Department of Distance Education

Punjabi University, Patiala

Class : B.A. I (Math) **Semester : 2**
Paper : 4 (Advanced Calculus) **Unit : I**
Medium : English

Lesson No.

1.1 : DOUBLE INTEGRATION

1.2 : TRIPLE INTEGRATION

Department website : www.pbidde.org

LESSON NO. 1.1 **Author : Dr. Chanchal**

DOUBLE INTEGRATION

Structure :

1.0 Objectives

The prime objective of this unit is to understand the concept of double and triple integration. During the study in this particular lesson, we will focus on the idea of double integration and surface integrals.

1.1 Introduction

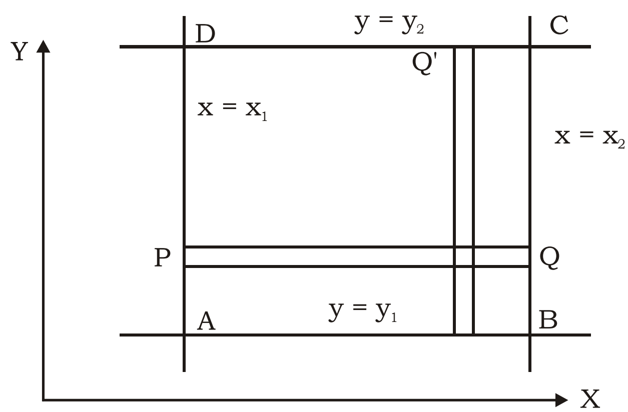
As we have already discussed about the limit and continuity of functions of two variables in lesson no. 1. Further, we have also gained the knowledge about partial derivatives and Jacobian of function of two variables in lesson no. 2 and 3. In this lesson, our aim is to study the double integration concerned with the functions of two variables. As we already know about the definite integral $\int_a^b f(x)\,dx$ in one-dimension only. In a similar way, a double integral is concerned with two dimensions or we can say, a region R in plane. Let f(x, y) be a bounded and single valued function of variables (x and y) in some region R of xy-plane. Then, the double integral of f(x, y) over the region R is denoted by $\iint_R f(x,y)\,dA$, which may be further written as

$\iint_R f(x,y)\,dx, dy$ or $\iint_R f(x,y)\,dy\,dx$. Here 'A' denotes the area of the region R.

1.2 How to Evaluate Double Integral

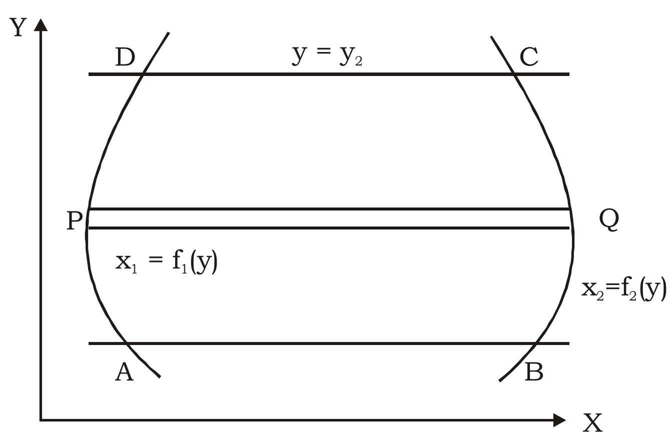
The evaluation of the double integral depends upon the nature of the curves. Suppose region R is bounded by the curves that the region R is bounded by the curves $x = x_1$, $x = x_2$ and $y = y_1$, $y = y_2$. Now, we have the following three possibilities :

I. If x_1, x_2, y_1, y_2 are constants. Here R is the rectangle ABCD. Here, the order of integration is immaterial, we can first integrate from the strip P to Q and then slide it from AB to CD or we can integrate first along the strip P'Q' and then slide it from AC to BD.



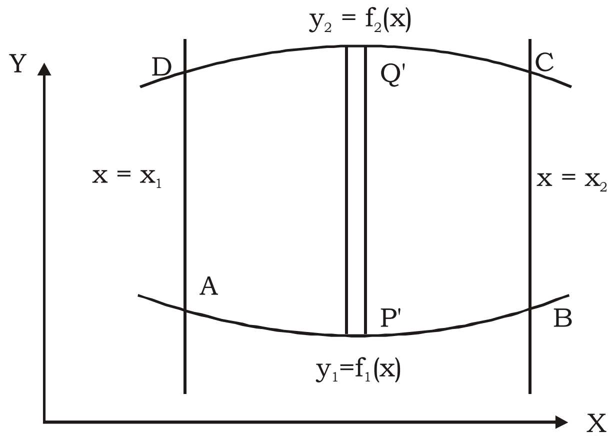
Thus, $\iint_R f(x, y)\,dx\,dy = \int_{y_1}^{y_2}\left(\int_{x_1}^{x_2} f(x, y)\,dx\right)dy = \int_{x_1}^{x_2}\left(\int_{y_1}^{y_2} f(x, y)\,dy\right)dx$.

II. If x_1, x_2 are the functions of y and y_1, y_2 are constants.
Let AD and BC are the curves $x_1 = f_1(y)$ and $x_2 = f_2(y)$. Take a hoirzontal strip PQ of width δy. Here the integral first evaluated w.r.t. x and then w.r.t. y of the resulting expression from y_1 to y_2.



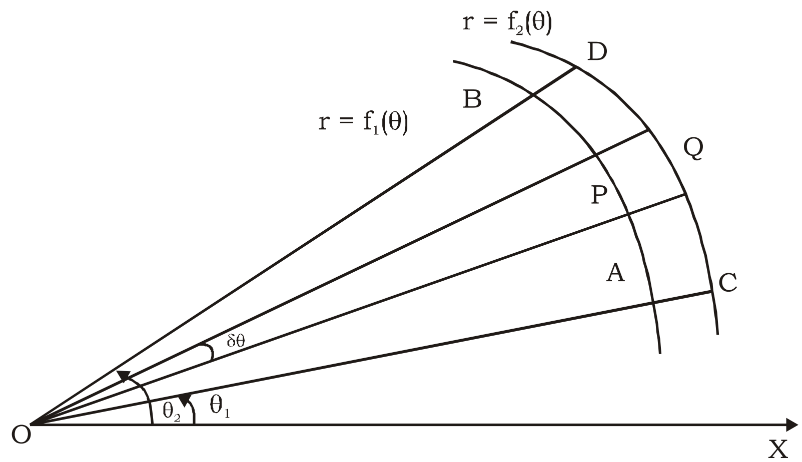
Thus $\iint_R f(x,y)\,dxdy = \int_{y_1}^{y_2}\left[\int_{x_1=f_1(y)}^{x_2=f_2(y)} f(x, y)\,dx\right]dy$.

