & 0 \end{vmatrix}$ = 4 (0 – 0) – 5 (0 – 0) + 0 (6 – 10) (Expansion by 1st Row) = 0

4. If any single row (column) of a matrix is multiplied by a scalar 'k', then the determinant is also multiplied by 'k'.

$\begin{vmatrix} kp & kq \\ r & s \end{vmatrix}$

= (kps – kqr) = k (ps – qr) =k $\begin{vmatrix} p & q \\ r & s \end{vmatrix}$

or $\begin{vmatrix} 4 & 3 \\ 2 & 3 \end{vmatrix}$ = 12 – 6 = 6

Suppose k = 2 and multiplying 1st column by 2

= $\begin{vmatrix} 8 & 3 \\ 4 & 3 \end{vmatrix}$ = (8 X 3 – 4 X 3) = 12 = 2 (6) = 2 $\begin{vmatrix} 4 & 3 \\ 2 & 3 \end{vmatrix}$

5. If one row (or column) is a multiple of another row (or column) the value of determinant will be zero.

For example $\begin{vmatrix} 2p & 2q \\ p & q \end{vmatrix}$ = 2pq – 2pq = 0

or $\begin{vmatrix} 4 & 20 \\ 2 & 10 \end{vmatrix}$ = 4 X 10 – 20 X 2 = 40 – 40 = 0

6. If two columns (or rows) are identical, the determinant will be zero.

$\begin{vmatrix} p & q \\ p & q \end{vmatrix}$ = (pq – pq) = 0 or $\begin{vmatrix} 13 & 11 \\ 13 & 11 \end{vmatrix} = 0$ or $\begin{vmatrix} 12 & 4 \\ 12 & 4 \end{vmatrix} = 0$

7. The addition (or subtraction) of a multiple of any row from another will leave the value of the determinant unaltered.

For example $\begin{vmatrix} p & q \\ r & s \end{vmatrix} = (ps - qr)$

And $\begin{vmatrix} p & q \\ r+p & s+q \end{vmatrix} = p\ (s+q) - (r+p)\ q = ps + pq - qr - qp = (ps - qr)$

Similarly $\begin{vmatrix} 4 & 5 \\ 4+3 & 5+2 \end{vmatrix} = 4\ (5+2) - 5\ (4+3) = -7$

(students should verify the result by subtraction)

8. If one row (or column) of a matrix is shifted to p place (up or below or left or right) then value of the determinant is unchanged if p is even; the value of determinant acquires a negative sign if p is odd :

Example :

$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 3 \end{vmatrix} = 6$, change first and second row, then

$|A) = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 1 & 5 & 3 \end{vmatrix} = 2\ (6-15) - 1\ (9-20) + 1\ (9-8)$

$= 2\ (-9) -1\ (-11) + 1 = -18 +11 +1 = - 18 +12 = -6$

9. If one row (or column) of a matrix is multiplied by λ another row (or column) by K then the value of the determinant is multiplied by λK:

Example :

$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 3 \end{vmatrix} = + 6$

Multiply 1st row by 2 and 3rd row by 3 :

$|A_1| = \begin{vmatrix} 2 & 4 & 6 \\ 2 & 3 & 4 \\ 3 & 15 & 9 \end{vmatrix} = 2\ (27 - 60) - 4\ (18 - 12) + 6\ (30 - 9)$

$= 2\ (-33) - 4\ (6) + 6\ (21)$

$= -66 - 24 + 126 = 126 - 90 = 36$

$= 2 \times 3\ |A|$

Thus these basic properties of determinants help in simplifying and evaluating them. Substraction and Interchange of rows (or columns) further help us in reducing our work.

Exercise : (a) Define determinant of a matrix

(b) Define minors and cofactors

2. Find the value of x if $\begin{vmatrix} 1 & 2 & 5 \\ 1 & x & 5 \\ 3 & -1 & 2 \end{vmatrix} = 0$

3. Show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)\ (b-c)\ (c-a)$

INVERSE AND RANK OF MATRICES

1. **Meaning of Inverse of Matrix**
2. **Gauss Elimination method**
3. **Adjoint Method**
4. **Properties of Inverse of matrix**
5. **Elementary Transformation or operations**
6. **Rank of Matrix**
7. **Questions**

There are two methods of finding the inverse of a given square matrix.

(i) Gauss Elimination (or Reduction) method.

(ii) Using Adjoint matrix/co-factor method.

i) Gauss Elimination (or Reduction) Method :

If A is a square matrix of order n, I is an identity matrix of order n, then a particular form of a new matrix : [A/I] (by placing I matrix by the side of matrix-A) is of order 2× 2. A has inverse only and only if [A/I] can be transformed to $[I/A^{-1}]$. The method thus consists in placing an identity matrix of the same order alongside the original matrix A which is required to be inverted. Then by performing the same row elementary transformations on both A and I portions, we can transform (reduce) A into an identity matrix, this transformed identify matrix will then become the inverse matrix.

Example 1:

Calculate inverse of matrix :

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$$

Solution :

Place I matrix of order 2 × 2 by the side of matrix A

$$[A/I] = \left(\begin{matrix} 2 & 3 \\ 4 & 2 \end{matrix} \middle/ \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right)$$

$$\sim \left[\begin{matrix} 1 & \frac{3}{2} \\ 4 & 2 \end{matrix} \middle/ \begin{matrix} \frac{1}{2} & 0 \\ 0 & 1 \end{matrix}\right] \text{ Applying } \frac{R_1}{2}$$

$$\sim \left[\begin{matrix} 1 & \frac{3}{2} \\ 0 & -4 \end{matrix} \middle/ \begin{matrix} \frac{1}{2} & 0 \\ -2 & 1 \end{matrix}\right] \text{ (Applying } R_2 - 4R_1)$$

$$\sim \left[\begin{matrix} 1 & \frac{3}{2} \\ 0 & 1 \end{matrix} \middle/ \begin{matrix} \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{4} \end{matrix}\right] \text{ Applying } \frac{R_2}{-4}$$

$$\sim \left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \middle/ \begin{matrix} -\frac{1}{4} & +\frac{3}{8} \\ \frac{1}{2} & -\frac{1}{4} \end{matrix}\right] \quad \text{Applying} \quad R_1 - \frac{3}{2}R_2$$

Which infact is $[I/A^{-1}]$

$$\therefore A^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{8} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$$

Check if we multiply AA^{-1} we get I_2

$$AA^{-1} = \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{4} & \frac{3}{8} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{3}{2} & \frac{6}{8} - \frac{3}{4} \\ \frac{4}{4} + \frac{2}{2} & \frac{3}{2} - \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

(i) **Using Adjoint of a Matrix** : First we will define adjoint of a matrix

Adjoint of A matrix

$$\text{Let } A = (a_{ij})_{n \times n} \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1i} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2i} & \ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{i1} & a_{i2} & \ldots\ldots & a_{ij} & \ldots\ldots & a_{in} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} & \ldots\ldots & a_{nn} \end{bmatrix}_{n \times n}$$

be a square matrix of order n, so that $|A|$ is the determinant of the nth order

$$\text{i.e. } |A| = \begin{vmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

Let the co-factor of $|A|$ be denoted by the corresponding capital letters A_{11}, A_{12} etc. Thus A_{ij} denote the cofactor of A_{ij} in $|A|$. Let us form the matrix of the cofactor of the corresponding capial letter (A) and denote it by C of (A) or C (A).

$$\text{Thus } C(A) = \begin{vmatrix} A_{11} & A_{12} & \ldots\ldots & A_{1n} \\ A_{21} & A_{22} & \ldots\ldots & A_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ A_{n1} & A_{n2} & \ldots\ldots & A_{nn} \end{vmatrix}$$

C (A) may be called a *cofactor matrix.*

Let us take the transpose of the matrix C (A) so that

$$C^1(A) = \begin{vmatrix} A_{11} & A_{12} & ..... & A_{1n} \\ A_{21} & A_{22} & ..... & A_{2n} \\ ..... & ..... & ..... & ..... \\ ..... & ..... & ..... & ..... \\ A_{n1} & A_{n2} & ..... & A_{nn} \end{vmatrix}'$$

$$= \begin{vmatrix} B_{11} & B_{12} & ..... & B_{1n} \\ B_{21} & B_{22} & ..... & B_{2n} \\ ..... & ..... & ..... & ..... \\ ..... & ..... & ..... & ..... \\ B_{n1} & B_{n2} & ..... & B_{nn} \end{vmatrix} \quad \text{(say)}$$

So that $C'(A) = (A_{ij})_{n \times n} = (B_{ij})_{n\times n}$ where B_{ij} (or A_{ij}) is the cofactor of a_{ij} in (A)

C′(A) is called the adjoint (or adjugate) or matrix A and is generally written as adj.

Let us now give a formal definition of adjoint of a square matrix.

Definition of Adjoint of Matrix : Let $A = [A_{ij}]_{n\times n}$ be a square matrix of order n, then the transpose of the cofactor of the corresponding small letters a_{ij} ′s in [A] is defined as the adjoint (for adjugate) of A and is briefly written as adj. A.

In simple language we can say that adj. A is the transpose of the matrix formed by the co-factors of the element of [A] or simply transpose of the cofactor matrix.

We shall illustrate the concept with examples :

Example 1 : Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Find adj. A

Solution :

Here $[A] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Cofactor of a = d

Cofactor of b = - c

Cofactor of c = - b

Cofactor of d = + a

$\therefore$ C (A) = Cofactor Matrix = $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

$\therefore$ adj A = C′ (A) = $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example 2 : Find adjoint of A = $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Solution :

Here $|A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix}$

Cofactors of the elements of the first row of A are

$+\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}, -\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, +\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$

i.e. 2 – 3 = -1, - (1 – 9) = 8, (1 – 6) = - 5

Cofactors of the elements of the second row of (A) are

$-\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, +\begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}, -\begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}$

i.e. -1 (1 – 2) = 1, (0 – 6) = - 6, – (0 – 3) = 3

Cofactors of the elements of the third row of |A| are

$+\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, -\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}, +\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$

i.e. (3 – 4) = – 1, (0 – 2) = + 2, (0 – 1) = –1

Now C (A) = Cofactor Matrix

$$\begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

$\therefore$ adj. A = C′ (A) = transpose of the cofactor matrix

$$\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

Properties of adjoint of A. Let A = $(a_{ij})_{n\times n}$

1. The product of the matrix and its adjoint is commutative
   i.e. A (Adj. A) = (adj. A) A = $|A|\ I_n$
2. If A is a squared singular matrix (i.e. |A| = o, then
   A (adj. A) = (Adj. A) A = 0 [Null matrix)
3. adj. (AB) = (adj B) (adj A) where A and B are n-squared matrices

4. If A is symmetric, then adj A is also symmetric.
5. If A is Hermitian then adj A is also hermitian.

All these properties can be verified by taking any square matrix. Students are advised to verify these by taking a 3×3 matrix.

Inverse of a Matrix

(Using Adjoint of a matrix)

Let A be any square matrix, then a matrix B if it exists, such that AB = BA = I, B is called the inverse of A. I being the unit matrix and we write $B = A^{-1}$. (A^{-1} to be read as 'A inverse').

Properties of Inverse of A

1. If a matrix A has an inverse, then it is unique.

Let A be n-squared matrix whose inverse exists. Let us suppose that B and C are two inverses of A, then by definition of inverse we have

(a) AB = BA = I B is inverse of A

also (b) AC = CA = I C is inverse of A

$\therefore$ From (a) and (b), it follows that

AB = I which implies C (AB) = CI = C ………………. (1)

And CA = I which implies (CA) B = IB = B ………………. (2)

but by the associative law, we know that

C (AB) = (CA) B

$\therefore$ from (1) and (2) we get

B = C, hence the inverse of matrix is unique.

2. Condition for a square matrix A to possess an inverse is that A is non Singular

i.e. $|A| \neq 0$

Let A be n-squared matrix and B be its inverse, then by definition, we have AB = I (unit matrix)

Taking determinants of both sides

$|AB| = |I|$

But $|AB| = |A| \times |B|$ and $|I| = 1$

$|A| \times |B| = I$

Since R.H.S. is non zero $|A|$ has to be non zero. Hence A is non-singular.