III. If y_1, y_2 are the functions of x and x_1, x_2 are constants.
Let AB and CD are the curves $y_1 = f_1(x)$ and $y_2 = f_2(x)$.
Take a vertical strip P'Q' with width δx. Here we first integrate w.r.t. y along the strip P'Q' and then the resulting expression w.r.t. x from x_1 to x_2.



Thus $\iint_R f(x,y)\,dxdy = \int_{x_1}^{x_2}\left[\int_{y_1=f_1(x)}^{y_2=f_2(x)} f(x, y)\,dy\right]dx$.

Similarly in polar co-ordinates the double integral $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\,dr\,d\theta$ is bounded by the lines $\theta = \theta_1$, $\theta = \theta_2$ and the curves $r = r_1$, $r = r_2$. We first integrate w.r.t. r and then w.r.t. θ between the limits $\theta = \theta_1$, $\theta = \theta_2$.



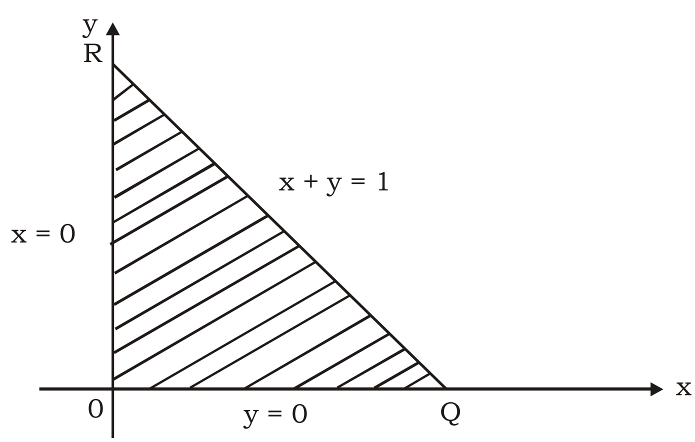
So, $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\,dr\,d\theta = \int_{\theta_1}^{\theta_2} f(\theta)\,d\theta \int_{r_1}^{r_2} f(r)\,dr$.

Example 4.1 : Evaluate $\iint x^{\frac{1}{2}} y^{\frac{1}{2}} (1-x-y)^3\,dx\,dy$ over the region A bounded by the triangle whose vertices are (0, 0), (1, 0) and (0, 1).

Sol. The vertices of the triangle OQR are (0, 0), (1, 0) and (0, 1).

$\therefore$ equation of line QR is $\frac{x}{1}+\frac{y}{1}=1$ or $x+y=1$

$\therefore$ $A = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le 1 - x\}$



where A is the region.

$$\therefore \quad \iint_A x^{\frac{1}{2}} y^{\frac{1}{2}} (1-x-y)^3 \, dx\, dy = \int_0^1 \left(\int_0^{1-x} x^{\frac{1}{2}} y^{\frac{1}{2}} (1-x-y)^3 dy \right) dx$$

Put $\quad y = (1-x)\, t, \ \therefore\ dy = (1-x)\, dt$

When $y = 0,\ t = 0$

When $y = 1 - x,\ t = 1$

$$\therefore \quad \iint_A x^{\frac{1}{2}} y^{\frac{1}{2}} (1-x-y)^3 \, dx\, dy = \int_0^1 \left(\int_0^1 x^{\frac{1}{2}} (1-x)^{\frac{1}{2}} t^{\frac{1}{2}} (1-x)^3 (1-t)^3) (1-x)\, dt \right) dx$$

$$= \int_0^1 \left(\int_0^1 x^{\frac{1}{2}} (1-x)^{\frac{9}{2}} t^{\frac{1}{2}} \ (1-t)^2 dt \right) dx = \int_0^1 x^{\frac{1}{2}} (1-x)^{\frac{9}{2}} dx \int_0^1 t^{\frac{1}{2}} (1-t)^3 \ dt$$

Let $\quad I_1 = \int_0^1 x^{\frac{1}{2}} (1-x)^{\frac{9}{2}} dx$

Put $\quad x = \sin^2\theta\,, \qquad \therefore\ dx = 2 \sin\theta \cos\theta\, d\theta$

When $x = 0,\ \theta = 0$

When $x = 1,\ \theta = \frac{\pi}{2}$

$$\therefore \quad I_1 = \int_0^{\pi/2} \sin\theta\, .\cos^9\theta . 2\sin\theta \cos\theta\, d\theta = 2 \int_0^{\pi/2} \sin^2\theta \cos^{10}\theta\, d\theta$$

$$= 2\, \frac{1.9.7.5.3.1}{12.10.8.6.4.2} \frac{\pi}{2} = \frac{945}{46080} \pi$$

$$I_2 = \int_0^1 t^{\frac{1}{2}} \ (1-t)^3 \ dt$$

Put $\quad t = \sin^2\phi$

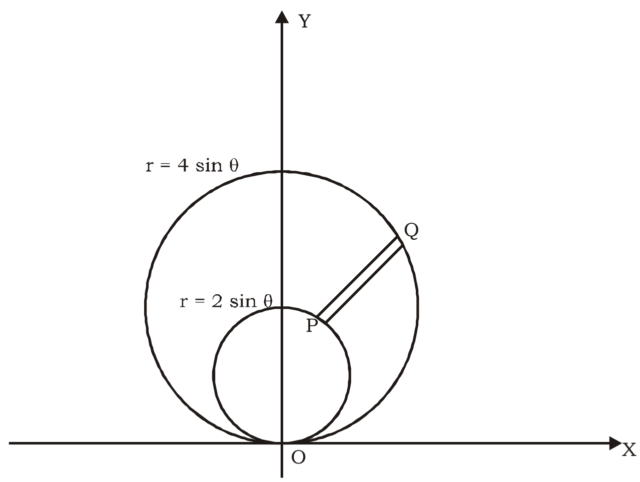
$$\therefore \quad I_2 = 2 \int_0^{\pi/2} \sin^2\phi \cos^7\phi\, d\phi = 2\, \frac{1.6.4.2}{9.7.5.3.1} = \frac{2 \times 48}{945} = \frac{96}{945}$$

$$\left[\textit{Here we are using the value of} \int_0^{\pi/2} \sin m\theta \cos n\theta d\theta \ \textit{ by reduction formulae} \right]$$

$\therefore \qquad \iint\limits_A x^{\frac{1}{2}} y^{\frac{1}{2}} (1-x-y)^3 \, dx\, dy = \frac{945}{46080}\pi \times \frac{96}{945} = \frac{\pi}{480}.$

Example 1.2 : Evaluate $\iint r^2 \, dr d\theta$ over the area included between the circles r = 2 sinθ and r = 4 sin θ.

Sol. $\quad I = \int_{-\pi/2}^{\pi/2} \int_{2\sin\theta}^{4\sin\theta} r^3 dr d\theta$



$$= \int_{-\pi/2}^{\pi/2} \frac{1}{4}\left[r^4\right]_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \frac{1}{4}\int_{-\pi/2}^{\pi/2} (256\sin^4\theta - 16\sin^4\theta)\, d\theta$$

$$= \frac{1}{4}\int_{-\pi/2}^{\pi/2} .240\sin^4\theta \,.\, d\theta = \frac{240}{4}\int_{-\pi/2}^{\pi/2} .\sin^4\theta \,.\, d\theta = 2 \times 60 \int_0^{\pi/2} \sin^4\theta d\theta$$

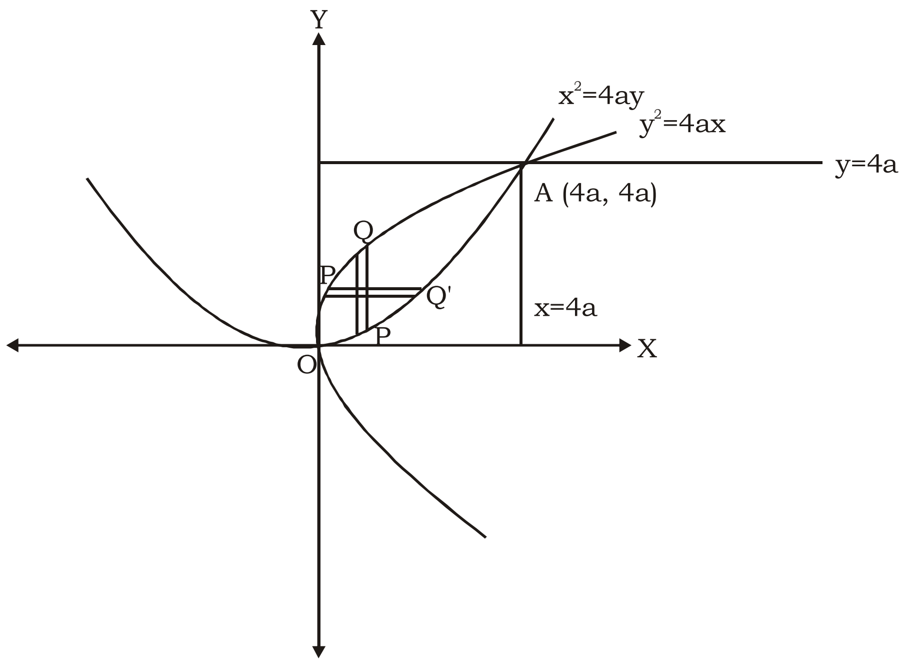
$$= 120 \times \frac{1\times 3}{4\times 2} \times \frac{\pi}{2} = \frac{45\pi}{2}.$$

Example 1.3 : Change the order of integration and evaluate the integral :

$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dydx$$

Sol. The given integral is $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dydx$

Here y varies from $\frac{x^2}{4a}$ to $2\sqrt{ax}$ and x varies from 0 to 4a.



Here integration is carried along the vertical strip PQ. When we change the order of integration the vertical strip must be changed to horizontal strip P'Q' where x varies from $x = \frac{y^2}{4a}$ to $x = 2\sqrt{ay}$ and y varies from 0 to 4a. The point of intersection of the cures is A (4a, 4a).

$$\therefore \qquad I = \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dydx = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dxdy = \int_0^{4a} [x]_{\frac{y^2}{4a}}^{2\sqrt{ay}}\, dy$$

$$= \int_0^{4a} \left[2\sqrt{ay} - \frac{y^2}{4a} \right] dy$$

$$= \left[2\sqrt{a} \cdot \frac{y^{3/2}}{3/2} - \frac{y^3}{3.4a} \right]_0^{4a} = \frac{4}{3}\sqrt{a}(4a)^{3/2} - \frac{64a^3}{12a} = \frac{32}{3}a^2 - \frac{16a^2}{3} = \frac{16a^2}{3}.$$

1.3 Change of Variables

If A is a simple closed subregion of the x y-plane which is mapped into the region B of the uv-plane by the transformations

$x = \phi\ (u, v),\ y = \psi\ (u, v)$

and (i) ϕ, ψ have continuous partial derivatives on B.

(ii) $$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0 \forall\ (u,v) \in B$$

then $\iint_A f\ (x,y)\, dx\, dy = \iint_B f\ (\phi(u,v),\ \psi\ (u,v))\,|\,J\,|\,du\, dv$

We accept this result without proof.