3. If A is non-singular and AB = AC then B = C

Since A is non-singular $\therefore$ A^{-1} exists

Since AB = AC

$\therefore A^{-1} (AB) = A^{-1} (AC)$

or $(A^{-1} A) B = (A^{-1} A) C$

or IB = IC

$\therefore B = C$

4. Reversal Law for the inverse of the product :

$(AB)^{-1} = A^{-1} B^{-1}$

i.e. the inverse of the product is the product of the inverse

5. The operations of transposing and inverting are commutative

i.e. $(A')^{-1} = (A^{-1})'$

6. $(A^0)^{-1} = (A^{-1})^0$ A being n × n singular matrix

and $A^0 = A'$ conjugate transpose of A

Remarks 1

Inverse of matrix A exists only if (i) the given matrix is a square matrix and (ii) the determinant of the given matrix is ≠ o, i.e. the matrix is non-singular. In other words:

(i) Every matrix need not have an inverse.

(ii) Every square matrix needn't have an inverse.

(iii) Every square non-singular matrix have an inverse.

Let A = diag (a, b, c) = $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

And B = diag $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right) = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$

Then clearly AB = BA = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ = 1

Hence B is the inverse of A.

Inverse of A diagonal matrix :

Let A = diag (a, b, c) .... Inverse of A is a

Diagonal matrix B = $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$

In general, inverse of a diagonal matrix

A = diag $(a_1, a_2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\, a_n)$ is the diagonal matrix.

B = diag $\left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots\ldots\ldots\ldots \frac{1}{a_n}\right)$

How to find Inverse of Matrix A?

We shall explain the method of finding inverse of a matrix with the help of adjoint. Let the matrix A has an inverse B so that

A is, by definition non-singualr i.e. $|A| \neq 0$

Now B will be inverse of A only

if it satisfies

AB = BA = 1

Let us choose B = $\frac{1}{|A|}$ (adj. A)

Since $|A| \neq 0$, our choosing B as above is justified.

Now AB = A $\left[\frac{1}{|A|}(\text{adj. A})\right]$

= $\frac{1}{|A|}$ A adj. A

= $\frac{1}{|A|}$ ($|A|$ I) $\quad\because$ (A adj A = $|A|$ I)

Similarly BA = I (prove it)

Hence AB = BA = I

which shows that B is the inverse of A.

of $\frac{1}{|A|}$ adj. A is the inverse of A

of $A^{-1} = \frac{1}{|A|}$ adj. A

Thus the necessary and sufficient condition for a matrix A to possess an inverse is that it is non-singular, i.e. |A| # 0

Remarks 2

For finding the inverse of a square matrix A, we shall first find the determinant of A. If $|A| = 0$, inverse doesn't exist. If $|A|$ # 0, we shall find the adjoint matrix and divide it by $|A|$ to get the inverse matrix.

Example 3 :

Find the inverse of A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Solution : We have already calculated adj A in example 1 above

i.e. adj A = $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

also $|A|$ = ad – bc

Assuming A to be non-zero i.e. ad$\neq$ bc

$$A^{-1} = \frac{adjA}{|A|} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Verification : AA^{-1} should be 1

Here $AA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$= \frac{1}{ad-bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & -ab+ab \\ cd-cd & -bc+ad \end{bmatrix}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} ad - bc & 0 \\ 0 & -bc+ad \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

Hence $A^{-1} = = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example 4 :

Find the inverse of the matrix : $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}_{3\times3}$

Solution : Let $|A| = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

$= 0 \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} -1 \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} +2 \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$

= 0 (2-30 -1 (1-9) + 2 (1-6)

= 0 + 8 – 10 = -2

Cofactor of different elements :

$C_{11} = (-1)^{1+1} = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = + (2 – 3) = -1$

$$C_{12} = (-1)^{1+2} = \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = (1 - 9) = +8$$

$$C_{13} = (-1)^{1+3} = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = +(1 - 6) = -5$$

$$C_{21} = (-1)^{2+1} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -(1 - 2) = +1$$

$$C_{22} = (-1)^{2+2} = \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = +(0 - 6) = -6$$

$$C_{23} = (-1)^{2+3} = \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = -(0 - 3) = +3$$

$$C_{31} = (-1)^{3+1} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = +(3 - 4) = -1$$

$$C_{32} = (-1)^{3+2} = \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = -(0 - 2) = +2$$

$$C_{33} = (-1)^{3+3} = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = +(0 - 1) = -1$$

The matrix of co-factor of different elements is

$$= \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

$$\text{Adj. A} = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}_{3\times3}$$

$$A^{-1} = \frac{\text{adj A}}{|A|} \begin{bmatrix} +\frac{-1}{-2} & \frac{1}{-2} & \frac{-1}{-2} \\ \frac{+8}{-2} & \frac{-6}{-2} & \frac{2}{-2} \\ \frac{-5}{-2} & \frac{3}{-2} & \frac{-1}{-2} \end{bmatrix}_{3\times3}$$

$$A^{-1} = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -4 & 3 & -1 \\ 2.5 & -1.5 & 0.5 \end{bmatrix}_{3\times3}$$

Example : 5

Find the adjoint of the following matrix. Hence or otherwise find the inverse of

$$A=\begin{bmatrix}1 & 0 & -1\\ 3 & 4 & 5\\ 0 & -6 & -7\end{bmatrix}$$

Solution :

$$\text{Here } A = \begin{bmatrix}1 & 0 & -1\\ 3 & 4 & 5\\ 0 & -6 & -7\end{bmatrix}$$

|A| = IC(1) – 0C + (-1) C (-1)

= 1 (-28 + 30) – 0 (-21 + 0) + (-1) (-18 + 0)

1 (2) + 0(21) + (-1) (-18)

= 2 + 0 + 18

= 20

we shall first find the cofactors of the elements of the first row of |A|.

$$+\begin{bmatrix}4 & 5\\ -6 & -7\end{bmatrix}, -\begin{bmatrix}3 & 5\\ 0 & -7\end{bmatrix}, +\begin{bmatrix}3 & 4\\ 0 & -6\end{bmatrix}$$

i.e. = 2, 21, 18

Cofactor of the elements of the second row of |A|

$$-\begin{bmatrix}0 & -1\\ -6 & -7\end{bmatrix}, +\begin{bmatrix}1 & -1\\ 0 & -7\end{bmatrix}, -\begin{bmatrix}1 & 0\\ 0 & -6\end{bmatrix}$$

i.e. 6 -7 +6

Cofactor of the elements of the third row of |A|

$$+\begin{bmatrix}0 & -1\\ 4 & 5\end{bmatrix}, -\begin{bmatrix}1 & -1\\ 3 & 5\end{bmatrix}, +\begin{bmatrix}1 & 0\\ 3 & 4\end{bmatrix}$$

i.e. 4 -8 4

Hence C (A) = Cofactor of A

$$= \begin{bmatrix}2 & 21 & -18\\ 6 & -7 & 6\\ 4 & -8 & 4\end{bmatrix}$$

and

adj A = C′ (A) = Transpose of the cofactor matrix

$$= \begin{bmatrix}2 & 6 & 4\\ 21 & -7 & -8\\ -18 & 6 & 4\end{bmatrix}$$

Since $|A| = 20$ [$\neq 0$]

$$A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{20}\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{20} & \frac{6}{20} & \frac{4}{20} \\ \frac{21}{20} & \frac{-7}{20} & \frac{-8}{20} \\ \frac{-18}{20} & \frac{6}{20} & \frac{4}{20} \end{bmatrix}$$

Verification :

AA^{-1} should be I

$$\text{Here } AA^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix} \frac{1}{20}\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} 2+0+18 & 6-0-6 & 4-0-4 \\ 6-84-90 & 18-28+30 & 12-32+30 \\ 0-126+126 & 0+42-42 & 0+48-28 \end{bmatrix}$$

$$= \frac{1}{20}\begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$$

$$\text{Here } A^{-1} = \frac{1}{20}\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{20} & \frac{6}{20} & \frac{4}{20} \\ \frac{21}{20} & \frac{-7}{20} & \frac{-8}{20} \\ \frac{-18}{20} & \frac{6}{20} & \frac{4}{20} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{10} & \frac{3}{10} & \frac{1}{5} \\ \frac{21}{20} & \frac{-7}{20} & \frac{-2}{5} \\ \frac{-9}{10} & \frac{3}{10} & \frac{1}{5} \end{bmatrix}$$

Example 6 :

Find the inverse of A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

Here $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$

= 1C (1) – 2C (2) + 3C (3)
= 1 (15 – 16) – 2 (10 – 12) + 3 (8 – 9)
= 1 (-1) – 2 (-2) + 3 (-1)
= -1 +4 – 3
= 0

Since $|A| = 0$ therefore A^{-1} doesn't exist.

Remark 3 :

Hence we needn't find adj A. It should be noted that we shall always find $|A|$ first. If $|A| \neq 0$. only then we should proceed further to find adj. A

Example 7 :

Find the inverse of A = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution :

Here $|A| = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$

$$C(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{and adj } A = C'(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A$$

In this example A has its own inverse.

Elementary Transformation or Operations :

We shall now discuss elementary transformation on a matrix which will neither change its order nor rank. If the transformation is applied to rows, it is called row transformation and if it is applied to columns, it is called column transformation. There are three types of transformation as given below :

1. Interchanging of any two rows (or columns)

Notation : R_{ij} stands for a row transformation in which ith and jth rows of a matrix are interchanged. Similarly C_{ij} stands for column transformation in which ith and jth column of matrix are interchanged.

For example if $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$

Applying R_{23} on A, we get

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 6 & 7 & 8 \\ 3 & 4 & 5 \end{bmatrix}$$

Applying C_{12} on B, we get

$$C = \begin{bmatrix} 2 & 1 & 0 \\ 7 & 6 & 8 \\ 4 & 3 & 5 \end{bmatrix}$$

Applying C_{12} on A, we get

$$D = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & 5 \\ 7 & 6 & 8 \end{bmatrix}$$

2. **Notation R_i (a)** stands for rows transformation in which elements of the ith row of the matrix are multiplied by a. Similarly Ci (a) stands for column transformation in which elements of the i^{th} columns of the matrix are multiplied by a.

For example, if $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$

Applying R_2 (4) on A, we get

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 3\times 4 & 4\times 4 & 5\times 4 \\ 6 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 12 & 16 & 20 \\ 6 & 7 & 8 \end{bmatrix}$$

3. **Addition to the elements of any row (or column) K times the corresponding elements of any other row (or column)_ where K is a non-zero scalar.**

Notation : R_{ij} (K) stands for row transformation in which the elements of the ith row multiplied by k and then added to the corresponding elements of the jth row. Similarly C_{ij} (k) stands for column transformation.

For example, if $A = \begin{vmatrix} 1 & 2 & 0 \\ 3 & -4 & 5 \\ 6 & 7 & -8 \end{vmatrix}$

Applying R_{12} (4) on A or R_1 + R_2(4), we get

$$A = \begin{vmatrix} 13 & -14 & 20 \\ 3 & -4 & 5 \\ 6 & 7 & -8 \end{vmatrix}$$

Applying C_{23} (-2) on A, we get

$$C = \begin{vmatrix} 1 & 2-0 & 0 \\ 3 & -4-10 & 5 \\ 6 & 7+16 & -8 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 \\ 3 & -14 & 5 \\ 6 & 23 & -8 \end{vmatrix}$$

Equivalent Matrices :

Two matrices A and B are said to be equivalent and written as A ~ B if one can be obtained from the other by the application of a number of elementary (row of column) transformations. If only row transformations are applied, then B is said to be row equivalent to A and written as ARB or A ~ B. Similarly if any column transformation is applied, then B is said to be column equivalent to A and is written as BCA or B ~ A.

The Inverse Elementary Transformation :

If by an elementary transformation on a matrix A we get an equivalent matrix B, then the elementary transformation which when applied on B gives the matrix A, will be called the inverse elementary transformation.

1. R_{ij} stands for interchanging ith and jth rows of A and we get B. Now if on B we apply R_{ji} we will get back A. Thus inverse transformation of R_{ij} is R_{ji}. Similarly inverse C_{ij} is C_{ji}.

2. $R_i(\alpha)$ stands for multiplying the ith row of A by α to get B. Now if on B, apply $R_i\left(\frac{1}{\alpha}\right)$ we will get back the matrix A. Thus inverse transformation of Ri, (α) is $R_i\left(\frac{1}{\alpha}\right)$, similarly inverse of $C_j(\alpha)$ is $C_j\left(\frac{1}{\alpha}\right)$.