Example 1.4 : Evaluate $\iint \sqrt{\dfrac{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}{1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}}}\, dx\, dy$ over the positive quadrant of the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \cdot$

Sol. Put $\frac{x}{a} = u,\ \frac{y}{b} = v$ i.e., $x = au,\ y = bv$

$\therefore$ $dx = a\, du,\ dy = b\, dv$

$\therefore$ ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the positive quadrant of xy-plane transforms into the circle $u^2 + v^2 = 1$ in the positive quadrant of the uv-plane.

$$\therefore \qquad J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

$$\therefore \qquad \iint \sqrt{\frac{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}{1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}}}\, dx\, dy = \iint_A \sqrt{\frac{1-u^2-v^2}{1+u^2+v^2}}\, .ab\, du\, dv$$

where A is the region $u \geq 0,\ v \geq 0,\ u^2 + v^2 \leq 1$

Put $u = r \cos \theta,\ v = r \sin \theta$

$$\therefore \qquad J = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\,(\cos^2\theta + \sin^2\theta) = r$$

Also $A = \left\{(r,\theta) : 0 \leq r \leq 1,\ 0 \leq \theta \leq \frac{\pi}{2}\right\}$

$$\therefore \qquad \iint \sqrt{\frac{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}{1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}}}\, dx\, dy = \int_0^{\pi/2}\int_0^1 \sqrt{\frac{1 - r^2\cos^2\theta - r^2\sin^2\theta}{1 + r^2\cos\theta + r^2\sin^2\theta}}\ ab.r\, dr\, d\theta$$

$$= ab \int_0^{\pi/2} d\theta \, . \int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\ r\, dr$$

$$= ab\, \frac{\pi}{2} \cdot \frac{1}{2}\left(\frac{\pi}{2} - 1\right) \qquad \text{(Proved above)}$$

$$= \frac{\pi\, ab}{8}(\pi - 2) \cdot$$

1.4 Calculation of Area

1. The area A of the region $\{(x, y) : a \le x \le b, f_1(x) \le y \le f_2(x)\}$ is given by

$$A = \int_a^b \int_{f_1(x)}^{f_2(x)} dy\, dx$$

2. The area A of the region $\{ (x, y) : c \le y \le d, g_1(y) \le x \le g_2(y)\}$ is given by

$$A = \int_c^d \int_{g_1(y)}^{g_2(y)} dx\, dy$$

3. The area A of the region $\{ (r, \theta) : a \le \theta \le \beta, f_1(\theta) \le r \le f_2(\theta)\}$ is given by

$$A = \int_\alpha^\beta \int_{f_1(\theta)}^{f_2(\theta)} r\, dr\, d\theta$$

4. The area A of the region $\{ (r, \theta) : r_1 \le r \le r_2, g_1(r) \le \theta \le g_2(r)\}$ given by

$$A = \int_{r_1}^{r_2} \int_{g_1(r)}^{g_2(r)} r\, d\theta\, dr$$

Example 1.5 : Find the area enclosed using double integration by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol. The equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

We know that ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is symmetrical about both the axes.

Also in first quadrant, $y = \frac{b}{a}\sqrt{a^2 - x^2}, 0 \le x \le a$

$\therefore$ required area $= 4\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dydx = 4\int_0^a [y]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx$

$$= \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\, dx = 4\frac{b}{a}\left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{4b}{a}\left[\left(a + \frac{a^2}{2}\sin^{-1} 1\right) - \left(0 + \frac{a^2}{2}\sin^{-1} 0\right)\right]$$

$$= \frac{4b}{a}\left[\frac{a^2}{2} \times \frac{\pi}{2}\right] = \pi\, ab\,.$$

1.4.1 Calculation of Surface Area

Let a function z = f (x, y) be continuous and having continuous first order partial derivations over a closed region A of the x y-plane, then the area of the surface z = f (x, y) of which the projection on xy plane is thearea A is given as

$$\text{Surface area } S = \iint_A \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\; dA\,.$$

1.5 Calculation of Centre of Gravity and Centroid in IR^2

Let a mass M be continuously distributed with density $\mu = \mu\,(x, y)$ over a region A of the xy plane.

If $(\bar{x}, \bar{y})$ is the centroid, then

(1) $M = \iint_A \mu\,(x, y)\, dx\, dy$

(2) $\bar{x} = \frac{1}{M}\iint_A x\, \mu\,(x, y)\, dx\, dy$

(3) $\bar{y} = \frac{1}{M}\iint_A y\, \mu\,(x, y)\, dx\, dy$

Example 1.6 : Find the centre of gravity of the area in the first quadrant lying between the curves $y^2 = x^3$ and $y = x$.

Sol. Given curves are

$y^2 = x^3$ … (1)

and $y = x$ … (2)

From (1) and (2), we get

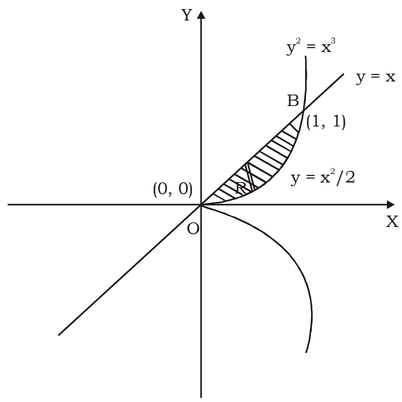
$y^2 = y^3$ or $y^2 (1 - y) = 0$

$\Rightarrow$ $y = 0, 1$

from (1), we get, $x = 0, 1$

$\therefore$ points of intersection are (0, 0) and (1, 1)

Let $(\bar{x}, \bar{y})$ be the required C.G. then



Now $\quad \bar{x} = \dfrac{1}{M} \iint \mu\, x\, dy\, dx$ ... (3)

and $\quad \bar{y} = \dfrac{1}{M} \iint \mu\, y\, dy\, dx$ ... (4)

Since μ is not given, $\therefore$ let $\mu = k$ (constant)

Now $M = \iint_R \mu\, dy\, dx,$ where R is the region shown in figure.

$\therefore$ region of integration is $R = \{(x, y) : 0 \le x \le 1 \text{ and } x^{3/2} \le y \le x\}$

$$\therefore \quad M = \int_0^1 \int_{x^{\frac{3}{2}}}^{x} k\, dy\, dx = k \int_0^1 [y]_{x^{3/2}}^{x} dx = k \int_0^1 \left(x - x^{3/2}\right) dx$$

$$= k\left[\frac{x^2}{2} - \frac{2}{5}x^{5/2}\right]_0^1 = k\left(\frac{1}{2} - \frac{2}{5}\right) = \frac{k}{10}$$

$$\therefore \quad \text{From (3), } \bar{x} = \frac{1}{M} \int_0^1 \int_{x^{\frac{3}{2}}}^{x} k\, x\, dy\, dx = \frac{k}{M} \int_0^1 x\,[y]_{x^{\frac{3}{2}}}^{x}\, dx$$

$$= \frac{k}{M} \int_0^1 x\left(x - x^{3/2}\right) dx = \frac{k}{M} \int_0^1 (x^2 - x^{5/2})\, dx = \frac{k}{M}\left[\frac{x^3}{3} - \frac{2}{7}x^{7/2}\right]_0^1$$

$$= \frac{k}{M}\left(\frac{1}{3}-\frac{2}{7}\right) = \frac{k}{M}\left[\frac{7-6}{21}\right] = \frac{k}{M}\left[\frac{1}{21}\right] = \frac{10}{21} \qquad \left[\because M = \frac{k}{10}\right]$$

Also, $\bar{y} = \dfrac{1}{M}\displaystyle\int_0^1\int_{x^{3/2}}^{x} ky\,dy\,dx = \dfrac{k}{M}\int_0^1\left[\dfrac{y^2}{2}\right]_{x^{3/2}}^{x} dx$

$$= \frac{k}{M}\cdot\frac{1}{2}\int_0^1 (x^2 - x^3)\,dx = \frac{k}{2M}\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1$$

$$= \frac{k}{2M}\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{k}{24M} = \frac{10}{24} = \frac{5}{12} \qquad \left[\because M = \frac{k}{10}\right]$$

$\therefore$ C.G. is $\left(\dfrac{10}{21}, \dfrac{5}{12}\right)$.

1.6 Moment of Inertia in IR²

The moment inertia (abbreviated M.I.) of a mass m about a fixed the l is defined a the product of mass m and square of the distance r of the mass from theline.

$\therefore$ M. I. = mr^2.

Let mass m be distributed continuously with density μ (x, y) in a region A. Then

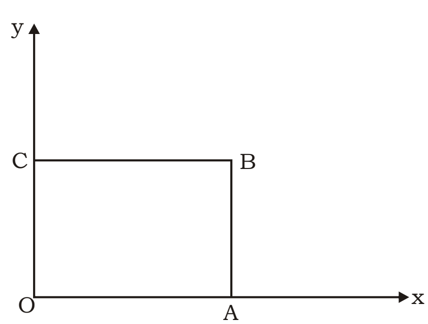
(i) Moment of inertia about x-axis $= \iint_A y^2 \mu(x, y)\,dx\,dy$ and is denoted by I_x.

(ii) Moment of inertia about y-axis $= \iint_A x^2 \mu(x, y)\,dx\,dy$ and is denoted by I_y.

Example 1.7 : Find the moment of inertia of a square region of unit density about one of its sides, the side being 2a.

Sol. Let OABC be the square. Take O as origin, OA as x-axis and OC as y-axis.

Let D denote the region of the square



$\therefore$ $D = \{(x, y) : 0 \le x \le 2a,\ 0 \le y \le 2a\}$ $\mu(x, y) = 1$

M.I. about OA i.e., x-axis

$$= \iint_D y^2 \mu(x, y)\, dx\, dy$$

$$= \int_0^{2a} \left\{ \int_0^{2a} y^2 dy \right\} dx = \int_0^{2a} \left[y^3 \right]^{2a} dx = \frac{8a^3}{3} \int_0^{2a} dx$$

$$= \frac{8a^3}{4} [x]_0^{2a} = \frac{8a^3}{3} \times 2a = \frac{16a^4}{3}.$$

1.7 Self Check Exercise

1. Let $f(x, y) = x + y$ be defined in the rectangle

   $A = \{(x, y) : 3 \le x \le 4,\ 1 \le y \le 2\}$

   then show that $\int_1^2 \left(\int_3^4 f(x, y)\, dx \right) dy = \int_3^4 \left(\int_1^2 f(x, y)\, dy \right) dx.$

2. Evaluate $\iint_A x\, dx\, dy$ where A is the region bounded by parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

3. Change the order of integration in the integral $I = \int_{-a}^{a} \int_0^{\sqrt{a^2 - y^2}} f(x, y)\, dx\, dy$.

4. Show that $\iint_A \frac{dx\, dy}{4 - x^2 - y^2} = \pi \log 3$ over the region A between the concentric circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 3$.

5. Find the area of the region bounded by $x = 0$, $y = 0$, $x^2 + y^2 = 1$, $y = \frac{1}{2}$.

1.8 Suggested Readings

R. K. Jain, SRK Lyengar	:	Advanced Engineering Mathematics (Narosa Publications)
JR Sharma	:	Advanced Calculus
Malik and Arora	:	Mathematical Analysis
Shanti Narayan	:	Mathematical Analysis
Thomas and Finney	:	Calculus and Geometry

LESSON NO. 1.2 **Author : Dr. Chanchal**

TRIPLE INTEGRATION

Structure :

2.0 Objectives

During the study in this particular lesson, we will focus on the idea of triple integration and volume integrals.

2.1 Introduction

As we have already studied about the double integration and surface integrals in the previous lesson. In this lesson, we will introduce about triple integration and volume integrals with its applications. Here, our function 'f' is a continuous function of three independent variables x, y and z and triple integration may be defined as –

Let f(x, y, z) be a continuous function of three independent variables, defined over a closed and bounded region enclosing a volume V in three dimensional space i.e. IR^3, then the volume integral is given by $\iiint_V f(x,y,z)\,dx\,dy\,dz$.

For evaluation, it can be expressed as the repeated integral

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)\,dxdydz \qquad \dots (1)$$

Where, the order of integration depends upon the limits.