Elementary Matrix

A Matrix obtained from a unit (or identity) matrix by subjecting it to any of the elementary transformation is called elementary matrix.

Thus if $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Applying R_{23} (or C_{23}) we get

$\therefore I_1 = E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$I_1 = E_{23}$ = called the elementary matrix, obtained by interchanging 2^{nd} and 3^{rd} rows (columns) of I.

Similarly $I_2 = E_2(3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $I_3 = E_{13}(4) = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

If the elementary matrix obtained by multiplying the 2nd row (column) of matrix by 3 is elementary matrix obtained by multiplying the 3rd row (column) of matrix by 4 and the adding it to the first row. These are the three types of elementary matrices.

Properties :

1. Every elementary row (column) transformation of a matrix (not unit matrix) be affected by pre-multiplication (post-multiplication) with the corresponding elementary matrix.

2. Two matrices A and B are equivalent if and only if there exists non singular matrices P × Q such that PAQ = B
3. Every Non singular square matrix can be expressed as the product of elementary matrix.
4. Inverse from Elementary Matrices : If A is reduced to 1 by a sequence of row transformation only, then A^{-1} is equal to the product of corresponding elementary matrices in reverse order.

Rank of Matrix

We shall first define minor of matrix.

Let A be a m × n matrix i.e.e $A = (a_{ij})_{m \times n}$

If we retain any r (r < m, n) rows and r columns of A we shall have a square sub-matrix of order r. The determinant of the square sub-matrix of order r is called a minor of A or order r. From the given matrix A, we can form square sub-matrices of order.

1, 2, 3, ...m if m < n and order n

1, 2, 3, ...n if n < m

Hence the minors of matrix A of order m × n will be of order 1, 2 ...., m if m < n and of order 1, 2, ...n, if n < m. Thus, if we have matrix A of order 3 × 4, then we can have minors of order 1, 2 and 3. We can't have minor of order 4.

For example let $A = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{vmatrix}_{3\times4}$

1. Minors of A order 1

Each element is a minor of order 1.

2. Minors of A of order 2

Retain any two rows/columns of A the determinant of the square sub-matrices or order 2, thus formed are called minors of order 2 of A.

e.g. $\begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix} \begin{vmatrix} 1 & 3 \\ 5 & 7 \end{vmatrix} \begin{vmatrix} 1 & 4 \\ 5 & 8 \end{vmatrix} \begin{vmatrix} 1 & 4 \\ 9 & 10 \end{vmatrix} \begin{vmatrix} 1 & 4 \\ 9 & 12 \end{vmatrix}$ etc.

are called minors of order 2 of A.

3. Minors of A of order 3

Retain any three rows and three columns of A and the determinant of square sub-matrices of order 3, thus formed are called minors of A of order 3.

e.g. $\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{vmatrix} \begin{vmatrix} 1 & 3 & 4 \\ 5 & 7 & 8 \\ 9 & 10 & 12 \end{vmatrix}$

$$\begin{vmatrix} 1 & 2 & 4 \\ 5 & 6 & 8 \\ 9 & 10 & 12 \end{vmatrix} \begin{vmatrix} 2 & 3 & 4 \\ 6 & 7 & 8 \\ 10 & 11 & 12 \end{vmatrix}$$

are minors of A of order 3.

Rank of a Matrix

Let $A = (a_{ij})_{m \times n}$ be a given matrix of the type of mxn. Then rank of A, to be written as φ (A), is defined to be $r \le \min(m, n)$ [i.e. r is less than or equal to minimum of m and n. if

(i) Every minor of order r + 1 of A is zero and

(ii) There exists at least one minor of order r, which is non-zero.

In other words φ (A) = r

(iii) Every square sub-matrix of order r + 1 of A is singular, and

(iv) There is at least one square sub-matrix of order r which is non-singular.

Remark 4

(a) If only (i) holds, it implies that $\varphi(A) \le r$

If only (ii) holds, it implies that $\varphi(A) \ge r$

If both (i) and (ii) holds simultaneously, we get $\varphi(A) = r$

(b) The rank of null i.e. (zero) matrix is zero.

(c) The rank of non singular matrix is always ≥ 1.

(d) The rank of a n-squared non-singular matrix is n.

(e) The rank of a singular square matrix of order n is less than n.

Working Rule for Rank of a Matrix

Calculate the minors of highest possible order of a given matrix A. If at least one of these is non zero then the order of minor is the rank. If all the minors are zero, then calculate minors of the next lower order. If at least one of them is non-zero, then this next lower order will be the rank. If however, all the minors of the next order are zero, then calculate minors or still next lower order and so on.

Example 8 : Find the rank of the following matrices :

(i) $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix}$ (ii) $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(iii) $C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ (iv) $D = \begin{bmatrix} 1 & 3 & 4 & -2 \\ 2 & 6 & 8 & -2 \\ 3 & 0 & 3 & 4 \end{bmatrix}$

Solution :

(i) Since A is 2×3, $\varphi(A) \le 2$

Minors of order 2 are

$M_1 = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$, $M_2 = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$, $M_3 = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$

$|M_1| = 0,\ |M_2| = 0,\ |M_3| = 0$

Since all the minors of order 2 are zero

$\therefore \varphi\ (A) \leq 2$

Since at least one minor of order 1 (say 1) is non zero

$\therefore \varphi\ (A) = 2$

(ii) $|B| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = I\ (\neq 0)$

Since $B_{3\times3}$ is non-singular $\therefore$(B) = 3

(iii) $|C| = 0 \ \therefore \varphi(C) \leq 2$ also, all minors of order 2 are also zero

$\therefore \varphi\ (C) \leq 2$

But the matrix C is a non-zero matrix and at least one minor of order 1 is non zero.

$\therefore \varphi\ (C) = 1$

(iv) Since D is of order $3 \times 4 \ \therefore\ p^1\ (D) \leq 3$

All the 3×3 minors of D are

$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & -2 \\ 2 & 6 & -2 \\ 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & -2 \\ 2 & 8 & -2 \\ 3 & 3 & 8 \end{bmatrix} \begin{bmatrix} 3 & 4 & -2 \\ 6 & 8 & -4 \\ 0 & 3 & 3 \end{bmatrix}$

i.e. $\Delta_1 = 0,\ \Delta_2 = 0,\ \Delta_3 = 0,\ \Delta_4 = 0$

since determinant of each of the 3×3 minor is 0, $\therefore\ \varphi\ (D) < 3$

now consider 2×2 minors of D.

There exist at least one non-zero minor of order 2 of, viz

$= \begin{bmatrix} 2 & 6 \\ 3 & 0 \end{bmatrix} = -18 \neq 0$ which is not zero, Hence $\varphi(D) = 2$.

Remarks 5 :

If all the second order minors of D had also been zero, then p(D) would have been one because D was a non-zero matrix.

Exercise :

(1) Find the adjoint, inverse and rank of the following matrices :

(i) $= \begin{bmatrix} 2 & 1 & 0 \\ 0 & -3 & 1 \\ -1 & -3 & 4 \end{bmatrix}$ (ii) $= \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ (iii) $= \begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & 0 \\ 2 & 6 & 5 \end{bmatrix}$

(2) Find the rank of the following matrices :

(i) $= \begin{bmatrix} 1 & 5 & 9 \\ 4 & 8 & 12 \\ 7 & 11 & 15 \end{bmatrix}$ (ii) $= \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 2 & 12 & 9 \\ -1 & -3 & -4 & -3 \end{bmatrix}$

(iii) $= \begin{bmatrix} 1 & 2 & -4 & 5 \\ 2 & -1 & 3 & 6 \\ 8 & 1 & 9 & 7 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 0 & -5 & 6 \\ 3 & -2 & 1 & 2 \\ 5 & -2 & -9 & 14 \\ 4 & -2 & -3 & 8 \end{bmatrix}$

The normal form of a matrix

By means of a elementary transformation every matrix A of order m × n and rank (>0) can be reduced to one of the following forms :

(i) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ $|I_r|$ $|0|$ $|I_r|$

Example : Reduce the matrix A to its normal form where

$$A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

and hence determine its rank.

Solution :

We shall indicate the transformations employed at every stage just below the equivalence sign. So

$$A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow C_{12}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 4 & 8 & 6 \\ 1 & 2 & 4 & 1 \end{bmatrix} \rightarrow C_{31}\ (1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 8 & 6 \\ 1 & 2 & 4 & 3 \end{bmatrix} \rightarrow C_{41}\ (2)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 8 & 6 \\ 1 & \frac{1}{2} & 4 & 3 \end{bmatrix} \rightarrow C_2\left(\frac{1}{4}\right) \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 8 & 6 \\ 1 & \frac{1}{2} & 4 & 3 \end{bmatrix} R_{31}(-1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 8 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow R_{32}\left(-\frac{1}{2}\right)$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} C_3\left(\frac{1}{8}\right), C_4\left(\frac{1}{6}\right)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} C_{32}(-1), C_{42}(-1)$$

$$\sim \begin{bmatrix} I_2 & 0 \\ 0 & 1 \end{bmatrix}$$ is the normal form of A.

Rank (A) = 2

1. **Exercise :** Using Gauss Reduction Method or Elementary Operation, find the inverse of the following matrices :

(a) $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix}$

2. Find rank of the following matrices

(i) $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 2 & 1 & 6 \\ 2 & 4 & 2 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$

SOLUTION OF SIMULTANEOUS EQUATIONS

I. Introduction
II. Objectives
III. Methods for solving simultaneous Equations.
  (i) Crammer's Rule
  (ii) Matrix Inverse Method
IV. Summary
V. Questions
VI. Suggested Readings

Introduction

The standard form of linear equations in two variables x any y are

$a_1 x + b_1 y + c_1 = 0$ .... (1)

$a_2 x + b_2 y + c_2 = 0$ .... (2)

These together are called simultaneous equations as there will be only one pair of values satisfying both the equations simultaneously

Now consider a system of n linear equations in n unknowns $x_1, x_2 \ldots\ldots X_n$.

$a_{11} x_1 + a_{12} x_2 + \ldots\ldots\ldots\ldots a_{1n} x_n = b_1$

$a_{21} x_1 + a_{22} x_2 + \ldots\ldots\ldots\ldots a_{2n} x_n = b_2$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$a_{n1} x_1 + a_{n2} x_2 + \ldots\ldots\ldots\ldots + a_{nn} x_n = b_n$

These equations may be written in the matrix form as AX = B

$$\text{Where } A = \begin{bmatrix} a_{11}a_{12}\ldots\ldots\ldots\ldots a_{1n} \\ a_{21}a_{22}\ldots\ldots\ldots\ldots a_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}a_{n2}\ldots\ldots\ldots\ldots a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \ldots\ldots \\ \ldots\ldots \\ x_n \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \ldots\ldots \\ \ldots\ldots \\ b_n \end{bmatrix}$$

Now if A is non singular

$X = A^{-1} B$

The result $X = A^{-1} B$ gives us a solution to the above set of simultaneous equations

Let us consider simple case of 3 simultaneous linear equations

$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$

$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$

$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3$

These equations can be expressed as

$$AX = B \text{ where } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

II. Objectives :

The main objective of this lesson is to find solution of Linear Simultaneous Equations by using two different methods.