Let $z_1 = f_1(x, y)$ and $z_2 = f_2(x, y)$; $y_1 = \phi_1(x)$ and $y_2 = \phi_2(x)$ and x_1, x_2 be constants say $x_1 = a$, $x_2 = b$.

Then, (1) can be written as

$$\int_{x=a}^{x=b}\left[\int_{y=\phi_1(x)}^{y=\phi_2(x)}\left(\int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x,y,z)\,dz\right)dy\right]dx$$

i.e. We integrate f (x, y, z) first w.r.t z (treating x, y as constants), then resulting expression w.r.t.y (keeping x constant) and finally w.r.t.x.

When x_1, x_2, y_1, y_2 and z_1, z_2 are constant then the order of the integration is immaterial i.e.

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)\,dx\,dy\,dz = \int_{x_1}^{x_2}\int_{z_1}^{z_2}\int_{y_1}^{y_2} f(x,y,z)\,dy\,dz\,dx$$

$$= \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dz\,dy\,dx$$

We clerify the concept with the help of an example.

Example 2.1 : Compute the integral $\iiint\limits_V xyz\,dx\,dy\,dz$ over a domain bounded by $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$.

Sol. Domain is bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$

$\because$ $x + y + z \le 1$

$\therefore$ $x \le 1$, $x + y \le 1$, $x + y + z \le 1$

$\therefore$ $x \le 1$, $y \le 1 - x$, $z \le 1 - x - y$

$\therefore$ $V = \{(x, y, z) : 0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le 1 - x - y\}$

$\therefore$ $$\iiint\limits_V xyz\,dx\,dy\,dz = \int_0^1\int_0^{1-x}\int_0^{1-x-y} xyz\,dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} xy\left(\int_0^{1-x-y} z\,dz\right)dy\,dx = \int_0^1\int_0^{1-x} xy\left[\frac{z^2}{2}\right]_0^{1-x-y} dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^{1-x} xy\,(1-x-y)^2\,dy\,dx = \frac{1}{2}\int_0^1 x\left\{\int_0^{1-x} y(1-x-y)^2\,dy\right\}dx$$

$$= \frac{1}{2}\int_0^1 x\left\{\left[y\,\frac{(1-x-y)^3}{-3}\right]_0^{1-x} - \int_0^{1-x} 1\,.\,\frac{(1-x-y)^3}{-3}\,dy\right\}dx$$

$$= \frac{1}{2}\int_0^1 x\left[(0-0)+\frac{1}{3}\int_0^{1-x}(1-x-y)^3\,dy\right]dx = \frac{1}{6}\int_0^1 x\left[\frac{(1-x-y)^4}{-4}\right]_0^{1-x} dx$$

$$= -\frac{1}{24}\int_0^1 x[(1-x-y)]_0^{1-x}\,dx = -\frac{1}{24}\int_0^1 x[0-(1-x)^4]\,dx$$

$$= -\frac{1}{24}\int_0^1 x(1-x)^4\,dx = \frac{1}{24}\left[x\left\{\frac{(1-x)^5}{-5}\right\}\right]_0^1 - \frac{1}{24}\int_0^1 1\,.\,\frac{(1-x)^5}{-5}\,dx$$

$$= -\frac{1}{120}(0-0)+\frac{1}{120}\left[\frac{(1-x)^6}{-6}\right]_0^1 = 0-\frac{1}{720}[(1-x)^6]_0^1 = -\frac{1}{720}(0-1) = \frac{1}{720}.$$

2.2 Change of Variables

Let V be a simple closed sub-region of the xyz space and $f : V \to R$ be integrable over V. If the region V of the xyz space is mapped on the region V' of the uvw space by the transformations $x = f_1(u, v, w)$, $y = f_2(u, v, w)$, $z = f_y, (u, v, w)$ and

(i) f_1, f_2, f_3 have continuous partial derivatives on V

(ii) $$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \neq 0 \ \forall\ (u,v,w) \in V'. \text{ then}$$

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_V f(f_1,f_2,f_3)\,|\,J\,|\,du\,dv\,dw.$$

Particular Case : (i) Change to Cylindrical Coordinates

Here, $x = r\cos\theta$, $y = r\sin\theta$, $z = Z$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$= r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r(1) = r$

$\therefore \quad \iiint_V f(x, y, z)\, dx\, dy\, dz = \iiint_V f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz.$

2. Change to Spherical Coordinates

Here, $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\phi$

$$\therefore \quad J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\phi \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} = r^2\sin\phi$$

$$\therefore \quad \iiint_V f(x, y, z)\, dx\, dy\, dz = \iiint_V f(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)\, r^2\sin\phi\, dr\, d\theta\, d\phi.$$

Example 2.2 : Evaluate $\iiint_R (ax^2 + by^2 + cz^2)\, dx\, dy\, dz$ where R is the region $x^2 + y^2 + z^2 \leq 1$.

Sol. Since the region of integration is a ball bounded by the sphere $x^2 + y^2 + z^2 = 1$, so we change to spherical coordinates by substituting

$x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\phi$

$\therefore \quad V = \{(r, \phi, \theta): 0 \leq r \leq 1, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}$

$$\therefore \quad \iiint_R (ax^2 + by^2 + cz^2)\, dx\, dy\, dz = a\int_0^1\int_0^{2\pi}\int_0^{\pi} (r\sin\phi\cos\theta)^2 \,.\, r^2\sin\phi\, dr\, d\theta\, d\phi$$

$$+ b\int_0^1\int_0^{2}\int_0^{\pi} (r\sin\phi\sin\theta)^2 \,.\, r^2\sin\phi\, dr\, d\theta\, d\phi$$

$$+ c\int_0^1\int_0^{2\pi}\int_0^{\pi} (r\cos\phi)^2 \,.\, r^2\sin\phi\, dr\, d\theta\, d\phi$$

$$= a\int_0^1 r^4 dt \int_0^{2\pi} \cos^2\theta\, d\theta \int_0^{\pi} \sin^3\phi\, d\phi + b\int_0^1 r^4 dr \int_0^{2\pi} \sin^2\theta\, d\theta \int_0^{\pi} \sin^3\phi\, d\phi$$