III. Methods for solving Simultaneous Equations

Two methods of solving simultaneous equations which are as follows:

(i) Crammer's Rule

(ii) Matrix Inverse Method

1. Crammer's Rule :

Let us consider simple case of 3 simultaneous equations

$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$ .... (i)

$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$ .... (ii)

$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3$ .... (iii)

Put these equations in matrix form as AX = B where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Let A_{ij} be the co-factor of a_{ij} in $|A|$

Multiply (i), (ii) and (iii) by A_{11}, A_{21}, A_{31} respectively and add

$x_1 (a_{11} A_{11} + a_{21} A_{21} + a_{31} A_{31}) + x_2 (a_{12} A_{11} + a_{22} A_{21} + a_{32} A_{31}) + x_3 (a_{13} A_{11} + a_{23} A_{21} + a_{33} A_{31}) = b_1 A_{11} + b_2 A_{21} + b_3 A_{31}$ .... (iv)

From the properties of determinant as we know that the sum of products of elements of a row (or column) with co-factors of elements of corresponding row (or column) is $|A|$ and sum of product of elements of a row (or column) with co-factors of elements of other row (or column) is zero

From (iv), $x_1 \, |A| = b_1 A_{11} + b_2 A_{21} + b_3 A_{31}$

$$\text{i.e. } x_1 \, |A| = b_1 \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} + b_2 \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} + b_3 \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

$$\text{i.e. } x_1 \, |A| = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$$

$$X_1 = \frac{1}{|A|} \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$$

$$x_1 = \frac{|A_1|}{|A|}, \ |A| \neq 0$$

Similarly $x_2 = \frac{1}{|A|}\begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & a_2 & a_{23} \\ a_{31} & a_3 & a_{33} \end{bmatrix} = \frac{|A_2|}{|A|}, \ |A| \neq 0$

$$x_3 = \frac{1}{|A|}\begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix} = \frac{|A_3|}{|A|}, \ |A| \neq 0$$

Note 1 : To obtain the numerator of values of x_1, x_2, x_3 we replace by b_1, b_2, b_3 the elements of first, second and third columns respectively in $|A|$, which forms the denominator in each case.

Note 2 : In general, we may say that if the determinant of the matrix of co-efficient of the system of following linear equations

$a_{11}\, x_1 + a_{12}x_2 + \text{..........................} + a_{1n}x_n = b_1$

....

....

$a_{n1}\, x_1 + a_{n2}x_2 + \text{..........................} + a_{nn}x_n = b_n$

is not zero, then unique solution of X_i is given by

$$X_i = \frac{\begin{bmatrix} a_{11}\text{.......}a_{1i-1} & b_1 & a_{1i+1}\text{.......}a_{1n} \\ \text{................} & \text{..........} & \text{..................} \\ \text{................} & \text{..........} & \text{..................} \\ a_{n1}\text{.......}a_{ni-1} & b_n & a_{ni+1}\text{.......}a_{nn} \end{bmatrix}}{|A|}$$

The result is known as Crammer's Rule

Thus Crammer's Rule is not applicable if $|A| = 0$

Example 1 : Solve the following equations in the matrix form as

AX = B

Where $A = \begin{bmatrix} 9 & 1 \\ 8 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $B = \begin{bmatrix} 13 \\ 16 \end{bmatrix}$

By Crammer's Rule

$$x_1 = \frac{|A_1|}{|A|}, \ x_2 = \frac{|A_2|}{|A|}$$

Now $|A| = \begin{bmatrix} 9 & 1 \\ 8 & 2 \end{bmatrix} = 18 - 8 = 10$

$|A_1| = \begin{bmatrix} 13 & 1 \\ 16 & 2 \end{bmatrix} = 26 - 16 = 10$

$|A_2| = \begin{bmatrix} 9 & 13 \\ 8 & 16 \end{bmatrix} = 9 \times 16 - 8 \times 13 = 144 - 104 = 40$

$x_1 = \dfrac{10}{10} = 1$

$x_2 = \dfrac{40}{10} = 4 \therefore x_1 = 1 \; x_2 = 4$

Example 2 :

Consider the following systems of equations :

$2x_1 - x_2 + 3x_3 = 9$

$x_2 - x_3 = -1$

$x_1 + x_2 - x_3 = 0$

Put these equations in matrix form as

$AX = B$

Where $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 9 \\ -1 \\ 0 \end{bmatrix}$

$|A| = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix} = 2\,(-1 + 1) + 1\,(+1) + 3\,(-1) = -2$

$|A_1| = \begin{bmatrix} 9 & -1 & 3 \\ -1 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} = 9\,(-1 + 1) + 1\,(1 - 0) + 3\,(-1) = -2$

$|A_2| = \begin{bmatrix} 2 & 9 & 3 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix} = 2\,(1) - 9\,(+1) + 3\,(+1) = 2 - 9 + 3 = -4$

$$|A_3| = \begin{vmatrix} 2 & -1 & 9 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix} = 2\ (0 + 1) + 1(0 + 1) + 9\ (0 - 1) = 2 + 1 - 9 = -6$$

$$x_1 = \frac{|A_1|}{|A|} = \frac{-2}{-2} = 1,\ x_2 = \frac{|A_2|}{|A|} = \frac{-4}{-2} = 2,\ x_3 = \frac{|A_3|}{|A|} = \frac{-6}{-2} = 3$$

$x_2 = 1,\ x_2 = 2,\ x_3 = 3$

(2) Matrix Inverse Method

Another method for solving system of simultaneous equations is known as matrix inverse method. As the name suggests we have to find the inverse of the matrix.

Consider system of three simultaneous Linear equations in three unknowns

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

The given simultaneous equations can be written in matrix form as

AX = B

$$\text{Where } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},\ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$X = A^{-1} B$ (provided $|A| \neq 0$)

Where A^{-1} is the inverse of the matrix

To obtain A^{-1} we use the result

$$A^{-1} = \frac{adj A}{|A|} = \frac{\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}}{|A|}$$

Now we will consider some examples using matrix inverse method to solve simultaneous linear equations.

Example 3 : Consider two simultaneous equations with two variables

$2x - 3y = 3$

$4x - y = 11$

The given simultaneous equations can be written in matrix form as :

AX = B

$$\text{Where } A = \begin{bmatrix} 2 & -3 \\ 4 & -1 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \end{bmatrix},\ B = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$$

Now $|A| = \begin{vmatrix} 2 & -3 \\ 4 & -1 \end{vmatrix} = -2 - (-12) = 10 \neq 0$

To obtain A^{-1} we use the result

$A^{-1} = \left[\dfrac{adj A}{|A|}\right]$

adj A = Transpose of the matrix of co-factors A_{ij}.

Now $A_{11} = (-1)^{1+1}(-1) = -1$

$A_{12} = (-1)^{1+2}(4) = -4$

$A_{21} = (-1)^{2+1}(-3) = 3$

$A_{31} = (-1)^{2+2}(2) = 2$

Adj A = $\begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -4 & 2 \end{bmatrix}$ $A^{-1} = \dfrac{1}{10}\begin{bmatrix} -1 & 3 \\ -4 & 2 \end{bmatrix}$

$X = A^{-1} B = \dfrac{1}{10}\begin{bmatrix} -1 & 3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 11 \end{bmatrix} = \dfrac{1}{10}\begin{bmatrix} -3+33 \\ -12+22 \end{bmatrix}$

Or $\begin{bmatrix} x \\ y \end{bmatrix} = \dfrac{1}{10}\begin{bmatrix} 30 \\ 10 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Hence x = 3, y = 1

Example 4 : Let us consider the following equations :

$5x - 6y + 4z = 15$

$7x + 4y - 3z = 19$

$2x + y + 6z = 46$

The above system in the matrix form can be written as

$$\begin{bmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$$

Now AX = B

$X = A^{-1} B$ .... (i)

We know that

$A^{-1} = \dfrac{adj A}{|A|}$, where $|A| = \begin{vmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{vmatrix}$

$= 5\begin{bmatrix} 4 & -3 \\ 1 & 6 \end{bmatrix} + 6\begin{bmatrix} 7 & -3 \\ 2 & 6 \end{bmatrix} + 4\begin{bmatrix} 7 & 4 \\ 2 & 1 \end{bmatrix}$

$= 5\ (24 + 3) + 6\ (42 + 6) + 4\ (7 - 8)$

$= 5\ (27) + 6\ (48) + 4(-1)$

$|A| = 135 + 288 - 4 = 419$

$$\text{adj A} = \text{transpose of} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Where :

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 4 & -3 \\ 1 & 6 \end{vmatrix} = 24 + 3 = 27$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 7 & -3 \\ 2 & 6 \end{vmatrix} = -\ (42 + 6) = -48$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 7 & 4 \\ 2 & 1 \end{vmatrix} = (7 - 8) = -1$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} -6 & 4 \\ 1 & 6 \end{vmatrix} = -(-36 - 4) = 40$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 5 & 4 \\ 2 & 6 \end{vmatrix} = (30-8) = 22$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 5 & -6 \\ 2 & 1 \end{vmatrix} = -(5 + 12) = -17$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} -6 & 4 \\ 4 & -3 \end{vmatrix} = (18 - 16) = 2$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 5 & 4 \\ 7 & -3 \end{vmatrix} = -(-15 - 28) = 43$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 5 & -6 \\ 7 & 4 \end{vmatrix} = (20 + 42) = 62$$

$$\therefore \text{Adj A} = \begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}$$

$$A^{-1} = \frac{adj\text{A}}{|\text{A}|} = \frac{1}{419} \begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}$$

From (1), we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{419}\begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}\begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{419}\begin{bmatrix} 27\times15 & +40\times19 & +2\times46 \\ -48\times15 & +22\times19 & 43\times46 \\ -1\times15 & -17\times19 & +62\times46 \end{bmatrix}$$

$$= \frac{1}{419}\begin{bmatrix} 1257 \\ 1676 \\ 2514 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$ Hence x = 3, y = 4, z = 6

IV. Summary :

In the present lesson we have studied two different methods of solving simultaneous linear equations Under Crammer's Rule we put the simultaneous equations in matrix form and then solve for the variables. Crammer's rule is not applicable if determinant of the matrix is equal to zero. Under matrix inverse method, again the system of linear equations are put in matrix form and then by finding inverse of the matrix we find the solutions.

V. Questions :

1. Solve the following equations by Crammer's Rule

   $x + y + z = 1$, $ax + by + cz = k$, $a^2x + b^2y + c^2z = k^2$

2. Solve the following equations using Matrix Method

   $5x - 6y + 4z = 15$

   $7x + 4y - 3z = 19$

   $2x + y + 6z = 46$

3. Solve using Matrix Inverse Method

   $2x - 4x + 3z = 3$

   $4x - 6y + 5z = 2$

   $-2x + y - z = 1$

4. Solve the following Linear Equations by (using Crammer's Rule) :

   $x + y - z + 2 = 0$

   $x - 2y + z - 3 = 0$

   $2x - y - 3z + 1 = 0$

5. Solve (using Crammes's Rule)

   $x + 2y + 3z = 11$

   $2x - y + 4z = 13$

   $3x + 4y - 5z = 3$

VI. Suggested Readings :

1.	An introduction to Mathematical Economics	:	D. Bose
2.	Quantitative Techniques for Management	:	G.C. Sharma and Madhu Jain
3.	Mathematics for students of Economics	:	Bhardwaj and Sabharwal
4.	Mathematics for Students of Economics	:	Aggarwal and Joshi

LESSON NO. 2.5 **AUTHOR : DR. VIPLA CHOPRA**

APPLICATION OF SIMULTANEOUS EQUATIONS IN ECONOMICS

I. Introduction

In the previous lesson we have studied two methods for solving simultaneous linear equations in two/three variables. Now we will solve economic problems using simultaneous equations.

II. Applications of Simultaneous Equations using :

1. Crammer's Rule

We have already studied Crammer's Rule in solving simultaneous equations in previous lesson. Now we will consider its application in solving some economic problems.

Example 1 : Consider the following national income determination model :

$Y = C + I + G$

$C = a + b\,(Y - T)$

$T = d + tY$ where Y (national income)

C (consumption expenditure) and T (tax collection) are endogenous variables; I (Investment) and G (Govt. Expenditure) are exogenous variables; t is income tax rate.

Solve for the endogeneous variables, using Crammer's Rule :

Sol. The Given equations are :

$Y - C - o.T. = I + G$ .... (i)

$bY - C - b.T = -a$ .... (ii)

$ty - o.C - T = -d$ .... (iii)

These can be written as

$$\begin{bmatrix} 1 & -1 & 0 \\ b & -1 & -b \\ t & 0 & -1 \end{bmatrix} \begin{bmatrix} y \\ C \\ T \end{bmatrix} = \begin{bmatrix} 1+G \\ -a \\ -d \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 1 & -1 & 0 \\ b & -1 & -b \\ t & 0 & -1 \end{vmatrix} = 1(1) + (-b + bt) \quad = 1 - b + bt$$

$$\Delta_1 = \begin{vmatrix} 1+G & -1 & 0 \\ -a & -1 & -b \\ -d & 0 & -1 \end{vmatrix} = I + G + a - bd$$

$$y = \frac{\Delta_1}{\Delta}$$

Applying Crammer's rule, we get

Y = [I +G + a – bd] / (1 – b + bt)

Similarly $C = \dfrac{a - bd + b\,(1-t)\,(1+G)}{1 - b + bt}$

and $T = \dfrac{at + d - bd + t\,(1+G)}{1 - b + bt}$

Example 2 :

Three products A, B, C are produced after being processed through three departments P_1, P_2 and P_3. The following data are available :

	In P_1	In P_2	In P_3
A	2	5	1
B	1	2	3
C	2	2	3
Max. Time available in hours	1100	1800	1400

Find by matrix method the number of unit produced for each product to have full utilization of capacity.

Let a, b, c denote the number of units of products A, B, C respectively to have full utilization of capacity, then

2a + b + 2c = 1100

5a + 2b + 2c = 1800

a + 3b + 3c = 1400

Its matrix form is

$$\begin{bmatrix} 2 & 1 & 2 \\ 5 & 2 & 2 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1100 \\ 1800 \\ 1400 \end{bmatrix}, \text{ i.e. } AX = B$$

$$Where A = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 2 & 2 \\ 1 & 3 & 3 \end{bmatrix}, X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$B = \begin{bmatrix} 1100 \\ 1800 \\ 1400 \end{bmatrix}$$

Now

$$|A| = \begin{vmatrix} 2 & 1 & 2 \\ 5 & 2 & 2 \\ 1 & 3 & 3 \end{vmatrix} = 2\ (6 - 6)\ -1\ (15 - 2)\ +2(15 - 2) = -\ 13 + 26 = 13$$

By Crammer's Rule :

$$a = \frac{1}{|A|}\begin{vmatrix} 1100 & 1 & 2 \\ 1800 & 2 & 2 \\ 1400 & 3 & 3 \end{vmatrix} = \frac{1}{13}\{(1100(0) - 1800(-3) + 1400\ (-2)\}$$

$$= \frac{1}{13}[5400 - 2800]$$

$$= \frac{2600}{13} = 200$$

$$b = \frac{1}{|A|}\begin{vmatrix} 2 & 1100 & 2 \\ 5 & 1800 & 2 \\ 1 & 1400 & 3 \end{vmatrix} = \frac{1}{13}\{(1100(13) + 1800(4) -1400\ (-6)\}$$

$$= \frac{1300}{13} = 100$$

$$b = \frac{1}{|A|}\begin{vmatrix} 2 & 1 & 1100 \\ 5 & 2 & 1800 \\ 1 & 3 & 1400 \end{vmatrix} = \frac{1}{13}\{(1100(13)\ -1800(5)\ +1400\ (-1)\}$$

$$= \frac{1}{13}(3900) = 300$$

a = 200, b = 100, c = 300

2. Matrix Inverse Method :

Example 3: The daily cost of operating of hospital C is a linear function of the number of in-patients I, and out-patients P, plus a fixed cost a i.e.

C = a + Pb + dI

Given the following data for 3 days, find the values of a, b by setting up a linear system of equations and using the matrix inverse.

Day	Cost (in Rs.)	No. of Inpatients I	No. of out Patients
1	6,950	40	10
2	6,725	35	9
3	7,100	40	12

Substituting the tabulated values in C = a + bP + d1,

We get the following equations :

a + 10b + 40d = 6,950

a + 9b + 35d = 6,725

a + 12b + 40d = 7,100

In the matrix notation, we write

$$\begin{bmatrix} 1 & 10 & 40 \\ 1 & 9 & 35 \\ 1 & 12 & 40 \end{bmatrix} \begin{bmatrix} a \\ b \\ d \end{bmatrix} = \begin{bmatrix} 6,950 \\ 6,725 \\ 7,100 \end{bmatrix}$$

Which is of the form AX = B where

$$A = \begin{bmatrix} 1 & 10 & 40 \\ 1 & 9 & 35 \\ 1 & 12 & 40 \end{bmatrix}, X = \begin{bmatrix} a \\ b \\ d \end{bmatrix} \quad B = \begin{bmatrix} 6,950 \\ 6,725 \\ 7,100 \end{bmatrix}$$

Now

$$|A| = \begin{vmatrix} 1 & 10 & 40 \\ 1 & 9 & 35 \\ 1 & 12 & 40 \end{vmatrix} = 1(360 - 420) - 10\,(40 - 35) + 40\,(12 - 9)$$

$$= -60-50+120=10$$

Since $X = A^{-1}B$

$$\begin{bmatrix} a \\ b \\ d \end{bmatrix} = A^{-1} \begin{bmatrix} 6,950 \\ 6,725 \\ 7,100 \end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

It can be easily verified that

$$\text{adj } A = \begin{bmatrix} 60 & -80 & 10 \\ 5 & 0 & -5 \\ -3 & 2 & 1 \end{bmatrix}$$

Substituting in (2), we get

$$\begin{bmatrix} a \\ b \\ d \end{bmatrix} = -\frac{1}{10} \begin{bmatrix} 60 & -80 & 10 \\ 5 & 0 & -5 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 6,950 \\ 6,725 \\ 7,100 \end{bmatrix}$$

$$= -\frac{1}{10} \begin{bmatrix} 60 \times 6950 - 80 \times 6725 + 7100 \times 10 \\ 5 \times 6950 - 0 \times 6725 - 5 \times 7100 \\ -3 \times 6950 + 2 \times 6725 + 1 \times 7100 \end{bmatrix}$$

$$= -\frac{1}{10}\begin{bmatrix}-50000\\-750\\-300\end{bmatrix} = \begin{bmatrix}5000\\75\\30\end{bmatrix}$$

a = 5000, b = 75, d = 30

Example 4 :

A, B and C have Rs. 480, Rs. 760 and Rs. 710 respectively. They utilized the amounts to purchase three types of shares of prices x, y and z respectively. A Purchases 2 shares of price x, 5 of price y and 3 of price z. B purchases 4 shares of x, 3 of price y and 6 of price z, C purchases 1 share of price x, 4 of price y and 10 of price z. Find x, y and z.

Solution :

The following set of simultaneous linear equations are constructed by writing given information :

$2x + 5y + 3z = 480$

$4x + 3y + 6z = 760$

$x + 4y + 10z = 710$

In the matrix form, we write

$$\begin{bmatrix}2 & 5 & 3\\4 & 3 & 6\\1 & 4 & 10\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}480\\760\\710\end{bmatrix} \quad \text{i.e. } AX=B$$

$$\text{or} \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2 & 5 & 3\\4 & 3 & 6\\1 & 4 & 10\end{bmatrix}^{-1}\begin{bmatrix}480\\760\\710\end{bmatrix} \quad \text{i.e.} X=A^{-1}B \qquad \text{.............} (1)$$

$$\text{Now } A^{-1} = \frac{adj A}{|A|} \text{ where } |A| = \begin{vmatrix}2 & 5 & 3\\4 & 3 & 6\\1 & 4 & 10\end{vmatrix} = -119 \qquad \text{.............} (2)$$

$$\text{and adj } A = \begin{bmatrix}+6 & -38 & +21\\-34 & +17 & 0\\+13 & -3 & -14\end{bmatrix} \qquad \text{.............} (3)$$

Using (1), (2) and (3) we get :

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = -\frac{1}{119}\begin{bmatrix}+6 & -38 & +21\\-34 & +17 & 0\\+13 & -3 & -14\end{bmatrix}\begin{bmatrix}480\\760\\710\end{bmatrix}$$

$$= -\frac{1}{119}\begin{bmatrix} 6\times 480 - 38\times 760 + 21\times 710 \\ -34\times 480 + 17\times 760 + 0\times 710 \\ 13\times 480 - 3\times 760 - 14\times 710 \end{bmatrix}$$

$$= = -\frac{1}{119}\begin{bmatrix} -11090 \\ -3400 \\ -5980 \end{bmatrix} = \begin{bmatrix} 11090/119 \\ 3400/119 \\ 5980/119 \end{bmatrix}$$

$$x = \frac{11090}{119},\ y = \frac{3400}{119},\ z = \frac{5980}{119}$$

Example 5 : An automobile company uses three types of steel S_1, S_2, S_3 for producing three types of cars C_1, C_2 and C_3. Steel requirements (in tons) for each types of car are given below :

Steel		**Cars**	
	C_1	C_2	C_3
S_1	2	3	4
S_2	1	1	2
S_3	3	2	1

Determine the number of cars of each types which can be produced using 29, 13 and 16 tons of steel of three types respectively.

Sol. The above information can be put in the following terms :

$2c_1 + 3c_2 + 4c_3 = 29$

$1c_1 + 1c_2 + 2c_3 = 13$

$3c_1 + 2c_2 + 1c_3 = 16$

Put in matrix form we have

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 29 \\ 13 \\ 16 \end{bmatrix}$$

AC = B

$$\text{Whre } A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix},\ C = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix},\ B = \begin{bmatrix} 29 \\ 13 \\ 16 \end{bmatrix}$$

Apply Crammer's Rule

$$\Delta_1 = \begin{vmatrix} 29 & 3 & 4 \\ 13 & 1 & 2 \\ 16 & 2 & 1 \end{vmatrix} = 29(1-4) - 3(13-32) + 4(26-16)$$

$= -87 + 47 + 40 = 10$

$$\Delta_2 = \begin{vmatrix} 2 & 29 & 4 \\ 1 & 13 & 2 \\ 3 & 16 & 1 \end{vmatrix} = 2\,(13 - 32) - 29\,(1 - 6) + 4\,(16 - 39)$$

$= -38 + 145 - 92 = 15$

$$\Delta_3 = \begin{vmatrix} 2 & 3 & 29 \\ 1 & 1 & 13 \\ 3 & 2 & 16 \end{vmatrix} = 2(16 - 26) - 3\,(16 - 39) + 29\,(2 - 3)$$

$= -20 + 69 - 29 = 20$

$$\Delta = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 1 & 2 \\ 3 & 2 & 1 \end{vmatrix} = 2\,(1 - 4) - 3\,(1 - 6) + 4\,(2 - 3)$$

$= 2\,(-3) - 3\,(-5) + 4\,(-1)$

$= -6 + 15 - 5 = 5$

$\therefore C_1 = \dfrac{\Delta_1}{\Delta} = \dfrac{10}{5} = 2$

$C_2 = \dfrac{\Delta_2}{\Delta} = \dfrac{15}{5} = 3$

$C_3 = \dfrac{\Delta_3}{\Delta} = \dfrac{20}{5} = 4$

$C_1 = 2,\ C_2 = 3,\ C_3 = 4$

Matrix Inverse Method : The above example can be solved by using matrix inverse method.

$AC = B$

$C = A^{-1} B$

$$\text{Now } A^{-1} = \frac{\text{Adj}A}{|A|},\ = \frac{\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T}{|A|}$$

$$\text{Now } A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$A_{11} = +(1 - 4) = -3,\ A_{21} = -(3 - 8) = 5,\ A_{31} = 6 - 4 = 2$

$A_{12} = -(1 - 6) = 5,\ A_{22} = (2 - 12) = -10,\ A_{32} = -(4 - 4) = 0$

$A_{13} = +(2 - 3) = -1,\ A_{23} = -(4 - 9) = +5,\ A_{33} = 2 - 3 = -1$

$$\text{Adj } A = \begin{bmatrix} -3 & 5 & -1 \\ 5 & -10 & 5 \\ 2 & 0 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 5 & -1 \\ 5 & -10 & 5 \\ 2 & 0 & -1 \end{bmatrix}^T$$

$$C = A^{-1}B$$

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} \frac{-3}{5} & 1 & \frac{2}{5} \\ 1 & -2 & 0 \\ \frac{-1}{5} & 1 & \frac{-1}{5} \end{bmatrix}\begin{bmatrix} 29 \\ 13 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} -\frac{3}{5}\times 29+13+\frac{2}{5}\times 16 \\ 29-26+0 \\ -\frac{29}{5}+13-\frac{1}{5}\times 16 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad \therefore \; C_1 = 2, \; C_2 = 3, \; C_3 = 4$$

Example 6 : Given the following equilibrium conditions in a market

$5p_1 + p_2 + p_3 = 1$
$2p_2 + 2p_3 - 2 = 0$
$3p_1 + p_2 + 4p_3 = 4$

Find equilibrium prices in each market using Crammer's rule and Gauss elimination method.