$$+c\int_0^1 r^4 dr \int_0^{2\pi} d\theta \int_0^{\pi} \cos^2\phi \sin\phi \, d\phi$$

$$= a\left[\frac{r^5}{5}\right]_0^1 . \, 4\int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta \, . \, 2\int_0^{\frac{\pi}{2}} \sin^3\phi \, d\phi$$

$$+b\left[\frac{r^5}{5}\right]_0^1 . \, 4\int_0^{\frac{\pi}{2}} \sin^2\theta \, d\theta \, . \, 2\int_0^{\frac{\pi}{2}} \sin^3\phi \, d\phi + c\left[\frac{r^5}{5}\right]_0^1 [\theta]_0^{2\pi} \, . \, 2\left[-\frac{\cos^3\phi}{3}\right]_0^{\frac{\pi}{2}}$$

$$= a\left(\frac{1}{5}-0\right) . \, 4\frac{1}{2}\times\frac{\pi}{2} . 2\times\frac{2}{3} + b\left(\frac{1}{5}-0\right) . \, 4\frac{1}{2}\times\frac{\pi}{2} . 2\times\frac{2}{3}$$

$$+c\left(\frac{1}{5}-0\right) . \left(2\pi-0\right) . \left(\frac{2}{3}\right)(0+1)$$

$$= \frac{a}{5}\times\pi\times\frac{4}{3} + \frac{b}{5}\times\pi\times\frac{4}{3} + \frac{c}{5}\times 2\pi\times\frac{2}{3} = \frac{4\pi}{15}(a+b+c) \cdot$$

2.3 Calculation of Volume

(i) In case of cartesian coordinates, $V = \iiint dx\,dy\,dz$

$\therefore$ element of volume is $\delta V = \delta x \, \delta y \, \delta z$

(ii) In case of cylindrical coordinates, $V = \iiint r\,d\theta\,dr\,dz$

$\therefore$ element of volume is $\delta V = r\delta\theta \, \delta r \, \delta z$

(iii) In case of spherical coordinates, $V = \iiint r^2 \sin\phi \, dr\,d\phi\,d\theta$

$\therefore$ element of volume is $\delta V = r^2 \sin\phi \, \delta r \, \delta\phi \, \delta\theta$

where V stands for volume of the region.

Example 2.3 : Prove that the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $\frac{4\pi}{3}abc$.

Sol. Required volume $= \iiint dx\,dy\,dz$

over the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Put $\frac{x}{a} = u, \frac{y}{b} = v, \frac{z}{c} = w$

$\Rightarrow$ dx = a du, dy = b dv, dz = c dw

$\therefore$ volume $= \iiint abc\, du\, dv\, dw$

over the sphere $u^2 + v^2 + w^2 = 1$

Changing to spherical coordinates by the relations

$u = r \sin\phi \cos\theta,\ v = r \sin\phi \sin\theta,\ w = r \cos\phi$

$\therefore$ volume $= abc \iiint r^2 \sin\phi\, dr\, d\phi\, d\theta$

over the region $\{(r, \phi, \theta) : 0 \le r \le 1,\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\}$

$$= abc \int_0^1 r^2 dr \int_0^\pi \sin\phi\, d\phi \int_0^{2\pi} d\theta = abc \left[\frac{r^2}{3}\right]_0^1 \left[-\cos\phi\right]_0^\pi \left[\theta\right]_0^{2\pi}$$

$$= abc\left(\frac{1}{3} - 0\right)(-\cos\pi + \cos 0)\,.\,(2\pi - 0)$$

$$= abc \times \frac{1}{3} \times 2 \times 2\pi = \frac{4\pi}{3} abc.$$

2.4 Calculation of Centre of Gravity and Centroid in IR³

Let a mass M be continuously distributed with density $\mu = \mu(x, y, z)$ over a region of the xyz-space.

If $(\bar{x}, \bar{y}, \bar{z})$ is the centroid, then

(1) $M = \iiint_A \mu(x, y, z)\, dx\, dy\, dz$

(2) $\bar{x} = \frac{1}{M} \iiint_A x\mu(x, y, z)\, dx\, dy\, dz$

(3) $\bar{y} = \frac{1}{M} \iiint_A y\mu(x, y, z)\, dx\, dy\, dz$

(4) $\bar{z} = \frac{1}{M} \iiint_A z\mu(x, y, z)\, dx\, dy\, dz$

Example 2.4 : Find the centre of mass a solid cubical box bounded by the six planes $x = -1$, $x = 1$; $y = -1$, $y = 1$; $z = -1$ and $z = 1$ and density $\mu(x, y, z) = x^2$.

Or

Show that for the cube $\{(x, y, z) : |x| \le 1, |y| \le 1, |z| \le 1\}$ with mass density given by $\mu(x, y, z) = x^2$ the centre of gravity is $(0, 0, 0)$.

Sol. Let A denote the region of the cubical box.

$\therefore$ $A = \{(x, y, z) : -1 \le x \le 1, -1 \le y \le 1, -1 \le z \le 1\}$

Also $\mu(x, y, z) = x^2$

$$\therefore \quad M = \iiint_A \mu(x,y,z)\, dx\, dy\, dz = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} x^2 dx\, dy\, dz$$

$$= \int_{-1}^{1} x^2 dx \int_{-1}^{1} dy \int_{-1}^{1} dz = \left[\frac{x^3}{3}\right]_{-1}^{1} [y]_{-1}^{1} [z]_{-1}^{1}$$

$$= \frac{1}{3}(1+1)(1+1)(1+1) = \frac{8}{3}$$

$$\bar{x} = \frac{1}{M}\iiint_A \mu(x,y,z)\, x\, dx\, dy\, dz = \frac{1}{M}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} x^3\, dx\, dy\, dz$$

$$= \frac{1}{M}\int_{-1}^{1} x^3 dx \int_{-1}^{1} dy \int_{-1}^{1} dz = \frac{1}{M}\left[\frac{x^4}{4}\right]_{-1}^{1} [y]_{-1}^{1} [z]_{-1}^{1}$$