Solution : Given

$5p_1 + p_2 + p_3 = 1$
$2p_2 + 2p_3 = 2$
$3p_1 + p_2 + 4p_3 = 4$

The above equation can be rewritten as

$5p_1 + p_2 + p_3 = 1$
$0p_1 + 2p_2 + 2p_3 = 2$
$3p_1 + p_2 + 4p_3 = 4$

Put in matrix form :

$$\begin{bmatrix} 5 & 1 & 1 \\ 0 & 2 & 2 \\ 3 & 1 & 4 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

Where A = $\begin{bmatrix} 5 & 1 & 1 \\ 0 & 2 & 2 \\ 3 & 1 & 4 \end{bmatrix}$, X = $\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$, B = $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

Using Crammer's Rule

$$|A| = \begin{vmatrix} 5 & 1 & 1 \\ 0 & 2 & 2 \\ 3 & 1 & 4 \end{vmatrix} = 5(8 - 2) -1\ (0 - 6) +1\ (0 - 6) = 30 + 6 - 6 = 30$$

$$|A_1| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 4 & 1 & 4 \end{vmatrix} = 1(8 - 2) -1\ (8 - 8) + 1\ (2 - 8) = 6 - 6 = 0$$

$$|A_2| = \begin{vmatrix} 5 & 1 & 1 \\ 0 & 2 & 2 \\ 3 & 4 & 4 \end{vmatrix} = 0$$

$$|A_3| = \begin{vmatrix} 5 & 1 & 1 \\ 0 & 2 & 2 \\ 3 & 1 & 4 \end{vmatrix} = 30$$

$$\therefore P_1 = \frac{|A_1|}{|A|} = 0,\ P_2 = \frac{|A_2|}{|A|} = 0,\ P_3 = \frac{|A_3|}{|A|} = \frac{30}{30} = 1$$

IV. Questions :

1. Solve the following system of equations :

$$2x_1 - x_2 + 3x_3 = 9$$
$$x_2 - x_3 = -1$$
$$x_1 + x_2 - x_3 = 0$$

by (i) by the adjoint method of calculating inverse of a matrix.

(ii) by making use of Crammer's Rule

2. Suppose we are given data on price (in Rs. per kg.) of Apples, Potatoes, Onions in the month of October, Nov. and Dec. as follows :

	Apples	**Potatoes**	**Onions**
Oct.	5	1	2
Nov.	3	1.5	1.5
Dec.	4	1	2.5

The family can spend Rs. 21, 18 and 20.5 in Oct. Nov. and Dec. on these items. If the family requires to by the same combination of Apples, Potatoes and Onions in each month, find the quantity of Applies, Potatoes and Onions which the family buys each of these months.

3. The equilibrium condition for the related markets is given by

$$11P_1 - P_2 - P_3 = 31$$
$$-P_1 + 6P_2 - 2P_3 = 26$$
$$-P_1 - 2P_2 + 7P_3 = 24$$

Find the equilibrium price for each market using crammer's rule.

4. Matrix A has X - rows and x+3 columns;
and matrix B has Y-rows and 9-Y columns.
Both AB and BA defined, Find X and Y.

LESSON NO. 2.6 AUTHOR : DR. VIPLA CHOPRA

ARITHMETIC PROGRESSION AND GEOMETRIC PROGRESSION

I. Introduction
II. Objectives
III. Arithmetic Progression (A.P.)
  1. Definition
  2. nth term of an A.P.
  3. Sum of n terms of an A.P.
  4. Arithmetic mean between two quantities
  5. n Arithmetic Means (AMs) between a and b.
IV. Geometric Progression (G.P.)
  1. Definition
  2. nth Term of a G.P.
  3. Sum of n terms of a G.P.
  4. Geometric mean between two quantities
  5. n geometric Means (GMs) between a and b.
V. Summary
VI. Questions
VII. Suggested Readings

I. INTRODUCTION

Before understanding arithmetic and geometric progression it becomes necessary to understand the terms like sequence, series.

A set of numbers arranged according to some definite law is called a sequence. For example $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$..........., $\frac{1}{n}$.............. is a sequence where reciprocals of all natural number have been written in succession.

An expression consisting of the terms of a sequence joined by the sings+ve and or-ve, is called a series. For example 2+4+6+8+ .... +2n+ .... is a series associated with the sequence 2, 4, 6, 8 ...., 2n, ... or simply (2n). The various members of the sequence are known as the terms of the series. Generally, T_n denotes the nth term of a series and S_n denotes the sum of n term of a series.

II. Objectives :

The objectives of present lesson are to

(i) define AP and GP,

(ii) find nth terms of AP and GP,

(iii) find sum of n terms of AP and GP and

(iv) find A.M. and G.M.

First we will learn about arithmetic progression (A.P.) and then geometric progression (G.P.)

III. Arithmetic Progression (A.P.)

Definition

A series in which terms increase or decrease by a common difference is called arithmetical progression. The following series are in Arithmetical Progression (A.P.) :

2, 4, 6, 8, 10 .... common difference = 2

9, 5, 1, -3, -7, ... common difference = -4

If first term is a and common difference is d, then series in A.P. is a, a + d, a+2d, ...

2. n^{th} Term of an A.P.

Let a be the first term and d, the common difference of arithmetical progression, then

First term = T_1 = a = a + (1 – 1) d

Second term = T_2 = a + d = a + (2 – 1) d

Third term = T_3 = a + 2d = a + (3 – 1) d

....

....

nth term = Tn = a + (n – 1) d

Generally, the n^{th} term of an A.P. is called its general term. If an A.P. contains n terms only, the nth term i.e. T_n is the last term of the series and is denoted by l.

$\therefore$ Tn = l = a + (n -1) d.

Example 1 : Find the 20^{th} term of the series 10, 8, 6, 4, ...

Here a = 10, d = 8 – 10 = -2, n = 20

20^{th} term i.e. T_{20} =a + (n – 1) d = 10 + (20 – 1) (-2) = -28

Example 2 : Which term of the series 5, 8, 11, 14, ... is 95?

Let 95 be the nth term : T_n = 95 .... (i)

We know $\therefore$ T_n = a + (n – 1)d .... (ii)

From (i) and (ii) 95 = 5 + (n – 1) 3 = 3n – 3 = 90

$\therefore$ n = 31

$\therefore$ 95 is the 31st term

3. Sums of n terms of an A.P.

Let a be the first term, d the common difference, l the last term and S the sum of n terms.

$\therefore$ S_n = a + (a+d) + (a +2d) + ... + (l – 2d) + (l – d) + l .... (i)

Also S_n = l + (l -d) + (l -2d) + ... + (a + 2d) + (a + d) + a .... (ii)

Add (i) and (ii) we get

$2\,S_n = (a + l) + (a + l) + \ldots\ldots\ldots\ldots\ldots$ n terms

$= n\,(a + l)$

$\therefore\ S_n = \frac{n}{2}(a + l)$ .... (I)

We know $l = a + (n - 1)\,d$

Substituting this value in (I)

$\therefore\ S_n = \frac{n}{2}[2a + (n - 1)\,d]$

Example 3 : Sum up the series

(i) .9 + .91 + .92 + ... to 100 terms

(ii) -6 -2 + 2 .... to 21 terms

(i) The given series is .9 + .91 + .92 + ... to 100 terms

Here $a = .9 = \frac{9}{10}$, $d = .91 - .9 = \frac{91}{100} - \frac{9}{10} = \frac{1}{100}$, $n = 100$

$\therefore\ S_n = \frac{n}{2}[2a + (n - 1)d]$

$$S_{100} = \frac{100}{2}\left[2\times\frac{9}{10}+(100-1)\times\frac{1}{100}\right] = \frac{279}{2}$$

(ii) The given series is

-6 -2 + 2 .... to 21 terms

Here a = -6, d = -2 -(-6) = 4

21st Term = $a + (21 - 1)\,d = a + 20d = -6 + 20 \times 4 = 74 = l$

Here $\therefore\ S_n = \frac{n}{2}(a + l)$

$= \frac{21}{2}(-6 + 74) = 21 \times 34 = 714$

Example 4 :

How many terms of the series 15 + 13 + 11 + 9 + ... be taken to make the sum 55?

Here a = 15

d = -2

Let $S_n = 55$. Find n

We know $S_n = \frac{n}{2}[2a + (n - 1)d]$

$\therefore\ 55 = S_n = \frac{n}{2}[2 \times 15 + (n - 1)(-2)]$

$$55 = \frac{n}{2}[30 - 2n + 2]$$

$$\text{or } 55 = \frac{n}{2}[32 - 2n]$$

$$n^2 - 16n + 55 = 0$$

$$n = \frac{-(-16) \pm \sqrt{(-16)^2 - (4)(55)}}{2.1}$$

$$= \frac{(16) \pm \sqrt{36}}{2} = \frac{22}{2}, \frac{10}{2}, = 11,5$$

Hence the number of terms is 5 or 11

The series is 15 + 13 + 11 ....

The sum of 11 terms will be

15 + 13 + 11 + 9 + 7 + 5 + 3 + 1 – 1 – 3 – 5 = 55

It is self evident that the sum of the last six terms is zero.

$\therefore$ The sum of 5 terms of 11 terms is the same.

4. Arithmetic mean between two quantities a and b :

When three quantities are in A.P., the middle one is said to be the Arithmetic Mean (A.M.) between the other two.

Let x be the A.M. then a, x, b are in A.P. so that $x - a = b - x \therefore x = \frac{a+b}{2}$

5. n A.M.S. between two quantities a and b :

Here n terms are to be inserted between a and b so that a is the first term, b the (n + 2) term.

b = a + (n + 2 – 1) d, if d is the common difference.

$$d = \frac{b-a}{(n+1)}$$

Hence the required means are :

$$a + \frac{b-a}{(n+1)}, \ a + \frac{2(b-a)}{(n+1)}, \ a + \frac{3(b-a)}{(n+1)} \ ...., \ a + \frac{n(b-a)}{(n+1)}$$

Example 5 : Insert 3 arithmetic means between - 18 and 4

Let A_1, A_2, A_3 be 3 A.M.'s between -18 and 4

-18, A_1, A_2, A_3, 4 are in A.P.

a = -18, T_5 = 4 ($\therefore$total number of terms = 5)

$\therefore$ a + 4d = 4

-18 + 4d = 4 $\qquad \therefore$ 4d = 22 $\qquad \therefore d = \frac{22}{4}$

Now $A_1 = a + d = -18 + \frac{22}{4} = -\frac{25}{2}$

$A_2 = a + 2d = -18 + 2\frac{22}{4} = -7$

$A_3 = a + 3d = -18 + 3\frac{22}{4} = -\frac{3}{2}$

IV. GEOMETRIC PROGRESSION (G.P.)

1. Definition :

A series is said to be in Geometrical Progression (G.P.) if the ratio of any term to the proceeding term is constant throughout. This constant factor is called the common ratio. Thus :

1, 2, 4, 8 ....

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$, ...

$1, -\frac{1}{3}, -\frac{1}{9}, -\frac{1}{27}$, ...

a, ar, ar^2,ar^3 .... are all geometric series.

2. nth term of a G.P.

Let a be the first term and r the common ratio. If T_n denotes the nth term of the series.

$T_1 = a = ar^{1-1}$

$T_2 = ar = ar^{2-1}$

$T_3 = ar^2 = ar^{3-1}$

$T_4 = ar^3 = ar^{4-1}$

....

....

$T_n = ar^{n-1} = ar^{n-1}$

Example 6 : The second term of a G.P. is 24 the 5th term is 81, find the series and the 12th term.

Sol : Now second term = 24 $\therefore$ $T_2 = 24$

and 5th term = 81 $T_5 = 81$

As $T_n = ar^{n-1}$ $\therefore T_2 = ar = 24$ .... (i)

$T_5 = ar^{5-1} = ar^4$, $ar^4 = 81$ .... (ii)

Divide (ii) by (i) $\frac{ar^4}{ar} = \frac{81}{24}$, $r^3 = \frac{27}{8} = \left(\frac{3}{2}\right)^3$, $r = \frac{3}{2}$

Put $r = \frac{3}{2}$ in (i), $a.x\ \frac{3}{2} = 24$, a = 16

$\therefore$ The series is 16, $16.\frac{3}{2}$, $16.\left(\frac{3}{2}\right)^2$ ....

16, 24, 36 ....

$T_{12} = a^{12-1} = ar^{11} = 16\left(\frac{3}{2}\right)^{11} = \frac{3^{11}}{3^{7}}$

Example 7 : if pth term of a G.P. is P and qth term is Q, show that nth term of G.P. is

$$\left[\frac{P^{n-q}}{Q^{n-p}}\right]^{\frac{1}{p-q}}$$

Sol. : $T_p = ar^{p-1} = p$ .... (1)

$T_q = ar^{q-1} = Q$ .... (2)

Dividing (1) by (2), $\frac{ar^{p-1}}{ar^{q-1}} = \frac{P}{Q}$,

$r^{p-q} = \frac{P}{Q}$, $r = \left(\frac{P}{Q}\right)^{\frac{1}{p-q}}$

Put the value of r in (1), a $\left(\frac{P}{Q}\right)^{\frac{p-1}{p-q}} = P$, $\therefore a = P\left(\frac{Q}{P}\right)^{\frac{p-1}{p-q}}$

$\therefore$ $T_n = ar^{n-1}$ put value of a and r

$T_n = P.\left(\frac{Q}{P}\right)^{\frac{p-1}{p-q}} \left(\frac{P}{Q}\right)^{\frac{n-1}{p-q}} = P.\left(\frac{P}{Q}\right)^{\frac{n-p}{p-q}}$

$= \frac{P^{\frac{n-q}{p-q}}}{Q^{\frac{n-p}{p-q}}} = \left[\frac{P^{n-q}}{Q^{n-p}}\right]^{\frac{1}{p-q}}$

3. Sum of n terms of a G.P.

Let a be the first term, r the common ratio

$\therefore$ The series is a, ar, ar^2, ...ar^{n-1}

Let S_n denotes the sum of n terms of the series

$S_n = a + ar + ar^2 + \ldots\ldots + ar^{n-2} + ar^{n-1}$ .... (i)

Multiply both sides by r.

$rS_n = ar + ar^2 + ar^3 + \ldots\ldots + ar^{n-2} + ar^{n-1} + ar^n$ .... (ii)

subtract (ii) from (i)

$S_n - rS_n = a - ar^n$, $S_n(1 - r) = a(1 - r^n)$

$S_n = \frac{a(1-r^n)}{(1-r)}$, if r < 1

$$S_n = \frac{a(r^n - 1)}{(r-1)}, \text{ if } r > 1$$

Note : If common ratio r is more than one, then we use $S_n = \frac{a(r^n - 1)}{(r-1)}$ and when r is less than one, then we use $S_n = \frac{a(1-r^n)}{(1-r)}$

Example 8 : Sum up the series

$1 + \sqrt{3} + 3$ .... To 8 terms

Here $a = 1$, $r = \frac{\sqrt{3}}{1} = \sqrt{3}$, $n = 8$

$$\therefore S_n = \frac{1\left(\left(\sqrt{3}\right)^8 - 1\right)}{\sqrt{3}-1} \cdot \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{\left(\sqrt{3}+1\right)}{3-1} \times 80 = 40\left(\sqrt{3}+1\right)$$

Example 9 : Find, without assuming any formula, the sum of $2 + 1\frac{1}{3} + \frac{8}{9}$ .... to n terms.

Sol. : Here $r = \frac{1\frac{1}{3}}{2} = \frac{4}{3} x \frac{1}{2} = \frac{2}{3}$

$$\therefore \text{ nth term} = 2\left(\frac{2}{3}\right)^{n-1} = \frac{2^n}{3^{n-1}}$$

$$S_n = 2 + \frac{4}{3} + \frac{8}{9} + \ldots\ldots\ldots\ldots\ldots + \frac{2^n}{3^{n-1}} \qquad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$\text{also } \frac{2}{3} S_n = \frac{4}{3} + \frac{8}{9} + \ldots\ldots\ldots\ldots + \frac{2^n}{3^{n-1}} + \frac{2^{n+1}}{3^n} \qquad \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

$$S_n - \frac{2}{3} S_n = 2 - \frac{2^{n+1}}{3^n} = 2\left[1 - \left(\frac{2}{3}\right)^n\right]$$

$$S_n = 6\left[1 - \left(\frac{2}{3}\right)^n\right]$$

4. Geometric Mean between two quantities

When three quantities are in G.P., the middle one is called the Geometric Mean (G.M.) between the other two.

If a, b, c be in G.P. then $\frac{b}{a} = \frac{c}{b}$ or $b^2 = ac$

$\therefore$ G.M. between a and c is $\sqrt{ac}$

Example 10 : If 2, 4, 8 are in G.P. then 4 is G.M. between 2 and 8.

5. n G.M.'s between two numbers a and b

Let $G_1, G_1, \ldots\ldots\ldots\ldots\ldots\ldots G_n$ be the n G.M.'s between a and b

$\therefore$ a, $G_1, G_1, \ldots\ldots\ldots\ldots\ldots\ldots G_n$ b are in G.P.

Number of terms = n + 2

$T_{n+2} = ar^{n+2-1} = ar^{n+1}$

$\therefore ar^{n+1} = b$

$$r^{n+1} = \frac{b}{a},$$

$$r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$G_1 = ar^2 = a.\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$$

$$G_2 = ar^3 = a.\left(\frac{b}{a}\right)^{\frac{3}{n+1}}$$

....

$$G_n = ar^n = a.\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Example 11. Show that the product of n G.Ms between a and b is equal to the nth power of the G.M. between them.

Let $G_1, G_2, \ldots\ldots\ldots\ldots\ldots\ldots G_n$ be n geometric means between a and b.

$$\therefore G_1 = a.\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, G_2 = a.\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, G_3 = a.\left(\frac{b}{a}\right)^{\frac{3}{n+1}} \ldots\ldots\ldots\ldots\ldots\ldots$$

Now product of n G.M. s = $G_1 . G_2 . G_3 \ldots\ldots\ldots\ldots\ldots G_n$

$$= a.\left(\frac{b}{a}\right)^{\frac{1}{n+1}} .a\left(\frac{b}{a}\right)^{\frac{2}{n+1}} .a\left(\frac{b}{a}\right)^{\frac{3}{n+1}} \ldots\ldots\ldots\ldots\ldots a.\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

$$= a^n\left(\frac{b}{a}\right)^{\frac{1}{n+1}+\frac{2}{n+1}+\cdots\cdots\cdots\frac{n}{n+1}}$$

$$= a^n\left(\frac{b}{a}\right)^{\frac{1+2+\cdots\cdots\cdots n}{n+1}}$$

$$= a^n\left(\frac{b}{a}\right)^{\frac{\frac{n(n+1)}{2}}{n+1}}$$

$$= a^n\left(\frac{b}{a}\right)^{\frac{n}{2}}$$

$$= a^n\left(\frac{b^{\frac{n}{2}}}{a^{\frac{n}{2}}}\right) = a^{\frac{n}{2}}.b^{\frac{n}{2}}$$

$$= (ab)^{\frac{n}{2}} = \left(\sqrt{ab}\right)^n$$

$G_1, G_2 \ldots\ldots\ldots G_n$ = nth power of G.M. between a and b

Example 12 : If A be the A.M. and G be the G.M. between two numbers show that the numbers are A $\pm\sqrt{A^2-G^2}$

Sol. : If a, b, be the two numbers,

then A = $\frac{a+b}{2}$, G = $\sqrt{ab}$, a+b=2A ....(i)

$\therefore$ $(a-b)^2 = (a+b)^2 - 4ab = 4A^2 - 4G^2$,

$\therefore$ $a - b = 2\sqrt{A^2-G^2}$ i.e. $\frac{a-b}{2} = \sqrt{A^2-G^2}$, $\frac{a-b}{2}= \sqrt{A^2-G^2}$ ....(ii)

Add (i) and (ii)

$\therefore$ $a = A + \sqrt{A^2-G^2}$, $b = A - \sqrt{A^2-G^2}$

Example13 : If a, b, c, d be in G.P. prove that :

$(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2$

Solution : Let $b = ar$, $c = ar^2$, $d = ar^3$

then $a^2 + b^2 + c^2 = a^2(1 + r^2 + r^4)$

$b^2 + c^2 + d^2 = a^2r^2(1 + r^2 + r^4)$

and $ab + bc + cd = a^2r(1 + r^2 + r^4)$

$\therefore$ $a^2(1 + r^2 + r^4).\, a^2r^2(1 + r^2 + r^4) = a^4r^2(1 + r^2 + r^4)^2$

Hence the result

V. SUMMARY :

The present lesson defines arithmetic and geometric progressions. It also describes the method for finding General Terms and sum of n terms in case of both the series separately Besides these arithmetic means, geometric means have also been calculated.

VI. Questions :

Short Questions :

1. Define arithmetic progression.
2. nth term of an A.P. is also called _________.
3. When three numbers are in A.P., then middle one is called _________ between first and last.
4. Define geometric progression.
5. nth term of G.P. is ___________.
6. If a is the first term and r is the common ratio then sum of n terms of G.P. is defined as S_n = ___________.
7. State the rule to find the G.M. between two numbers.
8. The product of n G.M.'s between two numbrs a and b is the __________of the G.M. between them.

Long Questions :

1. The sum of first 15 terms of an A.P. is 270. Find the first term and the common difference if the 15th term is 39.
2. If S_1, S_2, S_3 be the sums of n, 2n, 3n terms respectively of an A.P., prove that $S_3=3(S_2 - S_1)$.
3. Prove that the sum of an odd number of terms of an A.P. is equal to middle term multiplied by the number of terms.
4. The 3rd term of a G.P. is $6\frac{1}{4}$ and the 7th term is the reciprocal of the 3rd term. Find the common ratio. Which term of the series is unity?
5. Find the value of n so that expression $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the G.M. between a and b.

VII. SUGGESTED READINGS

1. Bhardwaj and Sabharwal - Mathematics for Students of Economics.
2. Aggarwal and Joshi - Mathematics for Students of Economics.

LESSON NO. 2.7 AUTHOR : DR. VIPLA CHOPRA

ECONOMIC APPLICATION OF ARITHMETIC PROGRESSION AND GEOMETRIC PROGRESSION

I. INTRODUCTION

Arithmetic and geometric progressions are frequently used in economic analysis. Economic variables like assets, population etc. may change over time either in arithmetic progression or in geometric progression. For example output of a firm may increase or decrease in arithmetic progression over successive time periods. Geometric progression occupies a prominent place in multiplier analysis. It is often assumed that population increases in a geometric progression. Geometric mean is preferred to other averages, in the measurement of relative changes such as changes in the price level. Thus, arithmetic and geometric progression have wide application in economics.

II. OBJECTIVES

In the previous lesson we have studied arithmetic progression and geometric progression in a simple way. In the present lesson our objective is to study economic application of A.P. and G.P. with illustrations

Economic application of (i) A.P. and (ii) G.P.

III. ECONOMIC APPLICATION OF ARITHMETIC PROGRESSION

Example 1 : A firm produces 550 T.V. during its first year. The sum total of the firm's production at the end of 5 years was 3000.

(i) Estimate by how many units, production increased each year and.

(ii) Forecast, based on the estimate of the annual increment in production, the level of output for the 10th year ?

Sol. : Let a denote first year's production and *d* the difference between units produced in two successive years.

$\therefore$ a = 550, S_5 = 3000, n = 5

(Given)

$S_n = \frac{n}{2}\ [2a + (n - 1)\ d]$

$S_5 = \frac{5}{2}\ [2 \times 550 + 4d]$

$= 5\ (550 + 2d)$

$= 2750 + 10d$

$\therefore\ 10d = 3000 - 2750 = 250$

$d = 25$

Therefore production increased each year by 25 units.

Level of output for 10th year

$T_{10} = a + (n - 1)\ d = 550 + 9 \times 25 = 550 + 225 = 775$

Example 2 :

A piece of equipment costs in certain factory Rs. 6,00,000. If it depreciates in value, 15% the first year, $13\frac{1}{2}$% the next year, 12% the third year, and so on, what will be its value at the end of 10 years, all percentages applying to the original cost ?