$$= \frac{1}{M}\frac{1}{4}(1-1)(1+1)(1+1) = \frac{3}{8} \times \frac{1}{4} \times 0 \times 2 \times 2 = 0$$

$$\bar{y} - \frac{1}{M}\iiint_A y\mu(x,y,z)\, dx\, dy\, dz = \frac{1}{M}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} x^2 y\, dx\, dy\, dz$$

$$= \frac{1}{M}\int_{-1}^{1} x^2 dx \int_{-1}^{1} y\, dy \int_{-1}^{1} dz = \frac{1}{M}\left[\frac{x^3}{3}\right]_{-1}^{1}\left[\frac{y^2}{2}\right]_{-1}^{1}[z]_{-1}^{1}$$

$$= \frac{1}{M}\frac{1}{3}(1+1) \cdot \frac{1}{2}(1-1)(1+1) = 0$$

$$\bar{z} = \frac{1}{M}\iiint_A z\, \mu(z,y,z)\, dx\, dy\, dz = \frac{1}{M}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} zx^2\, dx\, dy\, dz$$

$$= \frac{1}{M}\int_{-1}^{1} x^2 dx \int_{-1}^{1} dy \int_{-1}^{1} z\, dz = \frac{1}{M}\left[\frac{x^3}{3}\right]_{-1}^{1} [y]_{-1}^{1} \left[\frac{z^2}{2}\right]_{-1}^{1}$$

$$= \frac{1}{M}\left[\frac{1}{3}+\frac{1}{3}\right][1+1]\left[\frac{1}{2}-\frac{1}{2}\right] = 0$$

$\therefore$ centre of mass is (0, 0, 0).

2.5 Moment of Inertia in IR³

Let mass m be distributed continuously with density μ (x, y, z) in a region A. Then

(i) $$I_x = \iiint_A \left(y^2 + z^2\right) \mu\,(x, y, z)\, dx\, dy\, dz$$

(ii) $$I_y = \iiint_A \left(z^2 + x^2\right) \mu\,(x, y, z)\, dx\, dy\, dz$$

(iii) $$I_z = \iiint_A \left(x^2 + y^2\right) \mu\,(x, y, z)\, dx\, dy\, dz$$

Note : If I_x and I_y be the moment of inertia of a plane lamina about the perpendicular lines x-axis and y-axis, then its M.I. about z-axis I_z is given by $I_z = I_x + I_y$.

Example 2.5 : Find the moment of inertia of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1$ with uniform mass density 1 about the x-axis.

Sol. Here μ (x, y, z) = 1

$$V = \left\{(x, y, z) : \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1\right\}$$

Put $\frac{x}{a} = u, \frac{y}{b} = v, \frac{z}{c} = w$

i.e., $x = au,\ y = bv,\ z = cw$

$\therefore$ $dx = a\,du,\ dy = b\,dv,\ dz = c\,dw \Rightarrow dx\,dy\,dz = abc\,du\,dv\,dw$

where $V = \{(u, v, w) : u^2 + v^2 + w^2 \le 1\}$

Changing to polar spherical coordinates by substituting

$u = r \sin\phi \cos\theta,\ v = r \sin\phi \sin\theta,\ w = r\cos\phi$

$\therefore$ $du\,d\theta\,dw = r^2 \sin\phi\, dr\, d\phi\, d\theta$

and $V = \{(r, \phi, \theta): 0 \le r \le 1,\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\}$

Now $I_x = \iiint\limits_V (y^2 + z^2)\,\mu\,(x, y, z)\,dx\,dy\,dz = abc \iiint\limits_V (b^2v^2 + c^2w^2)\,du\,dv\,dw$

$$= abc \int_0^1 \int_0^{2\pi} \int_0^{\pi} (b^2 .\, r^2 \sin^2\phi \sin^2\theta + c^2r^2\cos^2\phi)\,.\, r^2 \sin\phi\, dr\, d\phi\, d\theta$$

$$= abc\left[b^2 \int_0^1 r^4 dr \int_0^{\pi} \sin^3\phi\, d\phi \int_0^{2\pi} \sin^2\theta\, d\theta + c^2 \int_0^1 r^4 dr \int_0^{\pi} \cos^2\phi \sin\phi\, d\phi\,.\int_0^{2\pi} d\theta \right]$$

$$= abc\left[b^2 \frac{1}{5}.2 \int_0^{\pi/2} \sin^3\phi d\phi.4 \int_0^{\pi/2} \sin^2\theta d\theta + c^2.\frac{1}{5}.2 \int_0^{\pi/2} \cos^2\phi \sin\phi d\phi. \int_0^{2\pi} d\theta \right]$$

$$= abc\left[b^2.\frac{1}{5}.2.\frac{2}{3.1}.4.\frac{1}{2}.\frac{\pi}{2} + c^2.\frac{1}{5}.2.\frac{1}{3}.2\pi \right] = abc\left[\frac{4\pi b^2}{15} + \frac{4\pi c^2}{15} \right]$$

$$= \frac{4\pi\, abc}{15}(b^2 + c^2)\,.$$

2.6 Self Check Exercise

1. Evaluate $\int_{-1}^{1} \int_0^z \int_{x-z}^{x+z} (x + y + z)\,dy\,dx\,dz.$

2. Show that $\iiint (x^2 + y^2 + z^2)\,dx\,dy\,dz = \frac{4\pi}{5}$ over the region $x^2 + y^2 + z^2 \le 1$.

3. Show that the entire volume of the solid $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} + \left(\frac{z}{c}\right)^{\frac{2}{3}} = 1$ is $\frac{4\pi\, abc}{35}$.

4. Find the centroid of a tetrahedron with mass density 1 and bounded by planes x = 0, y = 0, z = 0, x + y + z = 2.

2.7 Suggested Readings

R. K. Jain, SRK Lyengar	:	Advanced Engineering Mathematics (Narosa Publications)
JR Sharma	:	Advanced Calculus
Malik and Arora	:	Mathematical Analysis
Shanti Narayan	:	Mathematical Analysis
Thomas and Finney	:	Calculus and Geometry