Solution : Let the cost of an equipment be Rs. 100 Now the percentages of depreciation at the end of 1st , 2nd, 3rd year are 15, $13\frac{1}{2}$, 12 .... Which are in A.P. with a = 15, and $d = -\frac{3}{2}$

Percentage of depreciation in the tenth year = $a + (10 - 1)\ d = 15 + 9\ \left(-\frac{3}{2}\right) = \frac{3}{2}$

Total value of depreciation in 10 years

$15 + 13\frac{1}{2} + 12 + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{3}{2} = \frac{10}{2}\left(15 + \frac{3}{2}\right) = \frac{165}{2}$ $\quad [s_n = \frac{n}{2}(a + l)]$

Hence the value of equipment at the end of 10 years = $100 - \frac{165}{2} = \frac{35}{2}$

The total cost being Rs. 6,00,000 its value at the end of 10 years

$= \frac{\text{Rs.}6{,}00{,}000}{100} \times \frac{35}{2}$ = Rs. 1,05,000

Example 3 : A man saves Rs. 16,500 in ten years. In each year after the first year he saved Rs. 100 more than he did in proceeding year. How much did he saved in the first year ?

Here we have to find savings in the first year ?

n = numbers of year = 10, d = 100, S_n = 16,500

Now $S_n = \frac{n}{2}$ [2a + (n – 1) d]

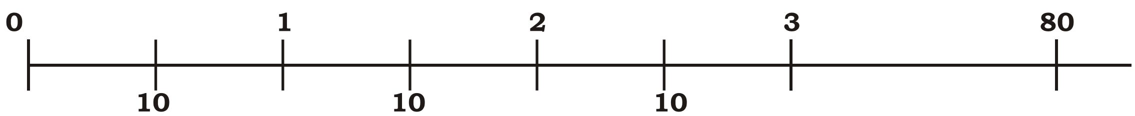
Or $16,500 = \frac{10}{2}$ [2a + (10 – 1) 100]

16,500 = 5 (2a + 900)

10a = 16,500 – 4,500 = 12,000

a = Rs. 1,200

Example 4 : 80 coins are placed in straight line on the ground. The distance between any two consecutive coins is 10 meters. How far must a person should travel to bring them one by one to a basket placed 10 meters behind the first coin ?



Suppose 1, 2, 3 ...., 80 represent the position of the coins and 0 that of the basket.

The distance covered in bringing the First coin = 10 + 10 = 20

The distance covered in bringing the Second coin = 20 + 20 = 40

The distance covered in bringing the Third coin = 30 + 30 = 60

and so on. This form an A.P. with a = 20 and d = 20

$\therefore$ Total distance covered in bringing 80 coins one by one.

$$= \frac{n}{2} [2a + (n - 1) d]$$

$$= \frac{80}{2} [2 \times 20 + (80 - 1) 20]$$

= 40 (40 + 1580) = 64,800 meters.

IV. ECONOMIC APPLICATION OF GEOMETRIC PROGRESSION

Example 5 :

At 10% per annum compound interest, a sum of money accumulates to Rs. 8650 in 5 years. Find the sum invested initially.

Let P be the principle.

Now amount of P after 5 years = $P \left(1+\frac{10}{100}\right)^5 = P \times \left(\frac{110}{100}\right)^5$

$$P = \frac{8650}{(1.1)^5} = \frac{8650}{1.4641} = 5976.37$$

Which is the required principal.

Example 6 : Given that the marginal propensity to consume is $\frac{2}{3}$, what will be the resultant increase in income for an autonomous intial investment of Rs. 10 lakhs? What will be the size of multiplier ?

Solution : The amount invested = Rs. 10 lakhs.

$\therefore$ Increase in income in first round = Rs. 10 lakhs.

Marginal propensity to consume = $\frac{2}{3}$

Consumption in first round = Rs. 10 × $\frac{2}{3}$

As one man's expenditure is another man's income. Increase in income in second round = Rs. 10 × $\frac{2}{3}$

Similarly increase in income in third round is Rs. 10 × $\left(\frac{2}{3}\right)^2$ and so on

$\therefore$ Total increase in income :

$$= \text{Rs. } \left(10 + 10 \times \frac{2}{3} + 10\left(\frac{2}{3}\right)^2 \ldots\ldots\ldots\right) \text{ Lakhs}$$

$$= \text{Rs. } 10 \left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 \ldots\ldots\ldots\right) \text{ Lakhs}$$

$$= \text{Rs. } 10 \left(\frac{1}{1 - \frac{2}{3}}\right) = 10 \times 3 = \text{Rs. } 30 \text{ Lakhs}$$

$$\text{Multiplier} = \frac{\text{Increase in income}}{\text{Increase in investment}} = \frac{30 \text{ lakhs}}{10 \text{ lakhs}} = 3$$

$\therefore$ Size of multiplier is 3

MULTIPLIER AND GEOMETRIC SERIES

In multiplier analysis, geometric series are frequently used by economists. With the help of geometric series the working of dynamic investment multiplier can be easily understandable.

Let there be repeated investment of Rs. 100 in each of the time periods t_1, t_2 .... t_n, Let us further assume that marginal propensity to consume is 5 and the consumption in any time period depends upon the income in the previous period. On these assumptions increase in income in the various time periods can be illustrates as.

Period	Change in Investment (in Rs.)	Induced Change in consumption			Total increase in income
t_1	100				100
t_2	100	50			150
t_3	100	50	25		175
t_4	100	50	25	12.5	187.5
–––	–––	–––	----		
–––	–––	–––	----		

In other words

Increase in income in period t_1 = Rs. 100

Increase in income in period t_2 = Rs. 100 + 100 $\left(\frac{1}{2}\right)$

Increase in income in period t_3 = Rs. $\left[100+100\times\left(\frac{1}{2}\right)+100\left(\frac{1}{2}\right)^2\right]$ and so on

$\therefore$ Increase in income in period t_n

$$= \text{Rs. } \left[100+100\times\left(\frac{1}{2}\right)+100\left(\frac{1}{2}\right)^2+\ldots\ldots\ldots\ldots 100\left(\frac{1}{2}\right)^{n-1}\right]$$

$$= \text{Rs. } 100\left[1+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2+\ldots\ldots\ldots\ldots\left(\frac{1}{2}\right)^{n-1}\right]$$

$$= \text{Rs. } 100\left[\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right]$$

$$= \text{Rs. } 200\left(1-\left(\frac{1}{2}\right)^n\right)$$

As n tends to infinity $\left(\frac{1}{2}\right)^n$ tends to zero and therefore resultant increase in income tends to Rs. 200.

In other words, as $n \to \infty$, $\left(\frac{1}{2}\right)^n \to 0$

$\therefore$ Permanent Increase in income = Rs. 200

Geometric Series and Population Growth

Let P_1 (i = 0, 1, 2 ....n) be the population of India in the ith year and let r be its rate of growth per year.

Population in the year $t_0 = P_0$

Population in the year $t_1 = P_0 + rP_0$, i.e... $P_1 = P_0 (1 + r)$

Similarly $P_2 = P_1 + rP_1 = P_1 (1 + r) = P_0 (1 + r)^2$

$P_3 = P_2 + rP_2 = P_2 (1 + r) = P_0 (1 + r)^3$

So in the year t_0, t_1, t_2, ... respectively, the population P_0, P_1, P_2,............ Form a G.P. whose first term is P_0 and common ratio is r

$\therefore$ Pn i.e. population in the nth year = $P_o(1 + r)^n$

Example 7 : The population of India was 54 crores in 1971. Predict the population in 1981 assuming an increase of 3 percent each year.

Solution : Population in year 1971 = 54 crores

Increase in population in each year = 3%

If P_0 be the population in the year t_0, r be the rate of growth per annum, population in the nth year = $P_0 (1 + r)^n$

$\therefore$ Population in the year 1981, $P_n = P_0 (1 + r)^n$

$P_n = 54 (1 + .03)^{10} = 54(1.03)^{10}$

Take log.

$\log P_n = \log 54 (1.03)^{10}$ [using $\therefore$ $\log mn = \log m + \log n$ and $\log m^n = n \log m$]

$\log P_n = \log 54 + 10 \log (1.03)$

$= 1.7324 + 10 \times (.0128) = 1.7324 + .1280 = 1.8604$

P_n = Antilog (1.8604)

P_n = 72.51 ($\because$ antilog (1.8604) =72.51)

$\therefore$ Population in 1981 is 72.51 crores.

Example 8 : A person deposits Rs. 1000 on 1st January 1990 and Rs. 1000 each year thereafter until 1st January 1998. Interest is at the rate of 8% compounded annually. How much will he have in his account immediately after he makes the payment on 1st January 1998 ?

Solution : Here n = 9, R = 1000, i = .08

Amount in his account

$$S = R\left(\frac{(1+i)^n - 1}{i}\right)$$

$$= 1000\left(\frac{(1.08)^9 - 1}{0.08}\right) = 1000 \times 12.4875$$

$= 12487.50$

Example 9 : A bond with a face value of Rs. 500 matures in 12 years. The nominal rate of interest on bond is 12% p.a., paid annually. What should be the price of the bond so as to yield an effective rate of return equal to 10% ?

Solution : We note that the given bond yields an income stream of Rs. 60 per year for 12 years. The present value of income stream at 10% p.a. rate of interest is

$$P = 60\left[\frac{1}{1.1} + \frac{1}{(1.1)^2} + \ldots\ldots\ldots\ldots + \frac{1}{(1.1)^{11}} + \frac{500}{(1.1)^{12}}\right]$$

$$= \frac{60}{1.1}\left[1 + (1.1)^{-1} + (1.1)^{-2}\ldots\ldots\ldots\ldots\ldots + (1.1)^{-11}\right] + \frac{500}{(1.1)^{12}}$$

$$= \frac{60}{1.1}\left[1 - (1.1)^{-12}\right] + \frac{500}{(1.1)^{12}}$$

$= 408.82 + 159.32 = \text{Rs. } 568.14$

V. **Example 10 :** A machine depreciates at the rate of 10 percent on book value. The original cost was Rs. 10,000 and ultimate scrap value Rs. 6561, find the effective life of the machine.

Solution : Original cost of machine = 10,000

Since machine depreciates at the rate of 10% on reducing balance,

Value of machine after one year = $10{,}000 \times \frac{90}{100}$

Value of the machine after two years = $10{,}000 \times \left(\frac{90}{100}\right)^2$

Thus, $10{,}000\left(\frac{90}{100}\right)$, $10{,}000\left(\frac{90}{100}\right)^2$ .... will form a G.P. series with

$a = 10000\left(\frac{90}{100}\right)$ and $r = \frac{9}{10}$

Let value of the machine after n years be Rs. 6561.

$$\Rightarrow T_n = ar^{n-1} \Rightarrow 6561 = 10{,}000\left(\frac{9}{10}\right)\left(\frac{9}{10}\right)^{n-1}$$

$$\Rightarrow \frac{6561}{10000} = \left(\frac{9}{10}\right)^n$$

$$\Rightarrow \left(\frac{9}{10}\right)^4 = \left(\frac{9}{10}\right)^n$$

$\Rightarrow$ n = 4

$\therefore$ n = 4

Hence the effective life of the machine is 4.

V. SUMMARY

In the present lesson we have studied various examples which deal with the determination of the growth of a given sum of money as well as the present value of a sum due at some future date. Often these examples involve the sum of either an arithmetical or geometrical progression.

VI. QUESTIONS

1. A person saves each year Rs. 50 more than the previous year, after ten years his savings amount to Rs. 3000, excluding the interest. Find the amount the saved in the first last year.
2. A price index was 9 in 1970 and 121 in 1980. Assuming an A.P., predict the price indeed for 1990. Also find the base year.
3. A firm produces 25 units of a commodity during the first year of its establishment and then increases the production by 12 units each year. Find the number of units produced in the 12th year of the firm's history. What wll be the total production then ?
4. India's population in 1950 and 1967 was 36 and 51.4 crore persons respectively. Find the annual arithmetic and geometric rates of growth.
5. The original cost of a machine is Rs. 10,000. If it depreciates at the compound rate of 10% every year, after how many years will it be valued at Rs. 6561 ?
6. The population of a city in 1982 is 8000. If it increases at the rate of 5% per annum, what will it be in 1985 ?
7. If the value of Fiat Car depreciated by 25 per cent annually, what will be its estimated value at the end of 8 years, if its present value is Rs. 2048 ?

VII. SUGGESTED READINGS

1. C.S. Aggarwal and R.C. Joshi - Mathematics for students of Economics.
2. G.C. Sharma and Madhu Jain - Quantitative Techniques for Management.
3. O.P. Bhardwaj and J.R. Sabharwal - Mathematics for students of Economics.